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SOLVING A SPECIAL CLASS OF MULTIPLE OBJECTIVE LINEAR FRACTIONAL PROGRAMMING PROBLEMS

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Abstract

In this paper a feasible direction method is presented to find all efficient extreme points for a special class of multiple objective linear fractional programming problems, when all denominators are equal. This method is based on the conjugate gradient projection method, so that we start with a feasible point and then a sequence of feasible directions towards all efficient adjacent extremes of the problem can be generated. Since methods based on vertex information may encounter difficulties as the problem size increases, we expect that this method will be less sensitive to problem size. A simple production example is given to illustrate this method.

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1. Introduction

Multiple objective linear fractional programming (MOLFP) problems have attracted considerable research and interest, since they are useful in corporate and financial planning, and also in planning of production, health care, and hospitals. For single objective linear fractional programming, the transformation of Charnes and Cooper [3] can be used to transform the problem into a linear programming problem. Some other approaches have been reported for solving MOLFP problems. Kormbluth and Steuer [18] considered this problem and presented a simplex-based solution procedure to find all weakly efficient vertices of the augmented feasible region. Benson [1] showed that the procedure suggested by Kormbluth and Steuer [18] for computing the numbers to find break points might not work all the time, and he proposed a failsafe method for computing these numbers. Geoffrion [10] introduced the notion of proper efficiency for MOLFP, and Choo [4] proved that every efficient solution for a MOLFP was properly efficient.

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On the other hand, the problem of multiple objective linear programming (MOLP) arises when the objective functions to be maximized are linear. Different approaches have been suggested for solving problems of this kind [5–8, 15]. The majority of previous methods depend on the canonical simplex tableau in multiple objective forms to find the efficient set of an MOLP. Recently, more novel work on solving general multiple objective mathematical programming has been done [9, 16, 17, 20, 21]. In this paper we develop a new method to find all efficient extreme points for MOLFP, when all the denominators are equal. This method provides us with a feasible direction of movement from an efficient extreme point to its adjacent one, and is based mainly on the conjugate gradient projection method. In Section 2 some notation and definitions of the MOLFP problem are given. In Section 3 we give the main result of this method together with a simple production example. Concluding remarks about this proposed method are given in Section 4.

2. Definitions and notation

MOLFP problems arise when several linear fractional objectives (that is, ratio objectives that have linear numerator and denominator) are to be maximized over a convex constraints polytope X, which is a special case of a polytope having the additional property that is also a convex set of points in the n-dimensional space R^n. The MOLFP can be formulated as

$$\begin{aligned} &\text{maximize} \quad Z(X) = (z_1(x), z_2(x), \ldots, z_k(x)) \\ &\text{subject to} \quad x \in X = \{x \in R^n \mid Ax \leq b\}, \\ &\text{where} \quad z_i(x) = \frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i} \quad \text{for } i = 1, 2, \ldots, k. \end{aligned}$$

Here c_i, d_i are vectors in R^n, α_i and β_i are scalars, A is an $(m+n) \times n$ matrix and $b \in R^{m+n}$. We point out that the nonnegativity condition is added to the set of constraints, and we also assume that X is a compact set and $d_i^T x + \beta_i > 0$, $i = 1, 2, \ldots, k$, for every $x \in X$. The set of all solutions of the above problem is denoted by E, and this set has the following definition [2, 4].

Definition 2.1. A solution x^0 is said to be efficient for MOLFP if $x^0 \in X$, and there is no $x \in X$ such that

$$\frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i} \geq \frac{c_i^T x^0 + \alpha_i}{d_i^T x^0 + \beta_i}.$$

Likewise, we define $z^0 = Z(x^0)$ to be nondominated if there is no $z = Z(x)$ such that $Z(x) > Z(x^0)$.

A characterization of an efficient solution of MOLFP has been made through the following lemma.

