Canad. Math. Bull. Vol. 53 (1), 2010 pp. 122–132
doi:10.4153/CMB-2010-026-8

A Class of Finsler Metrics with Bounded Cartan Torsion

Xiaohuan Mo and Linfeng Zhou

Abstract. In this paper, we find a class of (α, β) metrics which have a bounded Cartan torsion. This class contains all Randers metrics. Furthermore, we give some applications and obtain two corollaries about curvature of this metrics.

1 Introduction

On a manifold M, Finsler metrics are Riemannian metrics without the quadratic restriction. They give Minkowski norms instead of inner products on each tangent space T_xM. So they are more colorful and more complicated than Riemannian metrics. We mention the fact that Finsler metrics also have a (flag) curvature, but the meaning of constancy of this curvature remains mysterious today.

Cartan torsion is one of the most fundamental non-Riemannian quantities. It was first introduced by P. Finsler [8] and emphasized by E. Cartan [5], and it measures a departure from a Riemannian manifold. Precisely, a Finsler metric is Riemannian if and only if it has vanishing Cartan torsion. Intuitively, if the norm of Cartan torsion is far from zero, then this Finsler manifold is very different being a Riemannian manifold. For example, J. Nash [11] proved that any n-dimensional Riemannian manifold can be isometrically imbedded into a higher dimensional Euclidean space. So one question in Finsler geometry is whether every Finsler manifold can be isometrically imbedded into a Minkowski space. The answer is negative, as Burago–Ivanov [4] showed that there exists a Finsler metric on any non-compact manifold M that cannot be imbedded into any finite dimensional Banach space. Shen [15] proved that a Finsler manifold with unbounded Cartan torsion cannot be isometrically imbedded into any Minkowski space.

Randers metrics, which can be viewed as a perturbation of a Riemannian metric [3], are the simplest non-Riemannian Finsler metrics having the form

$$F = \alpha + \beta \quad (\|\beta\|_\alpha < 1)$$

where $\alpha := \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta := b_i(x)y^i$ is a 1-form. It is proved that the Cartan torsion of Randers metrics is uniformly bounded by $3/\sqrt{2}$. The bound for two dimensional Randers metrics was verified by B. Lackey and was extended to higher dimensions by Z. Shen [2, 13].

Received by the editors December 18, 2006; revised October 19, 2007.
Published electronically December 4, 2009.
This work was supported by the National Natural Science Foundation of China 10771004.
AMS subject classification: **58E20**.
Keywords: Finsler manifold, (α, β) metric, Cartan torsion, R-quadratic, flag curvature.

A natural task for us is to find other Finsler metrics which have bounded Cartan torsion. In this paper we explicitly construct a class of Finsler metrics (including Randers metrics and Berwald's famous example [12]) which have the following form

$$F = \frac{(\alpha+\beta)^s}{\alpha^{s-1}} \quad (s \in [1,\, 2]).$$

We will prove the following theorem.

Theorem 1.1 *Suppose that*

$$F = \frac{(\alpha+\beta)^s}{\alpha^{s-1}},$$

with $1 \leq s < 2$ *and* $\|\beta\|_\alpha < 1$, *is an* (α, β) *metric on a manifold* M, *where* $\alpha := \sqrt{a_{ij}(x)y^i y^j}$ *is a Riemannian metric and* $\beta := b_i(x)y^i$ *is a* 1*-form. Then the Cartan torsion of* F *is bounded.*

As an application, we obtain the following rigidity results.

Corollary 1.2 *Let* (M, F) *be a positively complete Finsler manifold. Suppose* F *is R-quadratic and has the form*

$$F = \frac{(\alpha+\beta)^s}{\alpha^{s-1}} \quad (s \in [1,2),\, \|\beta\|_\alpha < 1)$$

where $\alpha := \sqrt{a_{ij}(x)y^i y^j}$ *is Riemannian metric and* $\beta := b_i(x)y^i$ *is a* 1*-form. Then* F *must be Berwaldian.*

