Proceedings of the Edinburgh Mathematical Society (2012) **55**, 513–541
DOI:10.1017/S0013091510001069

ON THE CONVERGENCE OF DIOPHANTINE DIRICHLET SERIES

T. RIVOAL*

Institut Fourier, CNRS UMR 5582, Université Grenoble 1, 100 rue des Maths, BP 74, 38402 Saint-Martin d'Hères cedex, France
(tanguy.rivoal@fourier.ujf-grenoble.fr)

(Received 4 August 2010)

Abstract We study various Dirichlet series of the form $\sum_{n\geqslant 1} f(\pi n\alpha)/n^s$, where α is an irrational number and $f(x)$ is a trigonometric function like $\cot(x)$, $1/\sin(x)$ or $1/\sin^2(x)$. The convergence is slow and strongly depends on the Diophantine properties of α. We provide necessary and sufficient convergence conditions using the continued fraction of α. We also show that any one of our series is equal to a related series, which converges much faster, defined in term of iterations of the continued fraction operator $\alpha \mapsto \{1/\alpha\}$.

Keywords: Diophantine approximation; Dirichlet series; continued fractions

2010 *Mathematics subject classification:* Primary 11J70; 11M41; 11K70

1. Introduction

We study the convergence of certain Diophantine Dirichlet series of the form

$$\sum_{n=1}^{\infty} \frac{f(\pi n\alpha)}{n^s}, \tag{1.1}$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $s \in \mathbb{R}$. We will consider functions f defined on $\mathbb{R} \setminus \mathbb{Z}$ and such that $\sup_{x\in\mathbb{R}\setminus\mathbb{Z}} |\sin^r(\pi x)f(x)| < \infty$ for some $r \geqslant 1$ but $\sup_{x\in\mathbb{R}\setminus\mathbb{Z}} |\sin^\rho(\pi x)f(x)| = +\infty$ for any $\rho < r$. Typically, f is a trigonometric function such as a power of the cotangent or cosecant functions. It is easy to find sufficient conditions on α and s that ensure convergence of such series for almost all real numbers α (see Proposition 3.1). These conditions are expressed in terms of the continued fraction of α, which explains the word 'Diophantine' in the title.

Sometimes, the sufficient conditions given in Proposition 3.1 are also necessary, but when they are not it is a difficult problem to find the exact convergence conditions. It is also difficult to accurately compute an approximate value of some specific example of (1.1), because such series usually converge slowly, with the speed of convergence

* Present address: Université de Lyon, CNRS et Université Lyon 1, Institut Camille Jordan, Bâtiment Braconnier, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France.

depending strongly on the Diophantine characteristics of α. An efficient alternative expression for (1.1) is thus desirable.

The main results of the paper address both issues for the series

$$\Phi_s(\alpha) := \sum_{n=1}^{\infty} \frac{\cot(\pi n\alpha)}{n^s} \quad \text{and} \quad \hat{\Phi}_s(\alpha) := \sum_{n=1}^{\infty} \frac{1}{n^s \sin^2(\pi n\alpha)}. \tag{1.2}$$

We note that $-\pi\hat{\Phi}_{s-1}(\alpha)$ is formally the derivative of $\Phi_s(\alpha)$ with respect to α, but $\Phi_s(\alpha)$ is in fact nowhere continuous, which is a source of difficulty in the study undertaken here. We will prove that, for $s > 2$, the series $\Phi_s(\alpha)$ and $\hat{\Phi}_s(\alpha)$ converge if and only if

$$\sum_{j=0}^{\infty} (-1)^j \frac{q_{j+1}(\alpha)}{q_j^s(\alpha)} \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{q_{j+1}^2(\alpha)}{q_j^s(\alpha)}, \tag{1.3}$$

respectively, converge, where $q_n(\alpha)$ is the denominator of the nth convergent to α. We will also prove alternative expressions for both series in (1.2); roughly speaking, these identities provide an exact expression for the difference between a series in (1.2) and a modified version of (1.3). The general results are presented in §§ 1.2 and 1.3, respectively.

In certain cases where s is an integer, these alternative expressions are quite simple and enable one to compute numerical approximations of both series much faster than with the series in (1.2). For example, where $G_3(x) = \pi^3(x^4 - 5x^2 + 1)/(90x)$ and $T^j(\alpha)$ is the jth iterate of $T(\alpha) := \{1/\alpha\}$, with $\{\cdot\}$ the fractional part function. The right-hand side is directly related to the series $\sum_{j\geqslant 0}(-1)^j q_{j+1}(\alpha)/q_j^3(\alpha)$.

In § 1.4, we obtain sufficient convergence conditions and acceleration identities for the three Diophantine Dirichlet series

$$\left.\begin{aligned}
\Psi_s(\alpha) &:= \sum_{n=1}^{\infty} (-1)^n \frac{\cot(\pi n\alpha)}{n^s}, \\
\hat{\Psi}_s(\alpha) &:= \sum_{n=1}^{\infty} \frac{1}{n^s \sin(\pi n\alpha)}, \\
\tilde{\Psi}_s(\alpha) &:= \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s \sin(\pi n\alpha)}.
\end{aligned}\right\} \tag{1.4}$$

These series are tied together in our study. Generalizations of (1.2) (in particular, a multivariate generalization where the Jacobi–Perron algorithm naturally appears) are briefly discussed in § 7.

The study of the convergence at irrational points of 'Diophantine' Dirichlet or trigonometric series is a classical subject, studied in particular by Hardy and Littlewood [**13**, **14**, **16**], Chowla [**7**], Davenport [**8**, **9**] and Walfisz [**29**]. In [**16**], the series $\hat{\Psi}_s(\alpha)$ and $\tilde{\Psi}_s(\alpha)$ are studied for $s = 1$ and $s = 0$ (when they diverge) for quadratic numbers $\alpha = \sqrt{a^2+1}$, a an odd integer; these series also played a role in the recent paper [**4**]. Our approach, which is different, might also be used in the situation of [**16**].

With $\sigma_s(n) = \sum_{d|n} d^s$, Wilton [**30**] proved that the trigonometric series

$$\sum_{n=1}^{\infty} \frac{\sigma_0(n)}{n} \cos(2\pi n\alpha) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sigma_0(n)}{n} \sin(2\pi n\alpha)$$

converge for a given irrational number α if and only if

$$\sum_{j=0}^{\infty} \frac{\log^2(q_{j+1}(\alpha))}{q_j(\alpha)} \quad \text{and} \quad \sum_{j=0}^{\infty} (-1)^j \frac{\log(q_{j+1}(\alpha))}{q_j(\alpha)},$$

respectively, converge. In principle, his method could even provide an explicit formula (like the above one for $\Phi_3(\alpha)$). Such questions have recently gained renewed interest, after the introduction in [**11**] of the P-summation in the study of 'Davenport's identities' (which in short ask 'for which irrational numbers are two not-everywhere convergent series equal?' [**8**, **9**]), a method subsequently used in [**21**] for a similar purpose. A study of the fine analytic properties of the Brjuno series

$$\sum_{j\geqslant 0} \frac{\log(q_{j+1}(\alpha))}{q_j(\alpha)}$$

was recently made in [**3**], in connection with a problem raised in [**2**]. In [**26**, **27**], it is proved that Bundschuh's series defined by $\sum_{n=1}^{\infty} (-1)^{[2n\alpha]}/n$ converges if and only if

$$\sum_{j\geqslant 0, 2\nmid q_j(\alpha)} (-1)^j \frac{\log(q_{j+1}(\alpha))}{q_j(\alpha)}$$

converges, and that its Fourier series is

$$\frac{4}{\pi} \sum_{n\geqslant 1} \frac{1}{n} \sigma_{\mathrm{odd}}(n) \sin(2\pi n\alpha),$$

where $\sigma_{\mathrm{odd}}(n)$ is the number of odd divisors of n and $[\cdot]$ is the integer part function. A 'Diophantine' expression for Bundschuh's series could probably be derived from the study in [**26**]. Finally, let us mention that in [**10**] Davenport gave a new proof of an identity due to Hardy and Littlewood [**15**] for Hecke's series $\sum_{n=1}^{\infty} [n\alpha]/n^s$ (in the spirit of the above expression for $\Phi_3(\alpha)$), as well as a proof of the related identity

$$\sum_{n=1}^{\infty} [n\alpha] x^n = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{x^{q_j(\alpha)+q_{j-1}(\alpha)}}{(1 - x^{q_j(\alpha)})(1 - x^{q_{j-1}(\alpha)})}.$$

This identity is useful for studying the arithmetic nature of the values of the power series on the left-hand side [**22**]. It is possible that the identities proved in this paper could also be used to determine the arithmetic nature of the values of the series $\Phi_s(\alpha)$.

1.1. Properties of continued fraction expansions

Before going further, we recall a few properties of continued fractions. A classical reference is [**17**].

For any real number α such that $0 < \alpha < 1$, we define the classical 'continued fraction operator' $T(\alpha) = \{1/\alpha\}$, which maps $(0,1)$ into itself. This operator generates the regular continued fraction

$$[a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$$

of α, where $a_j := [T^j(\alpha)]$. We also define $p_n(\alpha)$ and $q_n(\alpha)$ as the numerator and denominator, respectively, of the nth convergent $[a_0; a_1, a_2, \dots, a_n]$ to α; in particular, $p_{-1}(\alpha) = 1$, $q_{-1}(\alpha) = 0$ and $p_0(\alpha) = 0$, $q_0(\alpha) = 1$. (For the sake of better readability, we will not always mention the dependence of $q_n(\alpha)$ and $p_n(\alpha)$ on α.) Both sequences satisfy the same recurrence relation: for any $n \geqslant 0$, $p_{n+1} = a_{n+1}p_n + p_{n-1}$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$. It is useful to have in mind that $p_n(T^k(\alpha)) = p_{n+k}(\alpha)$ and $q_n(T^k(\alpha)) = q_{n+k}(\alpha)$ for any integers $k, n \geqslant 0$, because the continued fraction of $T^k(\alpha)$ is $[a_k; a_{k+1}, a_{k+2}, \dots]$. We also have

$$T^k(\alpha) = \frac{q_k\alpha - p_k}{p_{k-1} - q_{k-1}\alpha} = \frac{|q_k\alpha - p_k|}{|q_{k-1}\alpha - p_{k-1}|}$$

and

$$\alpha T(\alpha) \cdots T^k(\alpha) = (-1)^k(q_k\alpha - p_k) = |q_k\alpha - p_k|.$$

Furthermore, $\frac{1}{2} \leqslant q_{k+1}|q_k\alpha - p_k| \leqslant 1$, so that, for any real numbers r, s,

$$\frac{|q_{k-1}\alpha - p_{k-1}|^s}{|p_k - q_k\alpha|^r} \asymp \frac{q_{k+1}^r}{q_k^s}. \tag{1.5}$$

For any irrational α, we have $q_j \gg 2^{j/2}$, hence for any real number $s > 0$, the series $\sum_k |q_k\alpha - p_k|^s$ is convergent. Finally, the function $T(\alpha)$ is differentiable at any irrational point α and at any rational point of $[0,1]$ that is not the inverse of an integer, where $T'(\alpha) = -1/\alpha^2$. It follows that $(T^j(\alpha))' = (-1)^j/(\alpha T(\alpha) \cdots T^{j-1}(\alpha))^2$ for any integer $j \geqslant 1$ (as induction easily shows), and that

$$(\alpha T(\alpha) \cdots T^j(\alpha))' = \alpha T(\alpha) \cdots T^j(\alpha) \sum_{k=0}^{j} (-1)^k \frac{T^k(\alpha)}{(\alpha T(\alpha) \cdots T^k(\alpha))^2} \tag{1.6}$$

at any irrational α, which follows by logarithmic differentiation and by the above formula for $(T^j(\alpha))'$.

1.2. The series $\boldsymbol{\Phi_s(\alpha)}$

Henceforth, $z^s = \exp(s\log(z))$, where $\log(z)$ is defined with its principal branch and $|\arg(z)| < \pi$.

