LAPLACE TRANSFORMS AND GENERALIZED LAGUERRE POLYNOMIALS

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1. Introduction. Various sets of necessary and sufficient conditions are known in order that a function $f(s)$, analytic for Re $s > 0$, be represented as the Laplace transform of a function in $L_p(0,\infty)$, $1 < p \leqslant \infty$. Most of these theories are based on the properties of some inversion operator for the transformation—see, for example, (**7**, chap. 7). However in the case $p = 2$ a number of representation theorems of a much simpler type are available. One of these is due to Shohat (**5**) who has in effect shown that a necessary and sufficient condition for such a representation, with $p = 2$, is that

$$\sum_{n=0}^{\infty} |q_n|^2 < \infty,$$

where

$$q_n = \sum_{r=0}^{n} \binom{n}{r} \frac{1}{r!} f^{(r)}(\tfrac{1}{2}).$$

Shohat's proof makes use of the Laguerre polynomials.

Recently the author has given (**4**) necessary and sufficient conditions that $f(s)$ be the Laplace transform of a function of the form $t^{\lambda}F(t)$, $F \in L_p(0, \infty)$, $1 < p \leqslant \infty$, $\lambda > -1/q$, where $p^{-1} + q^{-1} = 1$. These conditions were given in terms of a particular inversion operator. In this paper we shall see that Shohat's theorem can be generalized, for $p=2$, to cover this more general case. This is done in § 2 below, using generalized Laguerre polynomials. We also obtain there an expression for $F(t)$ which we shall use in § 3 to obtain some results about Hankel transforms. For convenience we write $\lambda = \frac{1}{2}\nu$ throughout the following.

2. Representation theorem. We start with a preliminary lemma.

LEMMA 1. *If* $f(s)$ *is analytic for* Re $s > 0$ *and* $\nu > -1$, *then*

$$f(s) = \frac{1}{(s+\frac{1}{2})^{\nu+1}} \sum_{n=0}^{\infty} q_n \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^n,$$

where

$$q_n = \sum_{r=0}^{n} \binom{n+\nu}{n-r} \frac{1}{r!} f^{(r)}(\tfrac{1}{2}),$$

the branch of $(s+\frac{1}{2})^{\nu+1}$ *that is positive when* $s + \frac{1}{2}$ *is positive being chosen.*

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Proof. Let

$$s = \tfrac{1}{2}\frac{1+z}{1-z},$$

and $f(s) = F(z)$. Then $F(z)$ is analytic in $|z| < 1$, and hence so is $F(z)/(1-z)^{\nu+1}$. Thus

$$F(z)/(1-z)^{\nu+1} = \sum_{n=0}^{\infty} q_n z^n, \qquad |z| < 1,$$

where if $r < 1$

$$\begin{aligned} q_n &= \frac{1}{2\pi i}\int_{|z|=r} (F(z)/z^{n+1}(1-z)^{\nu+1})dz \\ &= \text{Residue}_{z=0}\,(F(z)/z^{n+1}(1-z)^{\nu+1}) \\ &= \text{Residue}_{s=\frac{1}{2}}\,(f(s)\,(s+\tfrac{1}{2})^{n+\nu}/(s-\tfrac{1}{2})^{n+1}) \\ &= \frac{1}{n!}\lim_{s\to\frac{1}{2}}\left\{\frac{d^n}{ds^n}(f(s)\,(s+\tfrac{1}{2})^{n+\nu})\right\} \\ &= \frac{1}{n!}\lim_{s\to\frac{1}{2}}\left\{\sum_{r=0}^{n}\binom{n}{r} f^{(r)}(s)\,\frac{\Gamma(n+\nu+1)}{\Gamma(r+\nu+1)}(s+\tfrac{1}{2})^{r+\nu}\right\} \\ &= \sum_{r=0}^{n}\binom{n+\nu}{n-r}\frac{1}{r!}f^{(r)}(\tfrac{1}{2}). \end{aligned}$$

Hence

$$f(s) = F(z) = (1-z)^{\nu+1}\sum_{n=0}^{\infty} q_n z^n = \frac{1}{(s+\frac{1}{2})^{\nu+1}}\sum_{n=0}^{\infty} q_n\left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^n.$$