Corollary 1.3 *Let* (M, F) *be a complete Finsler manifold. Suppose* F *has constant flag curvature* K *and has the form*

$$F = \frac{(\alpha+\beta)^s}{\alpha^{s-1}} \quad (s \in [1,2),\, \|\beta\|_\alpha < 1)$$

where $\alpha := \sqrt{a_{ij}(x)y^i y^j}$ *is a Riemannian metric and* $\beta := b_i(x)y^i$ *is a* 1*-form.*

(i) *If* $K < 0$, *then* F *must be Riemannian.*

(ii) *If* $K = 0$, *then* F *must be locally Minkowskian.*

2 Cartan Torsion and Curvature

A Finsler metric on a manifold M is a C^∞ function on $TM\backslash\{0\}$ having the following properties.

- $F(x, y) \geq 0$ for any $y \in T_xM$ and $F(x, y) = 0$ if and only if $y = 0$;
- $F(x, \lambda y) = \lambda F(x, y)$ for any $y \in T_xM$ and $\lambda > 0$;
- For each $y \in T_xM$, the following bilinear symmetric form $\mathbf{g}_y$ is positive definite:

$$\mathbf{g}_y(u, v) := \tfrac{1}{2}[F^2(y + su + tv)]|_{s,t=0} \quad u, v \in T_xM.$$

https://doi.org/10.4153/CMB-2010-026-8 Published online by Cambridge University Press

A *Riemannian metric* is a special case such that at each point $x \in M$ the fundamental tensor $\mathbf{g}_y$ is independent of the tangent vector $y \in T_xM\backslash\{0\}$. To measure the non-Riemannian feature of F, define $\mathbf{C}_y\colon T_xM \times T_xM \times T_xM \to R$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}[\mathbf{g}_{y+tw}(u, v)]|_{t=0}, \quad u, v, w \in T_xM.$$

This trilinear symmetric form on the pullback bundle π^*TM(over $TM\backslash\{0\}$) is called *Cartan torsion*. E. Cartan got this quantity when he introduced his metric-compatible connection. Obviously F is a Riemannian metric if and only if $\mathbf{C}_y = 0$.

In fact, the *mean Cartan torsion* can also characterize the Riemannian metric. It is defined by

$$I_y(u) := g^{ij}(y)\mathbf{C}_y\Big(u, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\Big).$$

Deicke [7] proved that F is a Riemannian metric if and only if the mean Cartan torsion satisfies $\mathbf{I}_y = 0$ for any $y \in T_xM\backslash\{0\}$.

The bound of Cartan torsion $\mathbf{C}$ *at a point* $x \in M$ is defined by

$$\|\mathbf{C}\|_x := \sup_{y,u\in T_xM} \frac{F(x, y)|\mathbf{C}_y(u, u, u)|}{\sqrt{\mathbf{g}_y(u, u)^3}}$$

and *the bound of Cartan torsion on* M is defined by $\|\mathbf{C}\| := \sup_{x\in M} \|\mathbf{C}\|_x$.

Let $r(t)\colon [0, 1] \to M$ be a piecewise C^∞ curve on a Finsler manifold (M, F). We can define *the length of* $r(t)$ by $L(r) := \int_0^1 F(r(t), r'(t))\, dt$. By the first variation of length we can see a geodesic must satisfy the equation

$$\frac{d^2r^i}{dt^2} + 2G^i(r(t), r'(t)) = 0,$$

where $G^i(x, y)$ are called *the spray coefficients of* F [12] and are given in local coordinates by

$$G^i := \frac{1}{4}g^{il}\Big(\frac{\partial^2F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big).$$

A Finsler manifold is *positive complete* if every geodesic $r(t)$ with $t \in [0, 1]$ can be extended to $[0, +\infty)$.

For a tangent vector $y \in T_xM\backslash\{0\}$, define a tensor on the pullback bundle π^*TM $\mathbf{B}_y\colon T_xM \otimes T_xM \otimes T_xM \to T_xM$ in local coordinates by

$$\mathbf{B}_y(u, v, w) := B^i_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i},$$

where $u = u^i\frac{\partial}{\partial x^i}|_x$, $v = v^i\frac{\partial}{\partial x^i}|_x$, $w = w^i\frac{\partial}{\partial x^i}|_x$, and

$$B^i_{jkl} := \frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}.$$

A Finsler metric is called a *Berwald metric* if $\mathbf{B} = 0$. This is equivalent to its spray coefficients G^i being quadratic in y at every point $x \in M$. A Riemannian metric is Berwaldian because in this case $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$, where Γ^i_{jk} are Christoffel symbols.