For any real numbers $s > 1$ and $\alpha \neq 0$, set $F_s(\alpha, z) := \pi \cot(\pi z) \cot(\pi \alpha z)/z^s$, which is a meromorphic function of z in the cut plane $\mathbb{C} \setminus (-\infty, 0]$. We also define

$$G_s(\alpha) := \frac{1}{2\mathrm{i}\pi} \int_{1/2+\mathrm{i}\infty}^{1/2-\mathrm{i}\infty} F_s(\alpha, z)\,\mathrm{d}z. \tag{1.7}$$

An explicit expression of $G_s(\alpha)$ is not known in general, except when $s = 2n+1$ is an odd integer greater than or equal to 3:

$$G_{2n+1}(\alpha) = (-1)^{n+1}(2\pi)^{2n+1} \sum_{j=0}^{n+1} \frac{B_{2j} B_{2n+2-2j}}{(2j)!(2n+2-2j)!} \alpha^{2j-1}, \tag{1.8}$$

where the B_n are the Bernoulli numbers.* See the end of § 4 for the proof of (1.8).

For any $s > 1$, we have

$$G_s(\alpha) = \frac{\zeta(s+1)}{\pi\alpha} + P_s(\alpha), \tag{1.9}$$

where the P_s are bounded functions of $\alpha \in [0, 1]$. This will also be proved in § 4.

Theorem 1.1. *We fix a real number $s > 2$ and an irrational number $\alpha \in (0, 1)$. The series $\Phi_s(\alpha)$ converges if and only if*

$$\sum_{j=0}^{\infty} (-1)^j \frac{q_{j+1}(\alpha)}{q_j^s(\alpha)} \tag{1.10}$$

converges. Furthermore, we have the identity

$$\sum_{n=1}^{\infty} \frac{\cot(\pi n \alpha)}{n^s} = \sum_{j=0}^{\infty} (-1)^j |q_{j-1}\alpha - p_{j-1}|^{s-1} G_s(T^j(\alpha)), \tag{1.11}$$

where both series converge (or diverge) simultaneously.

Remarks. Equation (1.9) implies that the right-hand side of (1.11) converges if and only if

$$\sum_{j=0}^{\infty} (-1)^j \frac{|q_{j-1}\alpha - p_{j-1}|^s}{|q_j\alpha - p_j|}$$

converges (because, as stated above, $\sum_{j=0}^{\infty} |q_{j-1}\alpha - p_{j-1}|^s$ converges for any $s > 0$). We will show that this is equivalent to the convergence of (1.10) when $s > 2$. For $s > 2$, it follows from (1.10) that $\Phi_s(\alpha)$ converges for almost all α because, for any $\varepsilon > 0$, the inequality $q_{j+1}(\alpha) \ll_{\alpha,\varepsilon} q_j(\alpha)^{1+\varepsilon}$ holds for all $j \geqslant 0$ for almost all α (see § 2.2 for details). If we assume that $s > 1$, it follows from Proposition 3.1 that $\Phi_s(\alpha)$ converges when $\sum_{j=0}^{\infty} q_{j+1}(\alpha)/q_j^s(\alpha)$ converges. In fact, it is likely that Theorem 1.1 holds with the weaker assumption that $s > 1$. However, we cannot prove this.

* The polynomials $R_{2n+1}(\alpha) := (-1)^{n+1}(2\pi)^{-(2n+1)}\alpha G_{2n+1}(\alpha)$ form the sequence of *Ramanujan polynomials*, whose analytic properties have recently been studied in [**23**]. As the name suggests, these polynomials first appeared in certain formulae in Ramanujan's notebooks on values of the zeta function at odd integers: see, for example, [**5**,**12**]. Ramanujan's formulae express $\zeta(4n+3)$ in terms of $\pi^{4n+3}R_{4n+3}(i)$ and of the series $\sum_{k\geqslant 1} k^{-(4n+3)}/(\mathrm{e}^{2\pi k} - 1)$. The presence of the polynomials $R_{2n+1}(\alpha)$ in the present paper is not a coincidence.

1.3. The series $\hat{\Phi}_s(\alpha)$

For any real numbers $s > 1$ and $\alpha \neq 0$, set

$$\hat{F}_s(\alpha, z) := \frac{\pi \cot(\pi z)}{z^s \sin^2(\pi\alpha z)} = -\frac{1}{\pi}\frac{\partial}{\partial\alpha}F_{s+1}(\alpha, z),$$

which is meromorphic in the cut plane $\mathbb{C} \setminus (-\infty, 0]$. We also define a function $\hat{G}_s(\alpha)$ by

$$\hat{G}_s(\alpha) := -\frac{1}{\pi}\frac{\partial G_{s+1}(\alpha)}{\partial\alpha},$$

so that

$$\hat{G}_s(\alpha) = \frac{1}{2\mathrm{i}\pi}\int_{1/2+\mathrm{i}\infty}^{1/2-\mathrm{i}\infty} \hat{F}_s(\alpha, z)\,\mathrm{d}z.$$

In general, no explicit evaluation of $\hat{G}_s(\alpha)$ is known, except when $s = 2n$ is an even integer greater than or equal to 2 (by (1.8)):

$$\hat{G}_{2n}(\alpha) = (-1)^n 2^{2n+1}\pi^{2n}\sum_{j=0}^{n+1}\frac{(2j-1)B_{2j}B_{2n+2-2j}}{(2j)!(2n+2-2j)!}\alpha^{2j-2}. \tag{1.12}$$

For any $s > 1$, we have

$$\hat{G}_s(\alpha) = \frac{\zeta(s+2)}{(\pi\alpha)^2} + Q_s(\alpha), \tag{1.13}$$

where the Q_s are bounded functions of $\alpha \in [0, 1]$.

Theorem 1.2. *We fix a real number $s > 2$ and an irrational number $\alpha \in (0, 1)$. The series $\hat{\Phi}_s(\alpha)$ converges if and only if*

$$\sum_{j=0}^{\infty}\frac{q_{j+1}^2(\alpha)}{q_j^s(\alpha)} \tag{1.14}$$

converges. Furthermore, we have the identities

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n^s\sin^2(\pi n\alpha)} \\
&\quad = -\frac{1}{\pi}\sum_{j=0}^{\infty}(-1)^j\frac{\partial}{\partial\alpha}(|q_{j-1}\alpha - p_{j-1}|^s G_{s+1}(T^j(\alpha))), && (1.15\,a) \\
&\quad = \sum_{j=0}^{\infty}\biggl(|q_{j-1}\alpha - p_{j-1}|^{s-2}\hat{G}_s(T^j(\alpha)) \\
&\qquad\qquad + (-1)^{j+1}\frac{s}{\pi}|q_{j-1}\alpha - p_{j-1}|^s G_{s+1}(T^j(\alpha))\sum_{k=0}^{j-1}\frac{(-1)^k T^k(\alpha)}{(q_k\alpha - p_k)^2}\biggr), && (1.15\,b)
\end{aligned}$$

where the three series converge (or diverge) simultaneously.

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

Remarks. From the first assertion, it follows that $\hat{\varPhi}_s(\alpha)$ converges for almost all α. By Lemma 3.2, $\hat{\varPhi}_2(\alpha)$ diverges for all α.

For any $j \geqslant 0$, the function $|q_{j-1}\alpha - p_{j-1}|^s G_{s+1}(T^j(\alpha))$ is differentiable at any irrational point $\alpha \in (0,1)$, so that the summand on the right-hand side of (1.15 a) is well defined for any irrational number. Equation (1.15 b) is obtained by means of (1.6). By definition, we also have

$$\hat{\varPhi}_s(\alpha) = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\partial}{\partial\alpha}\left(\frac{\cot(\pi n\alpha)}{n^{s+1}}\right),$$

which means that (1.15 a) is the result of a formal differentiation applied to (1.11) (with s changed to $s+1$). However, neither $\varPhi_s(\alpha)$ nor the right-hand side of (1.11) are continuous at a single point of $(0,1)$ and there does not seem to be any general analytic result enabling us to quickly deduce (1.15 a) from (1.11). Instead, we use an ad hoc method.

1.4. The series $\varPsi_s(\alpha)$, $\hat{\varPsi}_s(\alpha)$ and $\tilde{\varPsi}_s(\alpha)$

The two series $\varPhi_s(\alpha)$ and $\hat{\varPhi}_s(\alpha)$ do not exhaust the possible use of our method. In this section, we discuss the three series defined by (1.4). We obtain only sufficient conditions for the convergence of the Diophantine Dirichlet series involved and prove acceleration identities similar to those proved in Theorems 1.1 and 1.2.

For any real numbers $s > 1$ and $\alpha \neq 0$, we set

$$P_s(\alpha, z) = \frac{\pi\cot(\pi\alpha z)}{z^s\sin(\pi z)} \quad \text{and} \quad Q_s(\alpha, z) = \frac{\pi}{z^s\sin(\pi z)\sin(\pi\alpha z)}.$$

We also define three functions:

$$U_s(\alpha) := \frac{1}{2\mathrm{i}\pi}\int_{1/2+\mathrm{i}\infty}^{1/2-\mathrm{i}\infty} P_s(\alpha, z)\,\mathrm{d}z,$$

$$V_s(\alpha) := \alpha^{s-1}U_s\left(\frac{1}{\alpha}\right),$$

$$W_s(\alpha) := \frac{1}{2\mathrm{i}\pi}\int_{1/2+\mathrm{i}\infty}^{1/2-\mathrm{i}\infty} Q_s(\alpha, z)\,\mathrm{d}z.$$

There are no known explicit evaluations of these integrals, except when $s = 2n+1$ is an odd integer greater than or equal to 3:

$$U_{2n+1}(\alpha) = (-1)^{n+1}\pi^{2n+1}\sum_{j=0}^{n+1}\frac{(2^{2n+1}-2^{2j})B_{2j}B_{2n+2-2j}}{(2j)!(2n+2-2j)!}\alpha^{2j-1}, \tag{1.16}$$

$$W_{2n+1}(\alpha) = (-1)^{n}\pi^{2n+1}\sum_{j=0}^{n+1}(2^{2j}-2)(2^{2n+1-2j}-1)\frac{B_{2j}B_{2n+2-2j}}{(2j)!(2n+2-2j)!}\alpha^{2j-1}. \tag{1.17}$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

Given a function $X_0 \in \{U_s, V_s, W_s\}$, we define a sequence $(X_j)_{j\geqslant 0}$ by the following five rules:

1. if $X_{j-1} = U_s$ and a_j is even, then $X_j = V_s$;
2. if $X_{j-1} = U_s$ and a_j is odd, then $X_j = W_s$;
3. if $X_{j-1} = W_s$ and a_j is even, then $X_j = W_s$;
4. if $X_{j-1} = W_s$ and a_j is odd, then $X_j = V_s$;
5. if $X_{j-1} = V_s$, then $X_j = U_s$.

(We recall that $a_k = [T^k(\alpha)]$.) This enables us to define three sequences of functions: $(U_{j,s})_{j\geqslant 0}$, $(V_{j,s})_{j\geqslant 0}$, $(W_{j,s})_{j\geqslant 0}$, with $U_{0,s} = U_s$, $V_{0,s} = V_s$ and $W_{0,s} = W_s$.

Theorem 1.3. *We fix a real number $s > 1$ and an irrational number $\alpha \in (0, 1)$. Each of the series $\Psi_s(\alpha)$, $\hat{\Psi}_s(\alpha)$, $\tilde{\Psi}_s(\alpha)$ converges if*

$$\sum_{j=0}^{\infty} \frac{q_{j+1}(\alpha)}{q_j^s(\alpha)} \tag{1.18}$$

converges. Furthermore, when (1.18) converges and $s > 2$, we have the identities

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cot(\pi n\alpha)}{n^s} = \sum_{j=0}^{\infty} (-1)^j |q_{j-1}\alpha - p_{j-1}|^{s-1} U_{j,s}(T^j(\alpha)), \tag{1.19}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s \sin(\pi n\alpha)} = \sum_{j=0}^{\infty} (-1)^j |q_{j-1}\alpha - p_{j-1}|^{s-1} V_{j,s}(T^j(\alpha)), \tag{1.20}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s \sin(\pi n\alpha)} = \sum_{j=0}^{\infty} (-1)^j |q_{j-1}\alpha - p_{j-1}|^{s-1} W_{j,s}(T^j(\alpha)). \tag{1.21}$$

Remarks. For example, if a_j is even for all $j \geqslant 1$, $W_{j,s} = W_s$ for all $j \geqslant 0$. If a_{2j+1} is even for all $j \geqslant 0$, $U_{2j,s} = U_s$ and $U_{2j+1,s} = V_s$ for all $j \geqslant 0$. If a_{2j+2} is even for all $j \geqslant 0$, $V_{2j,s} = U_s$ and $V_{2j+1,s} = V_s$ for all $j \geqslant 0$.