THEOREM 1. *A necessary and sufficient condition that a function $f(s)$, analytic for* Re $s > 0$, *be the Laplace transform of a function of the form $t^{\frac{1}{2}\nu}F(t)$, with $F \in L_2(0, \infty)$ and $\nu > -1$, is that*

$$\sum_{n=0}^{\infty}\frac{n!}{\Gamma(\nu+n+1)}|q_n|^2 < \infty$$

where

$$q_n = \sum_{r=0}^{n}\binom{n+\nu}{n-r}\frac{1}{r!}f^{(r)}(\tfrac{1}{2}).$$

In this case

$$F(t) = \underset{r\to\infty}{\text{l.i.m.}}\, t^{\frac{1}{2}\nu}e^{-\frac{1}{2}t}\sum_{n=0}^{r}\frac{n!}{\Gamma(\nu+n+1)}q_n L_n^{(\nu)}(t),$$

and

$$\sum_{n=0}^{\infty}\frac{n!}{\Gamma(\nu+n+1)}|q_n|^2 = \int_0^{\infty}|F(t)|^2\,dt.$$

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Proof of necessity. Suppose

$$f(s) = \int_0^\infty e^{-st} t^{\frac{1}{2}\nu} F(t)dt, \qquad F \in L_2(0, \infty), \nu > -1.$$

Let

$$\phi_n(t) = \left(\frac{n!}{\Gamma(\nu + n + 1)}\right)^{\frac{1}{2}} e^{-\frac{1}{2}t} t^{\frac{1}{2}\nu} L_n^{(\nu)}(t).$$

Then, as is well known, $\{\phi_n\}$ is a complete orthonormal sequence in $L_2(0, \infty)$. We have, using **(2**, §10.12(7)**)** and **(1**, chap. 3, §2**)**,

$$\begin{aligned}(F, \phi_n) &= \left(\frac{n!}{\Gamma(\nu + n + 1)}\right)^{\frac{1}{2}} \int_0^\infty e^{-\frac{1}{2}t} t^{\frac{1}{2}\nu} L_n^{(\nu)}(t)\, F(t)\, dt \\ &= \left(\frac{n!}{\Gamma(\nu + n + 1)}\right)^{\frac{1}{2}} \sum_{r=0}^{n} \binom{n+\nu}{n-r} \frac{1}{r!} \int_0^\infty e^{-\frac{1}{2}t}(-t)^r t^{\frac{1}{2}\nu} F(t)dt \\ &= \left(\frac{n!}{\Gamma(\nu + n + 1)}\right)^{\frac{1}{2}} \sum_{r=0}^{\infty} \binom{n+\nu}{n-r} \frac{1}{r!} f^{(r)}(\tfrac{1}{2}) = \left(\frac{n!}{\Gamma(\nu + n + 1)}\right)^{\frac{1}{2}} q_n.\end{aligned}$$

Hence

$$\begin{aligned}F(t) &= \underset{r\to\infty}{\text{l.i.m.}} \sum_{n=0}^{r} (F, \phi_n)\, \phi_n(t) \\ &= \underset{r\to\infty}{\text{l.i.m.}}\, t^{\frac{1}{2}\nu} e^{-\frac{1}{2}t} \sum_{n=0}^{r} \frac{n!}{\Gamma(\nu + n + 1)} q_n L_n^{(\nu)}(t),\end{aligned}$$

and from the Parseval relation

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(\nu + n + 1)} |q_n|^2 = \sum_{n=0}^{\infty} |(F, \phi_n)|^2 = \int_0^\infty |F(t)|^2\, dt < \infty .$$

Proof of sufficiency. Since

$$\sum_{n=0}^{\infty} \frac{n!}{\Gamma(\nu + n + 1)} |q_n|^2 < \infty ,$$

by the Riesz-Fischer theorem there is a function $F \in L_2(0, \infty)$ such that

$$(F, \phi_n) = q_n \left(\frac{n!}{\Gamma(\nu + n + 1)}\right)^{\frac{1}{2}}.$$

Let $G(t) = t^{\frac{1}{2}\nu} e^{-\bar{s}t}$, Re $s > 0$. Then $G \in L_2(0, \infty)$, and from **(3**, §4.11(28)**)**,

$$\begin{aligned}(G, \phi_n) &= \int_0^\infty t^{\frac{1}{2}\nu} e^{-\bar{s}t} \phi_n(t)\, dt \\ &= \left(\frac{n!}{\Gamma(\nu + n + 1)}\right)^{\frac{1}{2}} \int_0^\infty e^{-(\bar{s}+\frac{1}{2})t} t^{\nu} L_n^{(\nu)}(t)\, dt \\ &= \left(\frac{\Gamma(\nu + n + 1)}{n!}\right)^{\frac{1}{2}} \frac{(\bar{s} - \frac{1}{2})^n}{(\bar{s} + \frac{1}{2})^{n+\nu+1}}.\end{aligned}$$