Using the above tensor $\mathbf{B}_y$, we can define another tensor

$$\mathbf{L}_y := -\tfrac{1}{2}\mathbf{g}_y(\mathbf{B}_y(u,v,w),y).$$

In local coordinates, $\mathbf{L}_y(u,v,w) := L_{ijk}(y)u^i v^j w^k$ where $u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$, $w = w^i \frac{\partial}{\partial x^i}|_x$, and $L_{ijk} = -\frac{1}{2}y^m g_{ml}(y)B^l_{ijk}(y)$. A Finsler metric is called a *Landsberg metric* if $\mathbf{L} = 0$. Obviously, Berwald metrics must be Landsberg metrics.

Now we recall the definition of *Riemannian curvature*. For a vector $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$, define $\mathbf{R}_y = R^i_j dx^j \otimes \frac{\partial}{\partial x^i} \colon T_xM \to T_xM$ by

$$R^i_j := 2\frac{\partial G^i}{\partial x^j} - y^k \frac{\partial^2 G^i}{\partial x^k \partial y^j} + 2G^k \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^k}\frac{\partial G^k}{\partial y^j}.$$

A Finsler metric is said to be *R-quadratic* if its Riemannian curvature $\mathbf{R}_y$ is quadratic in $y \in T_xM$. All Berwald metrics are R-quadratic. The notion of R-quadratic metric is weaker than that of Berwald metric. See the example in Section 4.

For any tangent plane $P = \operatorname{span}\{y,u\} \subset T_xM$ define

$$K(P,y) := \frac{g_y(R_y(u),u)}{g_y(y,y)g_y(u,u) - g_y(y,u)g_y(y,u)};$$

K is called *flag curvature*. Usually, $K(P,y)$ depends on the direction $y \in P$. In the Riemannian case, $K(P,y)$ is independent of $y \in P$. So flag curvature generalizes sectional curvature in Riemannian geometry. If flag curvature K is constant, we say *F has constant flag curvature*. It is easy to see that F has constant flag curvature if and only if [2]

$$R^i{}_j = KF^2\left(\delta^i_j - \frac{y^i}{F}F_{y^j}\right).$$

3 A Class of (α,β) Metrics with Bounded Cartan Torsion

A Finsler metric F on a manifold is called an *(α,β)-metric* if it is in the form $F = \alpha\phi(\beta/\alpha)$,where $\alpha := \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta := b_i(x)y^i$ is a 1-form, $\phi = \phi(t)$ is a positive C^∞ function on some interval $(-r,\ r)$, and $\|\beta\|_\alpha < r$.

Lemma 3.1 *$F = \alpha\phi(\beta/\alpha)$ is a Finsler metric for any Riemannian α and 1-form β with $\|\beta\|_\alpha < r$ if and only if $\phi = \phi(t)$ satisfies the following conditions*

$$\phi(t) > 0, \quad (\phi(t) - t\phi'(t)) + (b^2 - t^2)\phi''(t) > 0,$$

where t and b are arbitrary numbers with $|t| \le b < r$.

For the proof, see [6] for more details.

Below is a family of special (α,β) metrics. Let $\phi\colon(-1,1)\to R$, $\phi(t)=(1+t)^s$, where $1\le s\le 2$ and $\|\beta\|_\alpha<1$. It is easy to see that

$$\phi'(t)=s(1+t)^{s-1},\qquad \phi''(t)=s(s-1)(1+t)^{s-2}.$$

It follows that

$$\phi(t)=(1+t)^s>0,\qquad \phi(t)-t\phi'(t)=(1+t)^{s-1}[1+t(1-s)]>0,$$
$$\phi''(t)=s(s-1)(1+t)^{s-2}\ge 0$$

for $|t|<1$. Thus $F=\alpha\phi\big(\frac{\beta}{\alpha}\big)=(\alpha+\beta)^s/\alpha^{s-1}$ is a Finsler metric.