2. Closed forms and the irrationality exponent

In this section, we present exact evaluations (or 'closed forms') of the series $\Phi_s(\alpha)$ and $\hat{\Phi}_s(\alpha)$, as well as some remarks on the convergence of these series for some classical constants. Of course, it would be possible to find exact evaluations for $\Psi_s(\alpha)$, $\hat{\Psi}_s(\alpha)$ and $\tilde{\Psi}_s(\alpha)$.

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

2.1. Closed forms

Since $|q_{j-1}\alpha - p_{j-1}| = \alpha T(\alpha)\cdots T^{j-1}(\alpha)$, it is easy to see that (1.11) and (1.15 a) respectively imply the identities

$$\Phi_s(\alpha) = -\alpha^{s-1}\Phi_s(T(\alpha)) + G_s(\alpha), \tag{2.1}$$

$$\hat{\Phi}_s(\alpha) = \alpha^{s-2}\hat{\Phi}_s(T(\alpha)) + \frac{s}{\pi}\alpha^{s-1}\Phi_{s+1}(T(\alpha)) + \hat{G}_s(\alpha), \tag{2.2}$$

where in (2.1) and (2.2), respectively, we assume that

$$\sum_{j\geqslant 0}(-1)^j q_{j+1}/q_j^s \quad \text{and} \quad \sum_{j\geqslant 0} q_{j+1}^2/q_j^s,$$

respectively, converge, so that all the involved series converge. By iteration of (2.1) and (2.2), we can obtain closed formulae for $\Phi_s(\alpha)$ and $\hat{\Phi}_s(\alpha)$ when $T^k(\alpha) = \alpha + j$ for some integers $k \geqslant 0$, $j \in \mathbb{Z}$. This can happen if and only if α is a quadratic number. Since the sequence of partial quotients $(a_j)_j$ of a quadratic number is periodic, the conditions of convergence in (2.1) and (2.2) are satisfied.

For example, if a quadratic number α satisfies the equation $T(\alpha) = \alpha$, we have

$$\Phi_s(\alpha) = \frac{G_s(\alpha)}{1+\alpha^{s-1}} \quad \text{and} \quad \hat{\Phi}_s(\alpha) = \frac{s\alpha^{s-1}G_{s+1}(\alpha) + (1+\alpha^s)\pi\hat{G}_s(\alpha)}{\pi(1+\alpha^s)(1-\alpha^{s-2})}.$$

For instance, we can take $\alpha = \sqrt{2}-1$ or $\alpha = \frac{1}{2}(\sqrt{5}-1)$. Of course, these identities are really closed forms when G_s (or G_{s+1}) and $\hat{G}_s$ are expressed as (1.8) and (1.12), which happens when s is odd for the evaluation of $\Phi_s(\alpha)$ and when s is even for the evaluation of $\hat{\Phi}_s(\alpha)$.

2.2. Irrationality exponent

We might want to compute an approximate value of $\Phi_s(\alpha)$ 'naively' using the definition of $\Phi_s(\alpha)$. The identity (4.9) clearly shows that the speed of convergence of the Nth partial sums $\Phi_{s,N}(\alpha)$ of $\Phi_s(\alpha)$ strongly depends on the speed of convergence to 0 of the sum

$$\sum_{j=m+1}^{\infty}(-1)^j\frac{q_{j+1}}{q_j^s},$$

where $q_m \leqslant N < q_{m+1}$. This is in accordance with the folklore observation that the value of $\Phi_{s,N}(\alpha)$ changes (relatively) quickly when N is the denominator of a convergent to α, but otherwise changes (relatively) more slowly between two such denominators.

For interesting numbers other than quadratic numbers, like e, log(2), π or real algebraic numbers of degree greater than or equal to 3, it does not seem possible to simplify further the right-hand side of the identities obtained for $\Phi_s(\alpha)$ and $\hat{\Phi}_s(\alpha)$ in Theorems 1.1 and 1.2. However, these right-hand sides converge much faster than the corresponding left-hand sides. When the functions $G_s(\alpha)$ or $\hat{G}_s(\alpha)$ can be evaluated quickly, this provides a fast method to compute $\Phi_s(\alpha)$ or $\hat{\Phi}_s(\alpha)$, in particular for αs whose continued fractions are well known.

We recall that an irrational number α is said to have *a finite irrationality exponent* $m(\alpha)$ if there exists a constant $c(\alpha) > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| \geqslant \frac{1}{c(\alpha)q^{m(\alpha)}}$$

for all integers p, q with $q \geqslant 1$. We denote by $\mu(\alpha)$ *the* irrationality exponent of α, defined as the infinimum of all possible $m(\alpha)$, regardless of the value of $c(\alpha)$. We always have $\mu(\alpha) \geqslant 2$. If $\mu(\alpha)$ is finite, its value provides a bound on the growth of the denominators $q_j(\alpha)$ of the continued fractions of α. Indeed, for any $\varepsilon > 0$ and any $j \geqslant 0$, we have

$$\frac{d(\alpha,\varepsilon)}{q_j^{\mu(\alpha)+\varepsilon}} \leqslant \left|\alpha - \frac{p_j}{q_j}\right| \leqslant \frac{1}{q_j q_{j+1}}$$

for some $d(\alpha,\varepsilon) > 0$. Hence, $q_{j+1} \leqslant d(\alpha,\varepsilon)^{-1} q_j^{\mu(\alpha)-1+\varepsilon}$. Furthermore, for almost all α, $\mu(\alpha) = 2$ (Dirichlet) so that, for all $\varepsilon > 0$ and almost all α, we have $q_{j+1} \ll_{\alpha,\varepsilon} q_j^{1+\varepsilon}$ for all $j \geqslant 0$. It follows that, for such an α, the series $\Phi_s(\alpha)$ and $\hat{\Phi}_s(\alpha)$ converge (at least) when $s > \mu(\alpha) - 1$ and $s > 2\mu(\alpha) - 2$, respectively. It is known that $\mu(\mathrm{e}) = 2$ [**1**], $\mu(\log(2)) \leqslant 3.5775$ [**20**], $\mu(\pi) \leqslant 7.6064$ [**25**] and $\mu(\sqrt[3]{2}) = 2$ [**24**]. For example, $\Phi_s(\mathrm{e})$ and $\hat{\Phi}_{2s}(\mathrm{e})$ converge for all $s > 1$, and $\Phi_4(\log(2))$ or $\hat{\Phi}_{14}(\pi)$ are convergent. But it is not yet possible to say if the series $\Phi_3(\log(2))$ or $\hat{\Phi}_{13}(\pi)$ are convergent. (It is conjectured that $\mu(\log(2)) = \mu(\pi) = 2$.)

3. A sufficient condition of convergence of the series (1.1)

In this section, we prove the following result.

Proposition 3.1. *Let f be a function defined on $\mathbb{R} \setminus \mathbb{Z}$ that takes real values and such that there exist a real number $r \geqslant 1$ and a constant $c > 0$ such that*

$$|f(x)| \leqslant \frac{c}{|\sin(x)|^r} \tag{3.1}$$

for any $x \in \mathbb{R} \setminus \mathbb{Z}$. Then, for any integer N such that $q_m \leqslant N < q_{m+1}$ for some $m \geqslant 1$, any integer $k \geqslant 0$, any real number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and any real number $s > r$, we have

$$\sum_{n=1}^{N} \left|\frac{f(\pi n T^k(\alpha))}{n^s}\right| \ll \sum_{j=k+1}^{k+m} \frac{q_{j+1}^r(\alpha)}{q_j^s(\alpha)}. \tag{3.2}$$

In particular, the series

$$\sum_{n=1}^{\infty} \frac{f(\pi n\alpha)}{n^s}$$

is convergent if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ satisfies the Diophantine condition

$$\sum_{j=0}^{\infty} \frac{q_{j+1}^r(\alpha)}{q_j^s(\alpha)} < \infty. \tag{3.3}$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

Under the condition $s > r$, the series $\sum_{n=1}^{\infty} f(\pi n\alpha)/n^s$ converges for almost all real numbers α.

Proposition 3.1 is a consequence of the following lemma, which is a generalization of some of Kruse's results [**18**], corresponding to the case $r = 1$. We use the standard notation that $\|x\|$ is the distance from x to $\mathbb{Z}$. See also [**28**, Exercise 168, p. 216] for related questions.

Lemma 3.2. *Let r, s be real numbers such that $s > r \geqslant 1$, and let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^s \|n\alpha\|^r} \tag{3.4}$$

converges if and only if $\sum_{j=0}^{\infty} q_{j+1}^r(\alpha)/q_j^s(\alpha)$ converges. More precisely, for any integer N such that $q_m \leqslant N < q_{m-1}$, we have

$$\sum_{n=1}^{N} \frac{1}{n^s \|n\alpha\|^r} \asymp \sum_{j=0}^{m} \frac{q_{j+1}(\alpha)^r}{q_j(\alpha)^s}, \tag{3.5}$$

where the two implicit constants are explicitly computable and depend at most on s, r, α but not on m or N.

Remark 3.3. It is also possible to consider the problem of the convergence/divergence of the series (3.4) where the condition $s > r \geqslant 1$ is relaxed, but since this is not directly connected to our study, we do not consider this question here.

Proof of Lemma 3.2. We fix the integer N and define m by the inequalities $q_m \leqslant N < q_{m-1}$. It is obvious that we have the lower bound

$$\sum_{n=1}^{N} \frac{1}{n^s \|n\alpha\|^r} \geqslant \sum_{j=1}^{m} \frac{1}{q_j^s \|q_j\alpha\|^r} \geqslant \sum_{j=1}^{m} \frac{q_{j+1}^r}{q_j^s} \tag{3.6}$$

because $\|q_j\alpha\| \leqslant 1/q_{j+1}$ for all j.

It is a little more difficult to obtain an upper bound of the same quality as (3.6). We write

$$\begin{aligned}\sum_{n=q_m}^{N} \frac{1}{n^s \|n\alpha\|^r} &= \sum_{\substack{n=q_m \\ n\not\equiv 0, q_{m-1}[q_m]}}^{N} \frac{1}{n^s \|n\alpha\|^r} + \sum_{\substack{n=q_m \\ n\equiv 0[q_m]}}^{N} \frac{1}{n^s \|n\alpha\|^r} + \sum_{\substack{n=q_m \\ n\equiv q_{m-1}[q_m]}}^{N} \frac{1}{n^s \|n\alpha\|^r} \\ &= S_1(q_m, N) + S_2(q_m, N) + S_3(q_m, N).\end{aligned}$$

The second and third sums can be bounded easily:

$$S_2(q_m, N) = \sum_{\substack{n=q_m \\ n\equiv 0[q_m]}}^{N} = \sum_{k=1}^{Q} \frac{1}{(kq_m)^s \|kq_m\alpha\|^r} \leqslant \sum_{k=1}^{Q} \frac{q_{m+1}^r}{(kq_m)^s} \leqslant \zeta(s) \frac{q_{m+1}^r}{q_m^s}$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

(where $Q = [N/q_m] < a_{m+1}$) because $\|kq_m\alpha\| \geqslant \|q_m\alpha\| \geqslant 1/q_{m+1}$ for any $k = 1, \dots, a_{m+1} - 1$. Similarly,

$$\begin{aligned} S_3(q_m, N) &= \sum_{\substack{n=q_m \\ n \equiv q_{m-1}[q_m]}}^{N} \frac{1}{n^s \|n\alpha\|^r} \\ &= \sum_{k=1}^{Q'} \frac{1}{(kq_m + q_{m-1})^s \|(kq_m + q_{m-1})\alpha\|^r} \\ &\leqslant \sum_{k=1}^{Q'} \frac{q_{m+1}^r}{(kq_m)^s} \leqslant \zeta(s) \frac{q_{m+1}^r}{q_m^s} \end{aligned}$$