Hence from Lemma 1 and Parseval's relation, if Re $s > 0$,

$$\begin{aligned} f(s) &= \frac{1}{(s+\frac{1}{2})^{\nu+1}} \sum_{r=0}^{\infty} q_n \left(\frac{s-\frac{1}{2}}{s+\frac{1}{2}}\right)^n \\ &= \sum_{n=0}^{\infty} \left\{ q_n \left(\frac{n!}{\Gamma(\nu+n+1)}\right)^{\frac{1}{2}} \right\} \left\{ \left(\frac{\Gamma(\nu+n+1)}{n!}\right)^{\frac{1}{2}} \frac{(s-\frac{1}{2})^n}{(s+\frac{1}{2})^{n+\nu+1}} \right\} \\ &= \sum_{n=0}^{\infty} (F, \phi_n)\, \overline{(G, \phi_n)} = (F, G) = \int_0^\infty e^{-st} t^{\frac{1}{2}\nu} F(t)\, dt. \end{aligned}$$

3. Application to Hankel transforms. For our purposes here we shall define the Hankel transform for $F \in L_2(0, \infty)$, $\nu > -1$, by

$$G(x) = \frac{d}{dx} \int_0^\infty k_\nu(xy)\, F(y)\, \frac{dy}{y}$$

where

$$k_\nu(x) = \int_0^x J_\nu(2\sqrt{y})\, dy.$$

Since the Mellin transform of $J_\nu(2\sqrt{y})$ is

$$\Gamma(s + \tfrac{1}{2}\nu)/\Gamma(\tfrac{1}{2}\nu - s + 1), \qquad -\tfrac{1}{2}\nu < \text{Re } s < 3/4,$$

it follows since $\nu > -1$ that the hypotheses of (**6**, Theorem 129) are satisfied so that if $F \in L_2(0, \infty)$, G exists and is in $L_2(0, \infty)$, and Parseval's equation holds. Further

$$F(x) = \frac{d}{dx} \int_0^\infty k_\nu(xy)\, G(y)\, \frac{dy}{y}.$$

Here we shall use the results of Theorem 1 to invert the Hankel transform. We first prove the following lemma (compare (**1**, chap 2 §16)).

LEMMA 2. *If* $F \in L_2(0, \infty)$, $\nu > -1$,

$$G(x) = \frac{d}{dx} \int_0^\infty k_\nu(xy)\, F(y)\, dy$$

where

$$k_\nu(x) = \int_0^x J_\nu(2\sqrt{y})\, dy,$$

$$f(s) = \int_0^\infty e^{-st} t^{\frac{1}{2}\nu} F(t) dt$$

and

$$g(s) = \int_0^\infty e^{-st} t^{\frac{1}{2}\nu} G(t) dt,$$

then

$$f(s) = \frac{1}{s^{\nu+1}}\, g(1/s).$$

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Proof. The Hankel transform of $t^{\frac{1}{2}\nu}e^{-st}$ is given, on using **(3**, §4.14(30)**)**, by

$$\begin{aligned}\frac{d}{dx}\int_0^\infty t^{\frac{1}{2}\nu-1}\, e^{-st}\, dt \int_0^{xt} J_\nu(2\sqrt{y})dy \\ &= \frac{d}{dx}\int_0^\infty t^{\frac{1}{2}\nu}\, e^{-st}\, dt \int_0^{x} J_\nu(2\sqrt{yt})dy \\ &= \frac{d}{dx}\int_0^x dy \int_0^\infty t^{\frac{1}{2}\nu}\, e^{-st}\, J_\nu(2\sqrt{yt})dt \\ &= \int_0^\infty t^{\frac{1}{2}\nu}\, e^{-st}\, J_\nu(2\sqrt{xt})dt \\ &= \frac{x^{\frac{1}{2}\nu}\, e^{-x/s}}{s^{\nu+1}},\end{aligned}$$

the interchange of the order of integrations being justified by Fubini's theorem.