Lemma 2.2. *Let $x^0 \in X$. Then x^0 is an efficient solution of (MOLFP) if and only if there exist $\lambda_i > 0$, for $i = 1, 2, \ldots, k$, and $u_r \geq 0$, for all $r \in I^0$, such that*

$$\sum_{i=1}^{k} \lambda_i \nabla z_i(x^0) = \sum_{r \varepsilon I^0} u_r A_r,$$

where I^0 is the set of indices of the binding constrains at x^0, and A_r is a sub-matrix of the given matrix A, containing only the coefficients defined at the binding constraints for the current point.

In this paper we are interested in solving the above MOLFP when the denominators are all equal. In this case, the MOLFP takes the form

$$\begin{aligned} &\text{maximize} \quad Z(X) = (z_1(x), z_2(x), \ldots, z_k(x)) \\ &\text{subject to} \quad x \in X = \{x \in R^n, Ax \leq b\}, \\ &\text{where} \quad z_i(x) = \frac{c_i^T x + \alpha_i}{d^T x + \beta} \quad \text{for } i = 1, 2, \ldots, k. \end{aligned} \tag{2.1}$$

If we assume that $\beta \neq 0$, then MOLFP (2.1) is equivalent to

$$\begin{aligned} &\text{maximize} \quad Z(X) = \Big(c_i^T - \frac{\alpha_i}{\beta} d^T\Big) \frac{x}{d^T x + \beta} + \frac{\alpha_i}{\beta} \quad \text{for } i = 1, 2, \ldots, k \\ &\text{subject to} \quad \Big(A + \frac{b}{\beta} d^T\Big) \frac{x}{d^T x + \beta} \leq \frac{b}{\beta}, \end{aligned}$$

and by defining $y = x/(d^T x + \beta) \geq 0$, it is further simplified to

$$\begin{aligned} &\text{maximize} \quad Z(y) = \Big(c_i^T - \frac{\alpha_i}{\beta} d^T\Big) y + \frac{\alpha_i}{\beta}, \quad i = 1, 2, \ldots, k \\ &\text{subject to} \quad \Big(A + \frac{b}{\beta} d^T\Big) y \leq \frac{b}{\beta}. \end{aligned} \tag{2.2}$$

MOLFP (2.2) can be written as a multiple objective linear programming problem:

$$\begin{aligned} &\text{maximize} \quad Z(y) = Cy + \frac{\alpha_i}{\beta} \quad \text{for } i = 1, 2, \ldots, k \\ &\text{subject to} \quad Gy \leq g, \end{aligned} \tag{2.3}$$

where C is a $k \times n$ matrix whose rows are those represented by $(c_i^T - (\alpha_i/\beta) d^T)$, $G = A + (b/\beta) d^T$ is the constraint matrix, and $g = b/\beta$.

From the definition of y above, $x = y(d^T x + \beta)$, and then premultiplying by the vector d^T in R^n on both sides, we have $d^T x = d^T y(d^T x + \beta)$. Simplifying further, this leads to

$$x = \beta \frac{y}{1 - d^T y}. \tag{2.4}$$

Remark 2.3. The assumption that $\beta \neq 0$ is essential in our algorithm, because if $\beta = 0$, we have $d^T y = 1$, and then equation (2.4) cannot be defined. Also, our transformation maintains the dimension of the given one unlike the classical transformation used by Charnes and Cooper [3], where the dimension of the defined problem is increased by 1.

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Based on the above definition of the MOLP problem (2.3), this problem can be considered as an equivalent of the MOLFP defined by equation (2.1), where all denominators of the objective functions are equal. Hence, due to the well-known theorem for MOLP [7, 8, 23], we can find $\lambda \in R^k$ and $\lambda > 0$ (weights) so that y will be an optimal solution for the linear program

$$\begin{aligned} &\text{maximize} \quad \lambda^T C y \\ &\text{subject to} \quad G y \le g. \end{aligned} \tag{2.5}$$

Note that our transformation also gives us the advantage of constructing the objective space in the form $Z = \{\eta \in R^k \mid Q\eta \le q\}$, which is done by defining the dual of equation (2.5) as

$$\begin{aligned} &\text{maximize} \quad u^T g \\ &\text{subject to} \quad u^T G = \lambda^T C \quad \text{and} \quad u \ge 0. \end{aligned} \tag{2.6}$$