(where $Q' = [(N - q_{m-1})/q_m] < a_{m+1}$) because $\|(kq_m + q_{m-1})\alpha\| \geqslant \|q_m\alpha\| \geqslant 1/q_{m+1}$ for any $k = 1, \dots, a_{m+1} - 1$. To bound $S_1(q_m, N)$, we write

$$\begin{aligned} S_1(q_m, N) &= \sum_{\substack{n=q_m \\ n \not\equiv 0, q_{m-1}[q_m]}}^{N} \frac{1}{n^s \|n\alpha\|^r} \\ &= \sum_{h=1}^{Q} \sum_{\substack{j=1 \\ j \neq q_{m-1}}}^{r_h} \frac{1}{(hq_m + j)^s \|(hq_m + j)\alpha\|^r} \\ &\leqslant \frac{1}{q_m^s} \sum_{h=1}^{Q} \frac{1}{h^s} \sum_{\substack{j=1 \\ j \neq q_{m-1}}}^{t_h} \frac{1}{\|(hq_m + j)\alpha\|^r}, \end{aligned}$$

where $t_h = q_m - 1$ if $1 \leqslant h < Q$, $t_h = N - Qq_m$ is $h = Q$. We now use a crucial remark of [**18**, (39), p. 241] to get

$$\sum_{\substack{j=1 \\ j \neq q_{m-1}}}^{t_h} \frac{1}{\|(hq_m + j)\alpha\|^r} \leqslant 2 \sum_{k=1}^{t_h} \frac{1}{(k/q_m)^r}$$

for any $h = 1, \dots, Q = [N/q_m]$. Hence,

$$S_1(q_m, N) = \sum_{\substack{n=q_m \\ n \not\equiv 0, q_{m-1}[q_m]}}^{N} \frac{1}{n^s \|n\alpha\|^r} \ll \frac{1}{q_m^{s-r}} \sum_{h=1}^{Q} \frac{\log(t_h)}{h^s} \leqslant \zeta(s) \frac{\log(q_m)}{q_m^{s-r}}.$$

(We use the hypothesis $r \geqslant 1$ to get the first inequality.) Therefore,

$$\begin{aligned} \sum_{n=1}^{N} \frac{1}{n^s \|n\alpha\|^r} &\leqslant \sum_{k=1}^{m} (S_1(q_k, q_{k+1} - 1) + S_2(q_k, q_{k+1} - 1) + S_3(q_k, q_{k+1} - 1)) \\ &\ll \sum_{k=1}^{m} \left(\frac{q_{k+1}^r}{q_k^s} + \frac{\log(q_k)}{q_k^{s-r}} \right) \ll \sum_{k=1}^{m} \frac{q_{k+1}^r}{q_k^s}, \end{aligned}$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

where the last inequality holds because the series $\sum_{k=1}^{\infty} \log(q_k)/q_k^{s-r}$ converges for all irrational α by the condition $s > r$. This completes the proof of (3.5). $\square$

Proof of Proposition 3.1. The domination condition (3.1) on $f(x)$ shows that it is sufficient to find conditions ensuring the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^s |\sin(\pi n T^k(\alpha))|^r}.$$

We have $|\sin(\pi n T^k(\alpha))| = \sin(\pi \|nT^k(\alpha)\|) \asymp \|nT^k(\alpha)\|$, where both implicit constants could be explicitly computed. Hence, it is enough to find an upper bound ensuring the convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^s \|nT^k(\alpha)\|^r}.$$

By Lemma 3.2,

$$\sum_{n=1}^{N} \frac{1}{n^s \|nT^k(\alpha)\|^r} \ll \sum_{j=1}^{m} \frac{1}{\hat{q}_j^s \|\hat{q}_j T^k(\alpha)\|^r} \ll \sum_{j=1}^{m} \frac{\hat{q}_{j+1}^r}{\hat{q}_j^s},$$

where $\hat{q}_j = q_j(T^k(\alpha))$ is the denominator of the jth convergent to $T^k(\alpha)$. Since $\hat{q}_j = q_{j+k}(\alpha)$, we obtain

$$\sum_{n=1}^{N} \frac{1}{n^s \|nT^k(\alpha)\|^r} \ll \sum_{j=k+1}^{k+m} \frac{q_{j+1}^r}{q_j^s},$$

as expected. $\square$

4. Proof of Theorem 1.1

4.1. Preliminary observations

It is necessary that

$$\lim_{k \to +\infty} \frac{q_{k+1}}{q_k^s} = 0$$

for the convergence of $\Phi_s(\alpha)$. Indeed, for $n = q_k$, we have

$$\begin{aligned}
\frac{\cot(\pi n \alpha)}{n^s} &= \frac{\cos(\pi q_k \alpha)}{q_k^s \sin(\pi q_k \alpha)} \\
&= \pm \frac{\cos(\pi \|q_k \alpha\|)}{q_k^s \sin(\pi \|q_k \alpha\|)} \\
&= \frac{\pm 1 + \mathcal{O}(\|q_k \alpha\|^2)}{q_k^s (\|q_k \alpha\| + \mathcal{O}(\|q_k \alpha\|^3))} \\
&= \frac{\pm 1}{q_k^s \|q_k \alpha\|} + \mathcal{O}\left(\frac{\|q_k \alpha\|}{q_k^s}\right).
\end{aligned}$$

Since $\|q_k \alpha\| \asymp 1/q_{k+1}$, the convergence of $\cot(\pi n \alpha)/n^s$ to 0 implies that of q_{k+1}/q_k^s to 0.

From (1.5), we deduce that

$$\sum_{j=0}^{\infty} \frac{q_{j+1}^r}{q_j^s} \quad \text{and} \quad \sum_{j=0}^{\infty} \frac{|q_{j-1}\alpha - p_{j-1}|^s}{|q_j\alpha - p_j|^r}$$

converge or diverge simultaneously for any real numbers r, s. But (1.5) is not sufficient to ensure immediately that, for $s > 2$, the series

$$\sum_{j=0}^{\infty}(-1)^j \frac{q_{j+1}}{q_j^s} \quad \text{and} \quad \sum_{j=0}^{\infty}(-1)^j \frac{|q_{j-1}\alpha - p_{j-1}|^s}{|q_j\alpha - p_j|}$$

converge or diverge simultaneously. This can be proved by means of a refinement of (1.5). For any $j \geqslant 0$, we have

$$q_j\alpha - p_j = (-1)^j|q_j\alpha - p_j| = \frac{(-1)^j}{q_{j+1} + x_j q_j},$$

where $x_j := [0; a_{j+2}, a_{j+3}, \ldots]$. In particular, $0 \leqslant x_j \leqslant 1/a_{j+2}$. Then, we have a chain of 'equalities':

$$\begin{aligned}
(-1)^j \frac{|q_{j-1}\alpha - p_{j-1}|^s}{|q_j\alpha - p_j|} &= (-1)^j \frac{q_{j+1} + x_j q_j}{(q_j + x_{j-1}q_{j-1})^s} \\
&= (-1)^j \frac{q_{j+1}}{(q_j + x_{j-1}q_{j-1})^s} + \mathcal{O}\biggl(\frac{1}{q_j^{s-1}}\biggr) \\
&= (-1)^j \frac{q_{j+1}}{q_j^s} + \mathcal{O}\biggl(\frac{x_{j-1}q_{j-1}q_{j+1}}{q_j^s}\biggr) + \mathcal{O}\biggl(\frac{1}{q_j^{s-1}}\biggr) \\
&= (-1)^j \frac{q_{j+1}}{q_j^s} + \mathcal{O}\biggl(\frac{q_{j-1}q_{j+1}}{a_{j+1}q_j^s}\biggr) + \mathcal{O}\biggl(\frac{1}{q_j^{s-1}}\biggr) \\
&= (-1)^j \frac{q_{j+1}}{q_j^s} + \mathcal{O}\biggl(\frac{q_{j-1}}{q_j^{s-1}}\biggr) + \mathcal{O}\biggl(\frac{1}{q_j^{s-1}}\biggr) \\
&= (-1)^j \frac{q_{j+1}}{q_j^s} + \mathcal{O}\biggl(\frac{1}{q_j^{s-2}}\biggr)
\end{aligned}$$

because $q_{j+1} \leqslant 2a_{j+1}q_j$. Since $\sum_j 1/q_j^{s-2}$ converges (for $s > 2$), the assertion follows. We deduce that, in the first assertion of Theorem 1.1, we can consider

$$\sum_{j=0}^{\infty}(-1)^j \frac{|q_{j-1}\alpha - p_{j-1}|^s}{|q_j\alpha - p_j|}$$

instead of

$$\sum_{j=0}^{\infty}(-1)^j \frac{q_{j+1}}{q_j^s}.$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

4.2. Proof of Theorem 1.1

Set $K = N + \frac{1}{2}$ for any integer $N \geqslant 1$. We consider the integral

$$I_s(\alpha) := \frac{1}{2\mathrm{i}\pi} \int_{\mathcal{R}_N} F_s(\alpha, z)\, \mathrm{d}z,$$

where $\mathcal{R}_N$ is the rectangle with sides $[\frac{1}{2} - \mathrm{i}N, K - \mathrm{i}N]$, $[K - \mathrm{i}N, K + \mathrm{i}N]$, $[K + \mathrm{i}N, \frac{1}{2} + \mathrm{i}N]$ and $[\frac{1}{2} + \mathrm{i}N, \frac{1}{2} - \mathrm{i}N]$. The function $F_s(\alpha, z)$ is holomorphic inside $\mathcal{R}_N$ and continuous on the boundary. Its poles inside $\mathcal{R}_N$ are

- at $z = k \in \mathbb{Z}$, $1 \leqslant k \leqslant N$, of order 1 with residue $\cot(\pi k\alpha)/k^s$,
- at $z = k/\alpha \in \mathbb{Z}$, $1 \leqslant k \leqslant N\alpha$, of order 1 with residue $\alpha^{s-1}\cot(\pi k/\alpha)/k^s$ (the assumption $0 < \alpha < 1$ is used here)Abbreviations.

By the residue theorem, we thus have the identity

$$\sum_{k=1}^{N} \frac{\cot(\pi k\alpha)}{k^s} = -\alpha^{s-1} \sum_{k=1}^{[N\alpha]} \frac{\cot(\pi k/\alpha)}{k^s} + I_s(\alpha). \tag{4.1}$$

We now proceed to bound the integral $I_s(\alpha)$. On the sides $C_1 := [\frac{1}{2} - \mathrm{i}N, K - \mathrm{i}N]$ and $C_2 := [K + \mathrm{i}N, \frac{1}{2} + \mathrm{i}N]$, it is clear that $|F_s(z)| \ll N^{-s}$, where the implicit constant is absolute. Hence,

$$\left| \frac{1}{2\mathrm{i}\pi} \int_{C_j} F_s(\alpha, z)\, \mathrm{d}z \right| \ll \frac{1}{N^{s-1}} \tag{4.2}$$

for $j = 1, 2$.

On the side $C_3 := [K - \mathrm{i}N, K + \mathrm{i}N]$, the estimate is a little more complicated: we have

$$\left| \frac{1}{2\mathrm{i}\pi} \int_{C_j} F_s(\alpha, z)\, \mathrm{d}z \right| \ll \begin{cases} \dfrac{1}{N^{s-2}} & \text{if } N \geqslant 1/\alpha, \\ \dfrac{1}{\alpha N^{s-1}} & \text{if } N \leqslant 1/\alpha. \end{cases}$$

The proof runs as follows. For any $z = K + \mathrm{i}y \in C_3$, we have

$$|F_s(\alpha, z)| \leqslant \left| \frac{\sinh(\pi y)\cosh(\pi\alpha y)}{\sinh(\pi\alpha y)\cosh(\pi y) N^s} \right|. \tag{4.3}$$

If $|y| \geqslant 1/\alpha$,

$$\left| \frac{\sinh(\pi y)\cosh(\pi\alpha y)}{\sinh(\pi\alpha y)\cosh(\pi y)} \right| = \mathcal{O}(1),$$

where the constant is absolute. On the other hand, if $|y| \leqslant 1/\alpha$, we have

$$\left| \frac{\sinh(\pi y)\cosh(\pi\alpha y)}{\sinh(\pi\alpha y)\cosh(\pi y)} \right| \leqslant \frac{1}{\alpha}.$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

Therefore, if $N \geqslant 1/\alpha$, we have

$$\begin{aligned}\left|\frac{1}{2\mathrm{i}\pi}\int_{C_j} F_s(\alpha,z)\,\mathrm{d}z\right| &\ll \int_0^{1/\alpha} |F_s(\alpha,K+\mathrm{i}y)|\mathrm{d}y + \int_{1/\alpha}^{N} |F_s(\alpha,K+\mathrm{i}y)|\mathrm{d}y \\ &\ll \frac{1}{\alpha^2 N^s} + \frac{N-1/\alpha}{N^s} \\ &\leqslant \frac{2}{N^{s-2}}.\end{aligned}$$

If $N \leqslant 1/\alpha$, we have

$$\left|\frac{1}{2\mathrm{i}\pi}\int_{C_j} F_s(\alpha,z)\,\mathrm{d}z\right| \ll \int_0^{N} |F_s(\alpha,K+\mathrm{i}y)|\mathrm{d}y \ll \frac{N}{\alpha N^s} = \frac{1}{\alpha N^{s-1}}.$$

This proves (4.3).