Example 3.2 Let ζ be an arbitrary constant and $\Omega=\mathrm{B}^n(r)$ where $r=1/\sqrt{-\zeta}$ if $\zeta<0$ and $r=+\infty$ if $\zeta\ge 0$. Define $F=\alpha+\beta\colon T\Omega\to[0,\infty)$ by

$$\alpha(x,y):=\frac{\sqrt{\kappa^2\langle x,y\rangle^2+\epsilon|y|^2(1+\zeta|x|^2)}}{(1+\zeta|x|^2)^p}\quad\text{and}\quad \beta(x,y):=\frac{\kappa\langle x,y\rangle}{(1+\zeta|x|^2)^p},$$

where ϵ is an arbitrary positive constant, and κ, p are arbitrary constants. A direct calculation yields

$$\|\beta\|^2_\alpha=\frac{\kappa^2|x|^2}{\epsilon+\varrho^2|x|^2}<1,$$

where $\varrho^2:=\epsilon\zeta+\kappa^2$ and α is a Riemannian metric.

Thus we get the following (α,β)-metrics.

$$\tilde F(x,y):=\frac{\Big(\sqrt{\kappa^2\langle x,y\rangle^2+\epsilon|y|^2(1+\zeta|x|^2)}+\kappa\langle x,y\rangle\Big)^q}{(1+\zeta|x|^2)^p\Big(\sqrt{\kappa^2\langle x,y\rangle^2+\epsilon|y|^2(1+\zeta|x|^2)}\Big)^{q-1}}.\tag{3.1}$$

Note that β is an exact form.

Remark We have several special cases of the above Example.

- When $p=q=1$, our metrics have been studied in [10]. Furthermore, if $\zeta=-1$, $\kappa=\pm1$ and $\epsilon=1$, they are reduced to the famous Funk metrics.
- When $p=q=2$, $\zeta=-1$, $\kappa=\pm1$ and $\epsilon=1$ (3.1) is reduced to Berwald's example. Its projective factor is exactly the Funk metric.

Proof of Theorem 1.1 Let us first consider the case of $\dim M=2$. There exists a local orthonormal coframe $\{\omega_1,\omega_2\}$ of Riemannian metric α. So α^2 can be written as $\alpha^2=\omega_1^2+\omega_2^2$. If we let $\alpha=\sqrt{a_{ij}y^iy^j}$ where $y=\sum_{i=1}^2 y^i\mathbf{e}_i$ and $\{\mathbf{e}_i\}$ is the dual frame of $\{\omega_i\}$, then $a_{ij}=\delta_{ij}$ and $a^{ij}=\delta^{ij}$. Adjust the coframe $\{\omega_1,\omega_2\}$

https://doi.org/10.4153/CMB-2010-026-8 Published online by Cambridge University Press

properly so that $\beta = \kappa\omega_1$. Then $b_1 = \kappa, b_2 = 0$ where $\beta = \sum_{i=1}^{2} b_i y^i$. Hence $\|\beta\|_\alpha := \sqrt{a^{ij}b_i b_j} = \kappa$.

For an arbitrary tangent vector $y = u\mathbf{e}_1 + v\mathbf{e}_2 \in T_pM$ we can obtain that

$$\alpha(p,y) = \sqrt{u^2+v^2}, \quad \beta(p,y) = \kappa u, \quad F(p,y) = \sqrt{u^2+v^2}\Big(1+\frac{\kappa u}{\sqrt{u^2+v^2}}\Big)^s.$$

Assume that $y^\perp$ satisfies

$$\mathbf{g}_y(y,y^\perp) = 0, \quad \mathbf{g}_y(y^\perp,y^\perp) = F^2(p,y). \tag{3.2}$$

Obviously $y^\perp$ is unique, because the metric is non-degenerate. The frame $\{y, y^\perp\}$ is called the *Berwald frame* [1].