Hence by the Parseval relation for the Hankel transform,

$$\begin{aligned}f(s) &= \int_0^\infty e^{-st}\, t^{\frac{1}{2}\nu}\, F(t)dt = \frac{1}{s^{\nu+1}}\int_0^\infty e^{-t/s}\, t^{\frac{1}{2}\nu}\, G(t)dt \\ &= \frac{1}{s^{\nu+1}}\, g\!\left(\frac{1}{s}\right).\end{aligned}$$

THEOREM 2. *If* $F \in L_2(0, \infty)$, $\nu > -1$,

$$G(x) = \frac{d}{dx}\int_0^\infty k_\nu(xy)\, F(y)\, \frac{dy}{y},$$

where

$$k_\nu(x) = \int_0^x J_\nu(2\sqrt{y})dy,$$

and

$$g(s) = \int_0^\infty e^{-st}\, t^{\frac{1}{2}\nu}\, G(t)dt,$$

then

$$F(t) = \underset{r\to\infty}{\text{l.i.m.}}\; t^{\frac{1}{2}\nu}\, e^{-\frac{1}{2}t} \sum_{n=0}^{r} q_n \frac{n!}{\Gamma(\nu+n+1)}\, L_n^{(\nu)}(t)$$

where

$$q_n = (-1)^n 2^{\nu+1} \sum_{r=0}^{n} \binom{n+\nu}{n-r} \frac{4^r}{r!}\, g^{(r)}(2).$$

Proof. By Theorem 1,

$$F(t) = \underset{r\to\infty}{\text{l.i.m.}}\; t^{\frac{1}{2}\nu}\, e^{-\frac{1}{2}t} \sum_{n=0}^{r} q_n \frac{n!}{\Gamma(\nu+n+1)}\, L_n^{(\nu)}(t)$$

where

$$q_n = \sum_{r=0}^{n} \binom{n+\nu}{n-r} \frac{1}{r!} f^{(r)}\left(\tfrac{1}{2}\right).$$

But in the proof of Lemma 1 we showed that

$$q_n = \text{Residue}_{s=\frac{1}{2}} \left(f(s)\,(s+\tfrac{1}{2})^{n+\nu}/(s-\tfrac{1}{2})^{n+1}\right),$$

and hence using Lemma 2

$$\begin{aligned} q_n &= \text{Residue}_{s=\frac{1}{2}} \left(\frac{1}{s^{\nu+1}}\, g\!\left(\frac{1}{s}\right)(s+\tfrac{1}{2})^{n+\nu}/(s-\tfrac{1}{2})^{n+1}\right) \\ &= \text{Residue}_{s=2} \frac{-1}{2^{\nu-1}} \left(g(s)\,(2+s)^{n+\nu}/(2-s)^{n+1}\right) \\ &= \frac{(-1)^n}{2^{\nu-1}\, n!} \lim_{s\to 2} \frac{d^n}{ds^n} \left(g(s)(2+s)^{n+\nu}\right) \\ &= \frac{(-1)^n}{2^{\nu-1}\, n!} \lim_{s\to 2} \sum_{r=0}^{n} \binom{n}{r} g^{(r)}(s) \frac{\Gamma(n+\nu+1)}{\Gamma(r+\nu+1)} (s+2)^{r+\nu} \\ &= (-1)^n\, 2^{\nu+1} \sum_{r=0}^{n} \binom{n+\nu}{n-r} \frac{4^r}{r!} g^{(r)}(2). \end{aligned}$$

COROLLARY. *Under the hypotheses of Theorem* 2, *if*

$$f(s) = \int_0^\infty e^{-st}\, t^{\frac{1}{2}\nu}\, F(t)dt,$$

then

$$G(t) = \underset{r\to\infty}{\text{l.i.m.}}\; t^{\frac{1}{2}\nu} e^{-\frac{1}{2}t} \sum_{n=0}^{r} q'_n \frac{n!}{\Gamma(\nu+n+1)} L_n^{(\nu)}(t)$$

where

$$q'_n = (-1)^n\, 2^{\nu+1} \sum_{r=0}^{n} \binom{n+\nu}{n-r} \frac{4^r}{r!} f^{(r)}(2).$$

Proof. This follows from Theorem 2 since the relation between F and G is reciprocal.

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