It remains to obtain an estimate of $C_4 := [\frac{1}{2}+\mathrm{i}N, \frac{1}{2}-\mathrm{i}N]$. We write

$$\frac{1}{2\mathrm{i}\pi}\int_{C_4} F_s(\alpha,z)\,\mathrm{d}z = G_s(\alpha) + \frac{1}{2\mathrm{i}\pi}\int_{1/2+\mathrm{i}N}^{1/2+\mathrm{i}\infty} F_s(\alpha,z)\,\mathrm{d}z + \frac{1}{2\mathrm{i}\pi}\int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}N} F_s(\alpha,z)\,\mathrm{d}z.$$

We have

$$\begin{aligned}\int_{1/2+\mathrm{i}N}^{1/2+\mathrm{i}\infty} F_s(\alpha,z)\,\mathrm{d}z &= \mathrm{i}\pi\int_N^{+\infty} \frac{\tanh(\pi t)\cot(\pi\alpha(\frac{1}{2}+\mathrm{i}t))}{(\frac{1}{2}+\mathrm{i}t)^s}\,\mathrm{d}t \\ &\ll \int_N^{+\infty} \frac{\tanh(\pi t)\coth(\pi\alpha t)}{|\frac{1}{2}+\mathrm{i}t|^s}\,\mathrm{d}t \\ &\ll \frac{N^{1-s}}{\alpha}\end{aligned}$$

because $0 \leqslant \tanh(\pi t)\coth(\pi\alpha t) \leqslant 1/\alpha$ for all $t \geqslant 0$. A similar bound holds on the interval $(\frac{1}{2}-\mathrm{i}\infty, \frac{1}{2}-\mathrm{i}N]$. Hence,

$$\frac{1}{2\mathrm{i}\pi}\int_{C_4} F_s(\alpha,z)\,\mathrm{d}z = G_s(\alpha) + \mathcal{O}\biggl(\frac{1}{\alpha N^{s-1}}\biggr). \tag{4.4}$$

Using (4.2)–(4.4) in (4.1), we deduce that

$$\sum_{k=1}^{N} \frac{\cot(\pi k\alpha)}{k^s} = -\alpha^{s-1}\sum_{k=1}^{[N\alpha]} \frac{\cot(\pi k/\alpha)}{k^s} + G_s(\alpha) + \mathcal{O}(E_N(\alpha)) \tag{4.5}$$

where the implicit constant depends at most on s and

$$E_N(\alpha) = \begin{cases} \dfrac{1}{N^{s-2}} & \text{if } N \geqslant 1/\alpha, \\ \dfrac{1}{\alpha N^{s-1}} & \text{if } N \leqslant 1/\alpha. \end{cases} \tag{4.6}$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

Let us define $\ell = \ell_N(\alpha)$ as the smallest integer such that $[\cdots[[N\alpha]T(\alpha)]\cdots T^{\ell}(\alpha)] = 0$. It is well defined because $[\cdots[[N\alpha]T(\alpha)]\cdots T^{\ell}(\alpha)] \leqslant N\alpha T(\alpha)\cdots T^{\ell}(\alpha) = N|q_\ell\alpha - p_\ell| \to 0$ as $\ell \to +\infty$. This inequality also implies that $\ell_{q_k}(\alpha) \leqslant k$ for any $k \geqslant 0$ because $q_k|q_k\alpha - p_k| < 1$. We now prove that, for any $j \leqslant \ell_N(\alpha) - 9$,

$$[\cdots[[N\alpha]T(\alpha)]\cdots T^{j}(\alpha)] \geqslant \tfrac{1}{2}N\alpha T(\alpha)\cdots T^{j}(\alpha) \tag{4.7}$$

and that $\ell_{q_k}(\alpha) \geqslant k - 8$. To do this, we first have to show that $q_{\ell_N(\alpha)} \leqslant N$. Indeed, $[\cdots[[N\alpha]T(\alpha)]\cdots T^{\ell_N(\alpha)-1}(\alpha)]$ is a positive integer (by definition of $\ell_N(\alpha)$) so that

$$1 \leqslant [\cdots[[N\alpha]T(\alpha)]\cdots T^{\ell_N(\alpha)-1}(\alpha)] \leqslant N|q_{\ell_N(\alpha)-1}\alpha - p_{\ell_N(\alpha)-1}| \leqslant \frac{N}{q_{\ell_N(\alpha)}}.$$

Now, for all $j \geqslant 0$,

$$\begin{aligned}[\cdots[[N\alpha]T(\alpha)]\cdots T^{j}(\alpha)] &\geqslant N\alpha T(\alpha)\cdots T^{j}(\alpha) - \sum_{k=1}^{j+1} T^{k}(\alpha)\cdots T^{j}(\alpha)\\ &= N\alpha T(\alpha)\cdots T^{j}(\alpha)\biggl(1 - \sum_{k=0}^{j}\frac{1}{N\alpha T(\alpha)\cdots T^{k}(\alpha)}\biggr)\\ &= N|q_j\alpha - p_j|\biggl(1 - \sum_{k=0}^{j}\frac{1}{N|q_k\alpha - p_k|}\biggr).\end{aligned}$$

Hence,

$$\begin{aligned}[\cdots[[N\alpha]T(\alpha)]\cdots T^{j}(\alpha)] &\geqslant N|q_j\alpha - p_j|\biggl(1 - 2\frac{q_1 + \cdots + q_{j+1}}{N}\biggr)\\ &\geqslant N|q_j\alpha - p_j|\biggl(1 - \frac{8q_{j+1}}{N}\biggr)\end{aligned} \tag{4.8}$$

by the inequality $q_1 + \cdots + q_{j+1} \leqslant 4q_{j+1}$. But we have $N \geqslant q_{\ell_N(\alpha)} \geqslant 16q_{\ell_N(\alpha)-8} \geqslant 16q_{j+1}$ if $j \leqslant \ell_N(\alpha) - 9$. Hence, for $j \leqslant \ell_N(\alpha) - 9$,

$$[\cdots[[N\alpha]T(\alpha)]\cdots T^{j}(\alpha)] \geqslant N|q_j\alpha - p_j|(1 - \tfrac{1}{2}) = \tfrac{1}{2}N\alpha T(\alpha)\cdots T^{j}(\alpha),$$

which proves the first assertion. For the second assertion, we use inequality (4.8) with $N = q_k$ and get

$$[\cdots[[q_k\alpha]T(\alpha)]\cdots T^{j}(\alpha)] \geqslant q_k|q_j\alpha - p_j|\biggl(1 - \frac{2q_{j+1}}{q_k}\biggr).$$

But, for any $j \leqslant k - 9$,

$$\frac{8q_{j+1}}{q_k} \leqslant \frac{8q_{j+1}}{16q_{k-8}} \leqslant \frac{8q_{j+1}}{16q_{j+1}} = \frac{1}{2}$$

and hence $[\cdots[[q_k\alpha]T(\alpha)]\cdots T^{j}(\alpha)] > 0$. Therefore $\ell_{q_k}(\alpha) \geqslant k - 8$.

We can now return to the proof of Theorem 1.1. By 1-periodicity of $\cot(\pi z)$, we can rewrite (4.5) as

$$\sum_{k=1}^{N}\frac{\cot(\pi k\alpha)}{k^s} = -\alpha^{s-1}\sum_{k=1}^{[N\alpha]}\frac{\cot(\pi kT(\alpha))}{k^s} + G_s(\alpha) + \mathcal{O}(E_N(\alpha)).$$

Since $T(\alpha)\in(0,1)$, we can iterate this equation to get

$$\begin{aligned}\sum_{k=1}^{N}\frac{\cot(\pi k\alpha)}{k^s} &= \sum_{j=0}^{\ell}(-1)^j(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-1}G_s(T^j(\alpha))\\ &\quad + (\alpha T(\alpha)\cdots T^{\ell}(\alpha))^{s-1}\sum_{k=1}^{[\cdots[[N\alpha]T(\alpha)]\cdots T^{\ell}(\alpha)]}\frac{\cot(\pi kT^{\ell+1}(\alpha))}{k^s}\\ &\quad + \mathcal{O}\biggl(\sum_{j=0}^{\ell}(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-1}E_{[\cdots[[N\alpha]T(\alpha)]\cdots T^{j-1}(\alpha)]}(T^j(\alpha))\biggr).\end{aligned}$$

In this expression, we adopt the conventions that, for $j=0$, $[\cdots[[N\alpha]T(\alpha)]\cdots T^{j-1}(\alpha)] = N$ and $\alpha T(\alpha)\cdots T^{j-1}(\alpha)=1$. By definition of ℓ, the sum

$$\sum_{k=1}^{[\cdots[[N\alpha]T(\alpha)]\cdots T^{\ell}(\alpha)]}\frac{\cot(\pi kT^{\ell+1}(\alpha))}{k^s}$$

is empty and vanishes, yielding the identity

$$\begin{aligned}\sum_{k=1}^{N}\frac{\cot(\pi k\alpha)}{k^s} &= \sum_{j=0}^{\ell_N(\alpha)}(-1)^j|q_{j-1}\alpha - p_{j-1}|^{s-1}G_s(T^j(\alpha))\\ &\quad + \mathcal{O}\biggl(\sum_{j=0}^{\ell_N(\alpha)}(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-1}E_{[\cdots[[N\alpha]T(\alpha)]\cdots T^{j-1}(\alpha)]}(T^j(\alpha))\biggr).\end{aligned}$$

(Recall that $\alpha T(\alpha)\cdots T^{j-1}(\alpha) = |q_{j-1}\alpha - p_{j-1}|$.)