Let $y = r\cos\theta\mathbf{e}_1 + r\sin\theta\mathbf{e}_2$, *i.e.*, $u = r\cos\theta$ and $v = r\sin\theta$. Plugging the above expression into (3.2) and computing with Maple (see Section A.1 below) yields

$$y^\perp = \frac{r\big(\sin\theta(-1-\kappa\cos\theta+s\kappa\cos\theta), \kappa(s-1)\cos^2\theta-\cos\theta-s\kappa\big)}{\sqrt{\cos^2\theta\kappa^2(1-s^2)+s\kappa^2(s-1)-\kappa\cos\theta(s-2)+1}}. \tag{3.3}$$

By the definition of the bound of Cartan torsion it is easy to show that for the Berwald frame $\{y, y^\perp\}$,

$$\|\mathbf{C}\|_p = \sup_{y\in T_pM\setminus\{0\}} \xi(p,y),$$

where

$$\xi(p,y) := \frac{F(p,y)|\mathbf{C}_y(y^\perp,y^\perp,y^\perp)|}{|\mathbf{g}_y(y^\perp,y^\perp)|^{3/2}}.$$

Again computing with Maple (see Section A.2 below), we obtain

$$\begin{aligned}&\xi(p,y)\\ &= \left|\frac{s\kappa\sin\theta\big(4(s^2-1)\kappa^2\cos^2\theta+\kappa(6s-9)\cos\theta+6s\kappa^2-4s^2\kappa^2-2\kappa^2-3\big)}{2\big(-\kappa^2(s^2-1)\cos^2\theta+\kappa(2-s)\cos\theta+s^2k^2-s\kappa^2+1\big)^{3/2}}\right|.\end{aligned} \tag{3.4}$$

Define two functions on $[0,1)\times[-1,1]$:

$$f(\kappa,x) := -\kappa^2(s^2-1)x^2+\kappa(2-s)x+s^2\kappa^2-s\kappa^2+1, \tag{3.5}$$

$$g(\kappa,x) := \frac{s\kappa\sqrt{1-x^2}}{2f(\kappa,x)^{\frac{3}{2}}}\big[4(s^2-1)\kappa^2x^2+\kappa(6s-9)x+6s\kappa^2-4s^2\kappa^2-2\kappa^2-3\big]. \tag{3.6}$$

Hence

$$\|\mathbf{C}\|_p = \max_{0\le\theta\le 2\pi}|g(\kappa,\cos\theta)|. \tag{3.7}$$

https://doi.org/10.4153/CMB-2010-026-8 Published online by Cambridge University Press

Notice that

$$\lim_{\kappa\to 1^-} g(\kappa,x)=\frac{s\sqrt{1-x}|4xs^2-4x-5-4s^2+6s|}{2|x-xs^2+s^2-s+1|},$$

for all $x\in[-1,1]$ with $2(-x+xs^2-s^2+s-1)\neq 0$. So $\lim_{\kappa\to 1^-} g(\kappa,x)$ is continuous and has an upper bound G. Then it follows that there exists $\delta>0$, such that when $1-\delta\le\kappa<1$ and $x\in[-1,1]$ we have $g(\kappa,x)\le G+1$. From (3.7) we get

$$\|\mathbf{C}\|_p\le G+1. \tag{3.8}$$

Now we consider the case of $0\le\kappa\le 1-\delta$. For a fixed $\kappa=\kappa_0$, $f(\kappa_0,x)$ is a parabola and opening downwards. So $f(\kappa_0,x)\ge\min\{f(\kappa_0,1),f(\kappa_0,-1)\}$. By a direct computation we obtain

$$f(\kappa_0,1)=(1+\kappa_0)[1-(s-1)\kappa_0],\quad f(\kappa_0,-1)=(1-\kappa_0)[1+(s-1)\kappa_0].$$

Since $1\le s<2$, we can choose an infinitesimal $\epsilon>0$ satisfying $1\le s<2-\epsilon$. As ϵ is infinitesimal and $0\le\kappa_0\le 1-\delta<1$, so

$$\begin{aligned}f(\kappa_0,1)&>(1+\kappa_0)[1-(1-\epsilon)\kappa_0]>(1+1)[1-(1-\epsilon)]=2\epsilon,\\ f(\kappa_0,-1)&\ge\delta[1+(s-1)(1-\delta)]>\delta.\end{aligned}$$

