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CHARACTER VARIETIES, EARTHQUAKES AND ESSENTIAL SURFACES

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We study low-dimensional manifolds through their character variety from two perspectives. Character varieties of low-dimensional manifolds are a rich area of study that reflect geometric and topological information of the underlying manifold. Given a manifold M and an algebraic group G, the G-character variety is loosely the space of conjugacy classes of representations from the fundamental group of M to G, with some conjugacy classes identified so that the resulting space is a variety. We are particularly interested in the situation when M is a hyperbolic surface or 3-manifold and G is SL_2. This is because of the relationship between elements of the character variety and the orientation preserving isometry groups of the hyperbolic plane ($\mathrm{PSL}_2(\mathbb{R})$) and hyperbolic space ($\mathrm{PSL}_2(\mathbb{C})$).

The first perspective uses Teichmüller space and earthquakes. Teichmüller space of a surface is the space of all finite-area complete hyperbolic structures of the surface and can be naturally identified with a component of the associated $\mathrm{SL}_2(\mathbb{R})$-character variety [5]. There is a natural action on Teichmüller space by earthquakes, which have proved to be powerful in the exploration of these hyperbolic structures [6]. We derive explicit forms of the earthquake deformations on Teichmüller space associated with simple closed curves of the once-punctured torus with both algebraic and geometric interpretations [2]. Examining the limiting behaviour gives insight into earthquakes about measured geodesic laminations, of which simple closed curves are a special case [2].

The second perspective uses essential surfaces and the theory of Culler and Shalen. Essential surfaces are orientable, incompressible and boundary-incompressible surfaces in a 3-manifold, and tell us about the inherent geometry and topology of the manifold. Thurston's hyperbolisation theorem implies these essential surfaces

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are intimately connected with hyperbolic structures. In the seminal work of Culler and Shalen, essential surfaces in 3-manifolds are associated to ideal points of their $SL_2(\mathbb{C})$-character varieties [1]. We lay a general foundation for this theory in arbitrary characteristic by using the same approach instead over the $SL_2(\mathbb{F})$-variety of characters for $\mathbb{F}$ an arbitrary algebraically closed field [4]. We provide several applications of the extended theory, including bounds on dimension of the character variety, the *A*-polynomial, prime decomposition and JSJ decomposition [3, 4]. We give examples where new essential surfaces are detected in positive characteristic and where an essential surface is never detected in any characteristic.

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