We set

$$\varepsilon_N := \sum_{j=0}^{\ell_N(\alpha)}(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-1}E_{[\cdots[[N\alpha]T(\alpha)]\cdots T^{j-1}(\alpha)]}(T^j(\alpha))$$

and we prove that ε_N tends to 0. In ε_N, we have to distinguish the cases $j\leqslant \ell_N(\alpha)-8$ and $j\geqslant \ell_N(\alpha)-7$. In the first case,

$$\begin{aligned}0 &\leqslant \sum_{j=0}^{\ell_N(\alpha)-8}(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-1}E_{[\cdots[[N\alpha]T(\alpha)]\cdots T^{j-1}(\alpha)]}(T^j(\alpha))\\ &\leqslant \sum_{j=0}^{\ell_N(\alpha)-8}\frac{(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-1}}{[\cdots[[N\alpha]T(\alpha)]\cdots T^{j-1}(\alpha)]^{s-2}}\end{aligned}$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

$$\ll \frac{1}{N^{s-2}} \sum_{j=0}^{\ell_N(\alpha)-4} \alpha T(\alpha)\cdots T^{j-1}(\alpha)$$
$$\ll \frac{1}{N^{s-2}},$$

where we use (4.7). In the second case, since $[\cdots[[N\alpha]T(\alpha)]\cdots T^{j-1}(\alpha)] \geqslant 1$ (because $j-1 \leqslant \ell_N(\alpha)-1$), we use the trivial upper bound deduced from (4.6):

$$0 \leqslant E_{[\cdots[[N\alpha]T(\alpha)]\cdots T^{j-1}(\alpha)]}(T^j(\alpha)) \leqslant \frac{1}{T^j(\alpha)}.$$

Hence, for $\ell_N(\alpha)-7 \leqslant j \leqslant \ell_N(\alpha)$,

$$|(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-1} E_{[\cdots[[N\alpha]T(\alpha)]\cdots T^{j-1}(\alpha)]}(T^j(\alpha))| \ll \frac{(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-1}}{T^j(\alpha)} \ll \frac{q_{j+1}}{q_j^s}.$$

Therefore,

$$\varepsilon_N = \mathcal{O}\biggl(\frac{1}{N^{s-2}}\biggr) + \mathcal{O}\biggl(\sum_{j=\ell_N(\alpha)-7}^{\ell_N(\alpha)} \frac{q_{j+1}}{q_j^s}\biggr)$$

and

$$\sum_{k=1}^{N} \frac{\cot(\pi k\alpha)}{k^s} = \sum_{j=0}^{\ell_N(\alpha)} (-1)^j |q_{j-1}\alpha - p_{j-1}|^{s-1} G_s(T^j(\alpha)) + \mathcal{O}\biggl(\frac{1}{N^{s-2}}\biggr) + \mathcal{O}\biggl(\sum_{j=\ell_N(\alpha)-7}^{\ell_N(\alpha)} \frac{q_{j+1}}{q_j^s}\biggr). \tag{4.9}$$

We can now deduce the theorem. The assumption $s > 2$ ensures that $\mathcal{O}(1/N^{s-2})$ tends to 0. We assume for the moment that

$$G_s(T^j(\alpha)) = \frac{\zeta(s+1)}{\pi T^j(\alpha)} + P_s(T^j(\alpha))$$

with $P_s(\alpha) = \mathcal{O}(1)$ for all $\alpha \in [0,1]$; this will be proved below. Since

$$T^j(\alpha) = \biggl|\frac{q_j\alpha - p_j}{q_{j-1}\alpha - p_{j-1}}\biggr|,$$

we have, for any integer $J \geqslant 0$,

$$\sum_{j=0}^{J} (-1)^j |q_{j-1}\alpha - p_{j-1}|^{s-1} G_s(T^j(\alpha)) = \frac{\zeta(s+1)}{\pi} \sum_{j=0}^{J} (-1)^j \frac{|q_{j-1}\alpha - p_{j-1}|^s}{|q_j\alpha - p_j|} + \sum_{j=0}^{J} (-1)^j |q_{j-1}\alpha - p_{j-1}|^{s-1} P_s(T^j(\alpha)).$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

On the right-hand side, the second sum converges for any irrational number α (because $P_s(T^j(\alpha)) = \mathcal{O}(1)$). Furthermore, since

$$\frac{|q_{j-1}\alpha - p_{j-1}|^s}{|q_j\alpha - p_j|} \asymp \frac{q_{j+1}}{q_j^s},$$

the convergence of

$$\sum_{j=0}^{\infty}(-1)^j|q_{j-1}\alpha - p_{j-1}|^{s-1}G_s(T^j(\alpha))$$

implies that

$$\lim_{N\to+\infty}\sum_{j=\ell_N(\alpha)-7}^{\ell_N(\alpha)}\frac{q_{j+1}}{q_j^s} = 0.$$

Thus, the convergence of

$$\sum_{k=1}^{\infty}\frac{\cot(\pi k\alpha)}{k^s}$$

follows, with the expected identity.

On the other hand, with $N = q_m(\alpha)$ for m large enough, the identity (4.9) becomes

$$\begin{aligned}\sum_{k=1}^{q_m}\frac{\cot(\pi k\alpha)}{k^s} &= \sum_{j=0}^{m}(-1)^j|q_{j-1}\alpha - p_{j-1}|^{s-1}G_s(T^j(\alpha)) \\ &\quad - \sum_{j=\ell_{q_m}(\alpha)+1}^{m}(-1)^j|q_{j-1}\alpha - p_{j-1}|^{s-1}G_s(T^j(\alpha)) \\ &\quad + \mathcal{O}\biggl(\frac{1}{q_m^{s-2}}\biggr) + \mathcal{O}\biggl(\sum_{j=\ell_{q_m}(\alpha)-7}^{\ell_{q_m}(\alpha)}\frac{q_{j+1}}{q_j^s}\biggr) \\ &= \sum_{j=0}^{m}(-1)^j|q_{j-1}\alpha - p_{j-1}|^{s-1}G_s(T^j(\alpha)) \\ &\quad + \mathcal{O}\biggl(\frac{1}{q_m^{s-2}}\biggr) + \mathcal{O}\biggl(\sum_{j=m-15}^{m}\frac{q_{j+1}}{q_j^s}\biggr)\end{aligned}$$

because $m - 8 \leqslant \ell_{q_m}(\alpha) \leqslant m$ and

$$\sum_{j=\ell_{q_m}(\alpha)+1}^{m}(-1)^j|q_{j-1}\alpha - p_{j-1}|^{s-1}G_s(T^j(\alpha)) = \mathcal{O}\biggl(\sum_{j=m-7}^{m}\frac{q_{j+1}}{q_j^s}\biggr) = \mathcal{O}\biggl(\sum_{j=m-15}^{m}\frac{q_{j+1}}{q_j^s}\biggr).$$

We have seen as a preliminary remark that the convergence of $\sum_{k=1}^{\infty}\cot(\pi k\alpha)/k^s$ implies that

$$\lim_{m\to+\infty}\frac{q_{m+1}}{q_m^s} = 0.$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

Hence, the series

$$\sum_{j=0}^{\infty}(-1)^j|q_{j-1}\alpha-p_{j-1}|^{s-1}G_s(T^j(\alpha))$$

converges. This completes the proof of the theorem.

We now prove (1.9), i.e. that $G_s(\alpha)=\zeta(s+1)/\pi\alpha+P_s(\alpha)$, where $P_s(\alpha)$ is bounded on $[0,1]$. The function

$$f(\alpha,t):=\cot(\pi\alpha(\tfrac{1}{2}+\mathrm{i}t))-\frac{1}{\pi\alpha(\frac{1}{2}+\mathrm{i}t)}$$

is continuous as a function of $\alpha\in[0,1]$ and $t\in\mathbb{R}$, and its modulus is bounded by 1 on $[0,1]\times\mathbb{R}$. In the definition (1.7) of $G_s(\alpha)$, we make the change of variable $z=\frac{1}{2}+\mathrm{i}t$, so that

$$G_s(\alpha)=-\frac{1}{2\pi\alpha}\int_{-\infty}^{\infty}\frac{\tanh(\pi t)}{(\frac{1}{2}+\mathrm{i}t)^{s+1}}\,\mathrm{d}t-\frac{1}{2}\int_{-\infty}^{\infty}\frac{\tanh(\pi t)f(\alpha,t)}{(\frac{1}{2}+\mathrm{i}t)^{s}}\,\mathrm{d}t.$$

Using the residue theorem on the rectangular contour $\mathcal{R}_N$ defined above, it is easy to prove that

$$-\frac{1}{2\pi\alpha}\int_{-\infty}^{\infty}\frac{\tanh(\pi t)}{(\frac{1}{2}+\mathrm{i}t)^{s+1}}\,\mathrm{d}t=\frac{\zeta(s+1)}{\pi\alpha}$$

for any $s>0$. This gives the desired result with

$$P_s(\alpha)=-\frac{1}{2}\int_{-\infty}^{\infty}\frac{\tanh(\pi t)f(\alpha,t)}{(\frac{1}{2}+\mathrm{i}t)^{s}}\,\mathrm{d}t,$$

which is obviously bounded for $\alpha\in[0,1]$.

Finally, the evaluation (1.8) of $G_s(\alpha)$ when $s=2n+1$ is a consequence of the fact that $-2G_{2n+1}(\alpha)$ is equal to the residue at $z=0$ of $F_{2n+1}(\alpha,z)$. This is proved by integrating $F_{2n+1}(\alpha,z)$ on the rectangular contour with sides $[\frac{1}{2}-\mathrm{i}N,\frac{1}{2}+\mathrm{i}N]$, $[\frac{1}{2}+\mathrm{i}N,-\frac{1}{2}+\mathrm{i}N]$, $[-\frac{1}{2}+\mathrm{i}N,-\frac{1}{2}-\mathrm{i}N]$, $[-\frac{1}{2}-\mathrm{i}N,\frac{1}{2}-\mathrm{i}N]$: in the limit $N\to+\infty$, the residue of $F_{2n+1}(\alpha,z)$ at $z=0$ is equal to

$$\frac{1}{2\mathrm{i}\pi}\int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty}F_{2n+1}(\alpha,z)\,\mathrm{d}z+\frac{1}{2\mathrm{i}\pi}\int_{-1/2+\mathrm{i}\infty}^{-1/2-\mathrm{i}\infty}F_{2n+1}(\alpha,z)\,\mathrm{d}z=-2G_{2n+1}(\alpha),$$

where we use the fact that $F_{2n+1}(\alpha,-z)=-F_{2n+1}(\alpha,z)$. The explicit computation of this residue is then done by means of the Laurent expansion

$$\cot(z)=\sum_{n=0}^{\infty}(-1)^n\frac{2^{2n}B_{2n}}{(2n)!}z^{2n-1}.$$

Note that Euler's formula

$$\zeta(2n)=(-1)^{n+1}\frac{2^{2n-1}B_{2n}}{(2n)!}\pi^{2n}$$

applied to (1.8) enables us to get another proof of (1.9) when $s=2n+1$.

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

5. Proof of Theorem 1.2

We first observe that, for any real number $s \geqslant 2$, $\hat{\Phi}_s(\alpha)$ converges if and only if

$$\sum_{j=1}^{\infty} \frac{q_{j+1}(\alpha)^2}{q_j(\alpha)^s}$$

converges. This is a consequence of Lemma 3.2 because, as we mentioned in its proof, we have $\sin^2(\pi n\alpha) \asymp \|n\alpha\|^2$. This proves the first assertion of Theorem 1.2.

We now turn to the proof of $(1.15\,a)$. The proof starts similarly to that of Theorem 1.1 but differs significantly in the details. Without loss of generality, we assume that

$$\sum_{j=0}^{\infty} \frac{q_{j+1}^2}{q_j^s} \tag{5.1}$$

is convergent. We also assume that $s > 2$: a condition that will be justified below. Differentiation of (4.1) with respect to α yields

$$\sum_{k=1}^{N} \frac{1}{k^s \sin^2(\pi k\alpha)} = \alpha^{s-2} \sum_{k=1}^{[N\alpha]} \frac{1}{k^s \sin^2(\pi k T(\alpha))} + \frac{s}{\pi}\alpha^{s-1} \sum_{k=1}^{[N\alpha]} \frac{\cot(\pi k T(\alpha))}{k^{s+1}} - \frac{1}{2\pi}\frac{\partial I_s(\alpha)}{\partial \alpha} \tag{5.2}$$

for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, where

$$-\frac{1}{\pi}\frac{\partial I_s(\alpha)}{\partial \alpha} = -\frac{1}{\pi}\frac{\partial}{\partial \alpha}\frac{1}{2\mathrm{i}\pi}\int_{\mathcal{R}_N} F_{s+1}(\alpha, z)\,\mathrm{d}z = \frac{1}{2\mathrm{i}\pi}\int_{\mathcal{R}_N} \hat{F}_s(\alpha, z)\,\mathrm{d}z$$

and $\mathcal{R}_N$ is the rectangular contour defined at the beginning of § 4. In particular, $K = N + \frac{1}{2}$.