Then $f(\kappa_0,x)>\zeta:=\min\{2\epsilon,\delta\}$. From (3.6) and (3.7), we have

$$\begin{aligned}\|\mathbf{C}\|_p&<\max_{0\le\theta\le 2\pi}\frac{\left|\begin{aligned}s\kappa\sin\theta\big(4(s^2-1)\kappa^2\cos^2\theta+\kappa(6s-9)\cos\theta\\ +6s\kappa^2-4s^2\kappa^2-2\kappa^2-3\big)\end{aligned}\right|}{4\zeta^{3/2}}\\ &\le\max_{0\le\theta\le 2\pi}\frac{\begin{aligned}s\kappa|\sin\theta|\big(4(s^2-1)\kappa^2\cos^2\theta+\kappa|(6s-9)\cos\theta|\\ +|6s\kappa^2-4s^2\kappa^2-2\kappa^2-3|\big)\end{aligned}}{4\zeta^{3/2}}\\ &\le\frac{s\kappa\big(4(s^2-1)\kappa^2+\kappa|6s-9|+|6s\kappa^2-4s^2\kappa^2-2\kappa^2-3|\big)}{4\zeta^{3/2}}\\ &\le\frac{2(4\times 3+3+9)}{4\zeta^{3/2}}=\frac{12}{\zeta^{3/2}}.\end{aligned}\tag{3.9}$$

From (3.8) and (3.9) we can draw a conclusion that the Cartan torsion is bounded. In higher dimensions, the definition of the Cartan torsion's bound at $p\in M$ is

$$\|\mathbf{C}\|_p:=\sup_{y,u\in T_pM}F(p,y)\frac{|\mathbf{C}_y(u,u,u)|}{|\mathbf{g}_y(u,u)^{3/2}|}.$$

Considering the plane $P=\operatorname{span}\{u,y\}$, from the above conclusion we obtain that $\|\mathbf{C}\|_p$ is bounded. Furthermore, the bound is independent of the plane $P\subseteq T_pM$ and the point $p\in M$. Hence the Cartan torsion is also bounded. ∎

https://doi.org/10.4153/CMB-2010-026-8 Published online by Cambridge University Press

Remark

- The second Cartan torsion [13] can also be computed and proved to be bounded in the similar way.
- When $s = 1$, the above metric is reduced to the well-known Randers metric, and (3.3) has been obtained in [12] by a direct calculation (see also [13]); (3.4) has been obtained in [13].

4 Some Applications

As applications of Section 3, we prove Corollary 1.2 and Corollary 1.3 concerning curvature of $F = (\alpha + \beta)^s/\alpha^{s-1}$, $(s \in [1, 2), \|\beta\|_\alpha < 1)$.

Proof of Corollary 1.2 By Theorem 1.1 we know that the Cartan torsion of $F = (\alpha + \beta)^s/\alpha^{s-1}$ must be bounded when $s \in [1, 2)$. Shen showed that if a positive complete Finsler metric with bounded Cartan torsion is R-quadratic, then it must be a Landsberg metric [14]. For this type of metric notice that $\phi(t) = (1 + t)^s$ and $\phi(t) \neq c_1\sqrt{1 + c_2t^2}$ for any constancy $c_1 > 0$ and c_2 when $s \geq 1$. Shen proved that a regular (α, β) metric is Landsbergian if and only if it is Berwaldian. This completes the proof. ■

Remark The condition of positive completeness in Corollary 1.2 cannot be omitted. Consider the following Randers metric defined near the origin in $\mathbb{R}^n$.

$$F := \frac{\left[|y|^2 - (|xQ|^2|y|^2 - \langle y, xQ\rangle^2)\right]^{1/2}}{1 - |xQ|^2} - \frac{\langle y, xQ\rangle}{1 - |xQ|^2},$$

where $Q = (q_j{}^i)$ is an anti-symmetric matrix. Then according to the classification theorem of Randers metrics with constant flag curvature in [3], F has zero flag curvature. Hence it is R-quadratic. It is obvious that F is not a Berwald metric when $Q \neq 0$.