Using the method of § 4, we obtain

$$\frac{1}{2\mathrm{i}\pi}\frac{\partial I_s(\alpha)}{\partial \alpha} = \hat{G}_s(\alpha) + \mathcal{O}\biggl(\frac{1}{N^{s-1}}\biggr) + \mathcal{O}\biggl(\frac{1}{\alpha N^{s-1}\|(2N+1)\alpha\|}\biggr). \tag{5.3}$$

With the same notation used in § 4 for the sides C_k, $k = 1, 2, 3, 4$, we have

$$\biggl|\frac{1}{2\mathrm{i}\pi}\int_{C_j} \hat{F}_s(\alpha, z)\,\mathrm{d}z\biggr| \ll \frac{1}{N^s}$$

for $k = 1, 2$. On C_3, we have

$$|\hat{F}_s(\alpha, z)| \leqslant \frac{1}{N^s}\biggl|\frac{\sin(\pi y)}{\sinh(\pi\alpha y)\sin(\pi\alpha(K + \mathrm{i}y))}\biggr| \leqslant \frac{1}{\alpha N^s \|K\alpha\|} \leqslant \frac{2}{\alpha N^s \|(2N+1)\alpha\|}$$

and (5.3) follows. (The inequality $\|K\alpha\| \geqslant \frac{1}{2}\|(2N+1)\alpha\|$ is easy.) On C_4, we have

$$\frac{1}{2\mathrm{i}\pi}\int_{C_j} \hat{F}_s(\alpha, z)\,\mathrm{d}z = \hat{G}_s(\alpha) + \mathcal{O}\biggl(\frac{1}{N^{s-1}}\biggr).$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

We have

$$\lim_{N} \frac{1}{N^{s-1}\|(2N+1)\alpha\|} = 0.$$

Indeed, if $2N+1$ is not a denominator of a convergent to α,

$$\frac{1}{N^{s-1}\|(2N+1)\alpha\|} \ll \frac{1}{N^{s-2}} \to 0,$$

whereas if $2N+1 = q_k$ for some k,

$$\frac{1}{N^{s-1}\|(2N+1)\alpha\|} \ll \frac{q_{k+1}}{q_k^{s-1}} \leqslant \frac{q_{k+1}^2}{q_k^s} \to 0,$$

because of condition (5.1). Note that it is just above that we use the assumption that $s > 2$.

Using all the previous estimates on the sides C_j, we deduce that

$$\lim_{N\to+\infty} \frac{1}{2\mathrm{i}\pi} \int_{\mathcal{R}_N} \hat{F}_s(\alpha, z)\,\mathrm{d}z = \hat{G}_s(\alpha). \tag{5.4}$$

Furthermore, under (5.1), the series

$$\sum_{k=1}^{\infty} \frac{1}{k^s \sin^2(\pi k\alpha)} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^s \sin^2(\pi k T(\alpha))}$$

are convergent (by Proposition 3.1), and this is also the case for the series

$$\sum_{k=1}^{\infty} \frac{\cot(\pi k T(\alpha))}{k^{s+1}},$$

which converges under an even weaker hypothesis. For future use, we observe here that, again by Proposition 3.1, the series

$$\sum_{k=1}^{\infty} \frac{\cot(\pi k T^j(\alpha))}{k^{s+1}} \tag{5.5}$$

is bounded by a constant independent of the integer $j \geqslant 0$.

Hence, under (5.1), we can pass to the limit $N \to +\infty$ in (5.2): using (5.4), we obtain

$$\sum_{k=1}^{\infty} \frac{1}{k^s \sin^2(\pi k\alpha)} = \alpha^{s-2} \sum_{k=1}^{\infty} \frac{1}{k^s \sin^2(\pi k T(\alpha))} + \frac{s}{\pi}\alpha^{s-1} \sum_{k=1}^{\infty} \frac{\cot(\pi k T(\alpha))}{k^{s+1}} + \hat{G}_s(\alpha).$$

Iterating this identity, we get

$$\begin{aligned}\hat{\Phi}_s(\alpha) = (\alpha T(\alpha)\cdots T^J(\alpha))^{s-2}\hat{\Phi}_s(T^{J+1}(\alpha)) + \sum_{j=0}^{J} (\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-2}\hat{G}_s(\alpha) \\ + \frac{s}{\pi}\sum_{j=0}^{J} (\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-2}(T^j(\alpha))^{s-1}\Phi_{s+1}(T^{j+1}(\alpha))\end{aligned} \tag{5.6}$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

for any integer $J \geqslant 0$. Both series

$$\sum_{j=0}^{\infty}(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-2}\hat{G}_s(\alpha)$$

and

$$\frac{s}{\pi}\sum_{j=0}^{\infty}(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-2}(T^j(\alpha))^{s-1}\Phi_{s+1}(T^{j+1}(\alpha))$$

are convergent by (1.13), (5.1) and by the comments around (5.5). Furthermore,

$$\lim_{J\to+\infty}(\alpha T(\alpha)\cdots T^J(\alpha))^{s-2}\hat{\Phi}_s(T^{J+1}(\alpha)) = 0$$

because, by Proposition 3.1, the quantity $\hat{\Phi}_s(T^{J+1}(\alpha))$ is bounded independently of J. Thus, passing to the limit $J \to +\infty$ in (5.6), we obtain

$$\hat{\Phi}_s(\alpha) = \sum_{j=0}^{\infty}(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-2}\hat{G}_s(\alpha) + \frac{s}{\pi}\sum_{j=0}^{\infty}(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-2}(T^j(\alpha))^{s-1}\Phi_{s+1}(T^{j+1}(\alpha)). \tag{5.7}$$

All that remains to do is to rearrange the terms in the second series on the right-hand side of (5.7). For this, we use the identity (1.11) with α replaced by $T^{j+1}(\alpha)$ and s replaced by $s+1$:

$$\Phi_{s+1}(T^{j+1}(\alpha)) = \sum_{m=0}^{\infty}(-1)^m(T^{j+1}(\alpha)\cdots T^{m+j}(\alpha))^s G_{s+1}(T^{m+j+1}(\alpha)),$$

which is an absolutely convergent series under (5.1). By Proposition 3.1, the series

$$\sum_{m=0}^{\infty}(T^{j+1}(\alpha)\cdots T^{m+j}(\alpha))^s|G_{s+1}(T^{m+j+1}(\alpha))|,$$

is bounded by a constant independent of j. Hence, the series with positive terms

$$\sum_{j=0}^{\infty}(\alpha T(\alpha)\cdots T^{j-1}(\alpha))^{s-2}\biggl(\sum_{m=0}^{\infty}(T^{j+1}(\alpha)\cdots T^{m+j}(\alpha))^s|G_{s+1}(T^{m+j+1}(\alpha))|\biggr)$$

is convergent and this justifies the exchange of summation from (5.8 a) to (5.8 b) below (for simplicity, we write T^p for $T^p(\alpha)$):

$$\sum_{j=0}^{\infty}(\alpha T\cdots T^{j-1})^{s-2}(T^j)^{s-1}\Phi_{s+1}(T^{j+1}) = \sum_{j=0}^{\infty}(\alpha T\cdots T^{j-1})^{s-2}(T^j)^{s-1}\sum_{m=0}^{\infty}(-1)^m(T^{j+1}\ldots T^{m+j})^s G_{s+1}(T^{m+j+1})$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

$$= \sum_{j=0}^{\infty}(\alpha T\cdots T^{j-1})^{s-2}(T^j)^{s-1}\sum_{k=j+1}^{\infty}(-1)^{k+j+1}(T^{j+1}\dots T^{k-1})^s G_{s+1}(T^k) \quad (5.8\,a)$$

$$= \sum_{k=1}^{\infty}(-1)^k G_{s+1}(T^k)(\alpha T\cdots T^{k-1})\sum_{j=0}^{k-1}\frac{(-1)^{j+1}T^j}{(\alpha T\cdots T^j)^2}. \quad (5.8\,b)$$

The last equality follows by simple manipulations after permutation of summations. Recalling that $\alpha T\cdots T^{k-1} = |q_{k-1}\alpha - p_{k-1}|$, we see that (5.8 b) shows that (5.7) is indeed equal to (1.15 a). The proof of Theorem 1.2 is complete.

Finally, the behaviour at $\alpha = 0$ of $G_s(\alpha)$ described by (1.13) can be obtained by the method used to prove (1.9).

6. Proof of Theorem 1.3

By Proposition 3.1, if $s > 1$, then the convergence of the series $\sum_{j=1}^{\infty} q_{j+1}/q_j^s$ implies the absolute convergence of the three series

$$\sum_{n=1}^{\infty}(-1)^n\frac{\cot(\pi n\alpha)}{n^s}, \quad \sum_{n=1}^{\infty}\frac{1}{n^s\sin(\pi n\alpha)} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^s\sin(\pi n\alpha)},$$

which proves the first assertion of the theorem.

We only sketch the proofs of the identities (1.19)–(1.21), because the argument is very similar to the proof of Theorem 1.2. We define the integrals

$$\tilde{I}_N(\alpha) := \frac{1}{2\mathrm{i}\pi}\int_{\mathcal{R}_N} P_s(\alpha, z)\,\mathrm{d}z,$$
$$\tilde{J}_N(\alpha) := \frac{\alpha^s}{2\mathrm{i}\pi}\int_{\mathcal{R}_N} P_s(1/\alpha, \alpha z)\,\mathrm{d}z,$$
$$\tilde{L}_N(\alpha) := \frac{1}{2\mathrm{i}\pi}\int_{\mathcal{R}_N} Q_s(\alpha, z)\,\mathrm{d}z$$

where $\mathcal{R}_N$ is the contour defined at the beginning of § 4. Evaluating the integrals by the residue theorem, we get

$$\sum_{n=1}^{N}(-1)^n\frac{\cot(\pi n\alpha)}{n^s} = \alpha^{s-1}\sum_{n=1}^{[N\alpha]}\frac{(-1)^{n[T(\alpha)]}}{n^s\sin(\pi n T(\alpha))} + U_s(\alpha) - \tfrac{1}{2}\tilde{I}_N(\alpha), \quad (6.1)$$

$$\sum_{n=1}^{N}\frac{1}{n^s\sin(\pi n\alpha)} = \alpha^{s-1}\sum_{n=1}^{[N\alpha]}(-1)^n\frac{\cot(\pi n/\alpha)}{n^s} + V_s(\alpha) - \tfrac{1}{2}\tilde{J}_N(\alpha) \quad (6.2)$$

$$\sum_{n=1}^{N}\frac{(-1)^n}{n^s\sin(\pi n\alpha)} = \alpha^{s-1}\sum_{n=1}^{[N\alpha]}\frac{(-1)^{n([T(\alpha)]+1)}}{n^s\sin(\pi n T(\alpha))} + W_s(\alpha) - \tfrac{1}{2}\tilde{L}_N(\alpha). \quad (6.3)$$

We now assume that $\sum_j q_{j+1}/q_j^s$ is convergent and that $s > 2$. By analytic estimates similar to those made during the proof of Theorem 1.2, the three integrals $\tilde{I}_N(\alpha)$, $\tilde{J}_N(\alpha)$

and $\tilde{L}_N(\alpha)$ tend to 0 as $N \to +\infty$; this is where the hypothesis $s > 2$ is used. We deduce that, for any integer $k \geqslant 0$, (6.1)–(6.3) yield the identities

$$\sum_{n=1}^{\infty} (-1)^n \frac{\cot(\pi n T^k(\alpha))}{n^s} = T^k(\alpha)^{s-1} \sum_{n=1}^{\infty} \frac{(-1)^{n a_{k+1}}}{n^s \sin(\pi n T^{k+1}(\alpha))} + U_s(T^k(\alpha)), \tag{6.4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^s \sin(\pi n T^k(\alpha))} = T^k(\alpha)^{s-1} \sum_{n=1}^{\infty} (-1)^n \frac{\cot(\pi n T^{k+1}(\alpha))}{n^s} + V_s(T^k(\alpha)), \tag{6.5}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^s \sin(\pi n T^k(\alpha))} = T^k(\alpha)^{s-1} \sum_{n=1}^{\infty} \frac{(-1)^{n(a_{k+1}+1)}}{n^s \sin(\pi n T^{k+1}(\alpha))} + W_s(T^k(\alpha)), \tag{6.6}$$

where we recall that the partial quotients a_k of the continued fraction of α are given by $a_k = [T^k(\alpha)]$.

To deduce (1.19), we start from (6.4) with $k = 0$.

(i) If a_1 is even, we use (6.5) with $k = 1$ and then we use (6.4) again but with $k = 2$.

(ii) If a_1 is odd, we use (6.6) with $k = 1$:

  (a) if a_2 is odd, we use (6.5) with $k = 2$ and then (6.4) again but with $k = 3$;

  (b) if a_2 is even, we use again (6.6) with $k = 2$.

And so on. This generates a sequence of identities which in the limit gives

$$\Psi_s(\alpha) = \sum_{j=0}^{\infty} (-1)^j |q_{j-1}\alpha - p_{j-1}|^{s-1} U_{j,s}(T^j(\alpha)),$$

with $U_{0,s} = U_s$ and $U_{j,s}$ as defined in Theorem 1.3. We get the identities for $\hat{\Psi}_s(\alpha)$ and $\tilde{\Psi}_s(\alpha)$ in a similar way. We conclude this 'proof' with the remark that it is important that not only are the three series on the left-hand sides of (6.4)–(6.6) convergent but their sums are bounded independently of k (this is ensured by Proposition 3.1).

We conclude this section by explaining the origin of the identities (1.16) and (1.17). In fact, the expressions $-2U_{2n+1}(\alpha)$ and $-2W_{2n+1}(\alpha)$ are just the residues at $z = 0$ of $P_{2n+1}(\alpha, z)$ and $Q_{2n+1}(\alpha, z)$, respectively. This follows from the Laurent expansions at the origin of $\cot(z)$ and

$$\frac{1}{\sin(z)} = \sum_{n=0}^{\infty} (-1)^{n+1} (2^{2n} - 2) \frac{B_{2n}}{(2n)!} z^{2n-1}.$$

7. Generalizations

7.1. A generalization of $\Phi_s(\alpha)$ and $\hat{\Phi}_s(\alpha)$

The series $\Phi_s(\alpha)$ and $\hat{\Phi}_s(\alpha)$ correspond to the cases $r = 0$ and $r = 1$ of the series

$$\sum_{n=1}^{\infty} \frac{\cot^{(r)}(\pi n \alpha)}{n^s}.$$

https://doi.org/10.1017/S0013091510001069 Published online by Cambridge University Press

We believe that the following statement, which would generalize Theorems 1.1 and 1.2, holds: *we fix an integer* $r \geqslant 2$, *a real number* $s > r+1$ *and an irrational number* $\alpha \in (0,1)$. *The series*

$$\sum_{n=1}^{\infty} \frac{\cot^{(r)}(\pi n\alpha)}{n^s}$$

converges if and only if

$$\sum_{j=0}^{\infty} (-1)^{(r+1)j} \frac{q_{j+1}^{r+1}(\alpha)}{q_j^s(\alpha)}$$

converges. Furthermore, we have the identity

$$\sum_{n=1}^{\infty} \frac{\cot^{(r)}(\pi n\alpha)}{n^s} = \frac{1}{\pi^r} \sum_{j=0}^{\infty} (-1)^j \frac{\partial^r}{\partial\alpha^r}(|q_{j-1}\alpha - p_{j-1}|^{s+r-1} G_{s+r}(T^j(\alpha))), \tag{7.1}$$

where both series converge or diverge simultaneously. The methods of the present paper show that (7.1) holds under the stronger assumption that the series

$$\sum_{j=0}^{\infty} \frac{q_{j+1}^{r+1}(\alpha)}{q_j^s(\alpha)}$$

converges. This follows from an adaptation of the proof of Theorem 1.2.

7.2. A multivariate generalization of $\boldsymbol{\Phi_s(\alpha)}$ and the Jacobi–Perron algorithm

A natural generalization of the function $\Phi_s(\alpha)$ is

$$\Phi_{d,s}(\boldsymbol{\alpha}) := \sum_{n=1}^{\infty} \frac{1}{n^s} \prod_{j=1}^{d} \cot(\pi n\alpha_j),$$

where $d \geqslant 1$, $s \in \mathbb{R}$ and $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_d) \in (0,1)^d$.

It is not difficult to see that $\Phi_{d,s}(\boldsymbol{\alpha})$ converges for almost all $\boldsymbol{\alpha} \in (0,1)^d$ when $s > d$. Indeed, by Hölder's inequality, we have

$$\sum_{n=1}^{N} \frac{1}{n^s} \prod_{j=1}^{d} |\cot(\pi n\alpha_j)| \ll \left(\sum_{n=1}^{N} \frac{1}{n^s \prod_{j=1}^{d} \|n\alpha_j\|} \right) \leqslant \prod_{j=1}^{d} \left(\sum_{n=1}^{N} \frac{1}{n^s \|n\alpha_j\|^d} \right)^{1/d}.$$

Lemma 3.2 implies that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s \|n\alpha\|^d}$$

converges for almost all α under the condition $s > d$. This proves that $\Phi_{d,s}(\boldsymbol{\alpha})$ converges for all $\boldsymbol{\alpha}$ in an (explicit) subset $\mathcal{C}_{d,s}$ of $(0,1)^d$ of measure 1. Another method to prove the almost-everywhere convergence of series like

$$\sum_{n=1}^{\infty} \frac{1}{n^s \prod_{j=1}^{d} \|n\alpha_j\|}$$

can be found in [**19**, proof of (6.3), pp. 94–95] for the case $s = 2$ and $d = 2$. It is non-constructive. It is an interesting problem to determine the exact conditions of convergence of $\Phi_{d,s}(\boldsymbol{\alpha})$. The above argument shows that convergence occurs if we consider the Diophantine behaviour of each α_j (through its continued fraction), independently of the others. However, the above set of convergence $\mathcal{C}_{d,s}$ is probably not the exact set of convergence of $\Phi_{d,s}(\boldsymbol{\alpha})$. In fact, as we now explain, this problem seems to be related to the Jacobi–Perron algorithm applied to the vector $\boldsymbol{\alpha}$.

We introduce the functions

$$F_{d,s}(\boldsymbol{\alpha}, z) := \frac{1}{z^s}\pi\cot(\pi z)\prod_{j=1}^{d}\cot(\pi\alpha_j z) \quad \text{and} \quad G_{d,s}(\boldsymbol{\alpha}) := \frac{1}{2\mathrm{i}\pi}\int_{1/2+\mathrm{i}\infty}^{1/2-\mathrm{i}\infty} F_{d,s}(\boldsymbol{\alpha}, z)\,\mathrm{d}z. \tag{7.2}$$

If s is an integer such that $s \equiv d[2]$, we have $G_{d,s}(\boldsymbol{\alpha}) := -\frac{1}{2}\,\mathrm{residue}(F_{d,s}(\boldsymbol{\alpha}, z), z = 0)$, which is a rational function in $\boldsymbol{\alpha}$ with coefficients in $\mathbb{Q}\pi^{s-1}$.

For simplicity, we explain the link with the two-dimensional Jacobi–Perron algorithm in the case $d = 2$ (where we write $\boldsymbol{\alpha} = (\alpha, \beta)$) but the general case can also be expressed in terms of the Jacobi–Perron algorithm in higher dimensions (see [**6**] for the theory). Integrating $F_{2,s}(\boldsymbol{\alpha}, z)$ on the contour $\mathcal{R}_N$ and letting $N \to +\infty$, we obtain the relation

$$\Phi_{2,s}(\alpha, \beta) = \alpha^{s-1}\Phi_{2,s}(T(\alpha/\beta), T(\alpha)) + \beta^{s-1}\Phi_{2,s}(T(\beta), T(\beta/\alpha)) + G_{2,s}(\alpha, \beta), \tag{7.3}$$

where we assume for simplicity that $s > 2$ and that $(\alpha, \beta) \in \mathcal{C}_{2,s}$ to ensure convergence of the three series.

We remark that the transformation of the square $(0, 1)^2$ to itself defined by $(x, y) \mapsto (T(x/y), T(x))$ is exactly the transformation used to run the two-dimensional Jacobi–Perron algorithm with some initial value (x_0, y_0). This algorithm is used to produce good simultaneous rational approximations of (x_0, y_0). In (7.3), we see that the Jacobi–Perron algorithm is run on (α, β) and (β, α). The convergence/divergence of $\Phi_{2,s}(\alpha, \beta)$ clearly depends on how badly or well the numbers α and β are simultaneously approximated by rational numbers. It is therefore natural to wonder if the conditions of convergence of $\Phi_{2,s}(\alpha, \beta)$ and of validity of (7.3) can be expressed in terms of the rational sequences generated by the Jacobi–Perron algorithm.

References

1. W. W. Adams, Asymptotic Diophantine approximations to e, *Proc. Natl Acad. Sci. USA* **55** (1966), 28–31.
2. L. Báez-Duarte, M. Balazard, B. Landreau and E. Saias, Étude de l'autocorrélation multiplicative de la fonction 'partie fractionnaire', *Ramanujan J.* **9**(2) (2005), 215–240.
3. M. Balazard and B. Martin, Comportement local moyen de la fonction de Brjuno, preprint (2010).
4. J. Beck, Randomness of the square root of 2 and the giant leap, Part 1, *Period. Math. Hungar.* **60**(2) (2010), 137–242.
5. B. Berndt, Modular transformations and generalizations of several formulae of Ramanujan, *Rocky Mt. J. Math.* **7** (1977), 147–189.

6. L. Bernstein, *The Jacobi–Perron algorithm, its theory and applications*, Lecture Notes in Mathematics, Volume 207 (Springer, 1971).
7. S. D. Chowla, Some problems of Diophantine approximation, I, *Math. Z.* **33** (1931), 544–563.
8. H. Davenport, On some infinite series involving arithmetical functions, I, *Q. J. Math.* **8** (1937), 8–13.
9. H. Davenport, On some infinite series involving arithmetical functions, II, *Q. J. Math.* **8** (1937), 313–320.
10. H. Davenport, Note on an identity connected with Diophantine approximation, *Proc. Camb. Phil. Soc.* **34** (1938), 27.
11. R. de la Bretèche and G. Tenenbaum, Séries trigonométriques à coefficients arithmétiques, *J. Analyse Math.* **92** (2004), 1–79.
12. E. Grosswald, Comments on some formulae of Ramanujan, *Acta Arith.* **21** (1972), 25–34.
13. G. H. Hardy and J. E. Littlewood, Some problems of Diophantine approximation, I, The fractional part of $n^k\Theta$, *Acta Math.* **37** (1914), 155–191.
14. G. H. Hardy and J. E. Littlewood, Some problems of Diophantine approximation, II, The trigonometrical series associated with the elliptic ϑ-functions, *Acta Math.* **37** (1914), 193–239.
15. G. H. Hardy and J. E. Littlewood, Some problems of Diophantine approximation: the analytic character of the sum of a Dirichlet's series considered by Hecke, *Hamb. Math. Abh.* **3** (1923), 57–68.
16. G. H. Hardy and J. E. Littlewood, Some problems of Diophantine approximation: a series of cosecant, *Bull. Calcutta Math. Soc.* **20** (1930), 251–266.
17. A. Ya. Khintchine, *Continued fractions* (Dover, New York, 1997).
18. A. H. Kruse, Estimates of $\sum_{k=1}^{N} k^{-s}\langle kx\rangle^{-t}$, *Acta Arith.* **12** (1966), 229–261.
19. M. Laczkovich, Equidecomposability and discrepancy: a solution of Tarski's circle-squaring problem, *J. Reine Angew. Math.* **404** (1990), 77–117.
20. R. Marcovecchio, The Rhin–Viola method for log(2), *Acta Arith.* **139**(2) (2009), 147–184.
21. B. Martin, Nouvelles identités de Davenport, *Funct. Approx. Comment. Math.* **37** (2007), 293–327.
22. K. Nishioka, *Mahler functions and transcendence*, Lecture Notes in Mathematics, Volume 1631 (Springer, 1996).
23. M. Ram Murty, C. Smyth and R. J. Wang, Zeros of Ramanujan polynomials, *J. Ramanujan Math. Soc.* **26** (2011), 107–125.
24. K. F. Roth, Rational approximations to algebraic numbers, *Mathematika* **2** (1955), 1–20 (see also corrigendum, p. 168).
25. V. Kh. Salikhov, On the irrationality measure of π, *Russ. Math. Surv.* **63** (2008), 570–572.
26. J. Schoissengeier, On the convergence of a series of Bundschuh, *Uniform Distribution Theory* **2** (2007), 107–113.
27. J. Schoissengeier and S. B. Tričković, On the divergence of a certain series, *J. Math. Analysis Applic.* **324** (2006), 238–247.
28. G. Tenenbaum, *Introduction à la théorie analytique et probabiliste des nombres*, 3rd edition (Belin, Paris, 2008).
29. A. Walfisz, Über einige trigonometrische Summen, *Math. Z.* **33** (1931), 564–601.
30. J. R. Wilton, An approximate functional equation with applications to a problem of Diophantine approximation, *J. Reine Angew. Math.* **169** (1933), 219–237.