Proof of Corollary 1.3 By Theorem 1.1, the Cartan torsion of F is bounded. According to the well-known Akbar–Zadeh theorem [2], we can get the result. ■

Remark Again, the condition of completeness cannot be omitted. Consider the following Randers metric as a counterexample:

$$F := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2},$$

where $x \in B^n(1)$ and $y \in T_x\big(B^n(1)\big)$. Then F is a Funk metric with flag curvature $K = -\frac{1}{4}$. However, it is not Riemannian. In fact, we know that the Funk metric is only positively complete.

A Maple Programs

A.1 Construction of the Berwald frame

```
> with(linalg):
> F:=sqrt(u^2+v^2)*(1+k*u/sqrt(u^2+v^2))^s:
> g:=simplify(1/2*hessian(F^2,[u,v])):
> gr:=simplify(subs(u=cos(theta),v=sin(theta),g)):
> y:=vector(2,[r*cos(theta),r*sin(theta)]);
```

$$y := [r\cos(\theta),\ r\sin(\theta)]$$

```
> yp:=vector(2):
> eq:=simplify(evalm(transpose(y)&*gr&*yp))=0:
> x:=solve(eq,yp[1]);
```

$$x := -\frac{yp_2\,(-k\cos(\theta) + s\,k\cos(\theta) - 1)\sin(\theta)}{-\cos(\theta)^2\,k + \cos(\theta)^2\,k\,s - k\,s - \cos(\theta)}$$

```
> ny:=simplify(r^2*subs(u=cos(theta),v=sin(theta),F^2));
```

$$ny := r^2\,(1 + k\cos(\theta))^{(2\,s)}$$

```
> yp[1]:=-(s*k*cos(theta)-k*cos(theta)-1)*sin(theta):
> yp[2]:=k*(s-1)*cos(theta)^2-cos(theta)-s*k:
> nyp:=simplify(evalm(transpose(yp)&*gr&*yp)):
> lambda:=simplify(sqrt(r^2*nyp/ny)/r):
> yp[1]:=yp[1]/lambda:
> yp[2]:=yp[2]/lambda:
> print(yp);
```

$$\left[-\frac{(-k\cos(\theta) + s\,k\cos(\theta) - 1)\sin(\theta)\,r}{\sqrt{-\cos(\theta)^2\,s^2\,k^2 + \cos(\theta)^2\,k^2 - s\,k^2 + s^2\,k^2 - s\,k\cos(\theta) + 2\,k\cos(\theta) + 1}},\right.$$
$$\left.\frac{(k\,(-1+s)\cos(\theta)^2 - \cos(\theta) - k\,s)\,r}{\sqrt{-\cos(\theta)^2\,s^2\,k^2 + \cos(\theta)^2\,k^2 - s\,k^2 + s^2\,k^2 - s\,k\cos(\theta) + 2\,k\cos(\theta) + 1}}\right]$$

- The method of computation:

  Step 1: Solve the equation $\mathbf{g}_y(y, y^{\perp}) = 0$

$$(x, yp_{[2]}) = \left(\frac{yp_{[2]}yp_{[1]}}{yp_{[2]}}, yp_{[2]}\right).$$

  Then $yp := (yp_{[1]}, yp_{[2]})$ is a particular solution.

  Step 2: Assume that $y^{\perp} = \frac{1}{\lambda}yp$ is the satisfied solution. Notice that when

$$\mathbf{g}_y(y^{\perp}, y^{\perp}) = F^2(y) := ny,$$

https://doi.org/10.4153/CMB-2010-026-8 Published online by Cambridge University Press

we can get

$$\lambda = \sqrt{\frac{nyp}{ny}}$$

where nyp is defined by $nyp := g_y(yp, yp)$.

Step 3: Plug these results into $y^{\perp}$; we get the Berwald frame $\{y, y^{\perp}\}$.

A.2 Computation of $\xi(p, y)$

```
> nyp:=simplify(evalm(transpose(yp)&*gr&*yp));
```

$$nyp := r^2\,(1 + k\cos(\theta))^{(2\,s)}$$

```
> bc:=abs(simplify(r^2*subs(t=0,q=0,p=0,diff(subs(u=cos(theta)
>    +t*yp[1]/r+q*yp[1]/r+p*yp[1]/r,
> v=sin(theta)+t*yp[2]/r+q*yp[2]/r+p*yp[2]/r,F^2/4),
>    [t,q,p])))/nyp);
```

$$bc := \frac{1}{2}\Big|\, s\,k\sin(\theta)\big(4\,s^2\,k^2\cos(\theta)^2 - 4\,k^2\cos(\theta)^2 - 9\,k\cos(\theta) + 6\,s\,k\cos(\theta) - 3 - 2\,k^2 - 4\,s^2\,k^2 + 6\,s\,k^2\big) \Big/ \big(k^2\cos(\theta)^2 - s^2\,k^2\cos(\theta)^2 + s^2\,k^2 - s\,k^2 - s\,k\cos(\theta) + 2\,k\cos(\theta) + 1\big)^{(3/2)}\Big|$$

- The method of computation: Let $nyp := \mathbf{g}_y(yp, yp) = \mathbf{g}_y(y^{\perp}, y^{\perp})$. Then compute

$$bc = \frac{F(y)c_y(y^{\perp}, y^{\perp}, y^{\perp})}{\mathbf{g}_y(y^{\perp}, y^{\perp})^{\frac{3}{2}}}.$$

This is prepared for estimating the bound of Cartan torsion.

Acknowledgements We would like to thank the referee for his careful reading of the manuscript and very helpful suggestions.

References

[1] D. Bao and S. S.Chern, *A note on the Gauss-Bonnet theorem for Finsler spaces*. Ann. Math. **143**(1996), no. 2, 233–252. doi:10.2307/2118643

[2] D. Bao, S. S.Chern, and Z. Shen, *An Introduction to Riemann-Finsler Geometry.* Graduate Texts inh Mathematics 200, Springer-Verlag, New York, 2000.

[3] D. Bao, C. Robles, and Z. Shen, *Zermelo navigation on Riemannian manifolds.* J. Differential Geom. **66**(2004), no. 3, 377–435.

[4] D. Burago and S. Ivanov, *Isometric embeddings of Finsler manifolds.* Algebra i Analiz **5**(1993), no. 1, 179–192 (in Russian); translation in St.Petersburg Math. J.**5**(1994), no. 1, 159–169.

[5] E. Cartan, *Les espaces de Finsler.* Actualités Scientifiques et Industrielles, no. 79, Hermann, Paris, 1934.

[6] S. S. Chern and Z. Shen, *Riemann-Finsler Geometry.* Nankai Tracts in Mathematics 6, World Scientific, Hackensack, NJ, 2005.

[7] A. Deicke, *Über die Finsler-Räume mit* $A_i = 0$, Arch. Math. **4**(1953), 45–51. doi:10.1007/BF01899750

[8] P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen.* Verlag Birkhäuser, Basel, 1951.

[9] K. Ernic, *A Guide to Maple.* Springer, 1999.

[10] X. Mo and C. Yang, *The explicit construction of Finsler metrics with special curvature properties.* Differential. Geom. Appl. **24**(2006), no. 2, 119–129. doi:10.1016/j.difgeo.2005.08.004

[11] J. Nash, *The immedding problem for Riemannian manifolds*, Ann. of Math. **63**(1956), 20–63. doi:10.2307/1969989

[12] Z. Shen, *Differential Geometry of Spray and Finsler Spaces.* Kluwer Academic Publishers, Dordrecht, 2001.

[13] ______, *Finsler metrics with $K = 0$ and $S = 0$.* Canad. J. Math. **55**(2003), no.1, 112–132.

[14] ______, *On R-quadratic Finsler spaces.* Publ. Math. Debrecen **58**(2001), no. 1-2, 263–274.

[15] ______, *On Finsler geometry of submanifolds.* Math. Ann. **311**(1998), no. 3, 549–576. doi:10.1007/s002080050200

Key Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University, Beijing 100871, P.R. China
e-mail: moxh@pku.edu.cn

Department of Mathematics, East China Normal University, 200241 Shanghai, P.R. China
e-mail: llfzhou@math.ecfnu.edu.cn