Since the set of constraints of the dual problem is in matrix form, we can multiply this set of constraints by a matrix $T = (T_1 \mid T_2)$, where $T_1 = C^T(CC^T)^{-1}\, T_1 = C^T(C^T \mid T_2)$, and the columns of the matrix T_2 constitute the bases of $N(C) = \{v; Cv = 0\}$, we get $u^T G T_1 = \lambda^T$, $u^T G T_2 = 0$ and $u \ge 0$, and hence we have the following two cases:

(1) If $k = n$, then $GT_2 = 0$, and the dual of equation (2.6) takes the form

$$\begin{aligned} &\text{maximize} \quad \lambda^T \eta \\ &\text{subject to} \quad G T_1 \eta \le b, \end{aligned}$$

where T_1 is the inverse of the given matrix C.

(2) If $k < n$, then $GT_2 \ne 0$; in this case an $l \times (m + n)$ matrix P of nonnegative entries is defined such that $PGT_2 = 0$. This matrix P can be considered as the polar matrix of the given matrix GT_2, and will play an important role for the construction of the objective space Z to be in the form $Z = \{\eta \in R^k \mid PGT_1\eta \le Pb\}$ or simply can be written as $Z = \{\eta \in R^k \mid Q\eta \le q\}$, where $Q = P\,GT_1$ and $q = Pb$. In this case the dual of equation (2.6) takes the form

$$\begin{aligned} &\text{maximize} \quad \lambda^T \eta \\ &\text{subject to} \quad Q\eta \le q. \end{aligned}$$

A sub-matrix $\overline{P}$ of the given matrix P satisfying

$$\overline{P} G T_1 = \lambda^T > 0$$

will play an important role in specifying the positive weights needed for detecting the nondominated point of this multiple objective linear programming problem (2.3). Also, our proposed method will depend mainly on the previously specified weights.

At a given optimal point of (2.5), we must find $u \ge 0$, $\lambda > 0$ such that

$$u^T \overline{G} = \lambda^T C,$$

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where $\overline{G}$ is an $n \times n$ sub-matrix of the given matrix G containing only the coefficients of the set of active constraints at the current point. Then, we get a similar result in the objective space if we consider the multiple objective linear programming MOLP of the form

$$\begin{aligned} \text{maximize} \quad & I\eta \\ \text{subject to} \quad & Q\eta \leq q, \end{aligned} \tag{2.7}$$

where I is the $k \times k$ identity matrix. Hence according to the objective space point of view, we have the following straightforward proposition.

Proposition 2.4. *A point η^0 is a nondominated point of (2.7) if there exist $u \geq 0$ and $\lambda > 0$ satisfying $u^T\overline{Q} = \lambda^T$, where $\overline{Q}$ represents the set of active constraints at the given point η^0.*

Corollary 2.5. *If Q'' represents the set of nonactive constraints at the given point η^0, then $u^T Q'' = 0$ has only the zero solution.*

Our main task now is to find the first efficient extreme point for (2.3) with the assumption that $\beta \neq 0$, which is done by considering the linear programming problem

$$\begin{aligned} \text{maximize} \quad & F(y) = e^T C y \\ \text{subject to} \quad & Gy \leq g, \end{aligned}$$

where $e \in R^k$ with all entries equal to 1. If $e^T C = p^T$, the above linear program is written as

$$\begin{aligned} \text{maximize} \quad & F(y) = p^T y \\ \text{subject to} \quad & Gy \leq g. \end{aligned} \tag{2.8}$$

This problem can also be written in the form

$$\begin{aligned} \text{maximize} \quad & F(y) = p^T y \\ \text{subject to} \quad & G_l^T y \leq g_l, \quad l = 1, 2, \ldots, m + n, \end{aligned} \tag{2.9}$$

where G_l^T represents the lth row of the given matrix G. Then, in the nondegenerate case, an extreme point (vertex) of $Y = \{y \in R^n \mid G_l^T y \leq g_l\}$, $l = 1, 2, \ldots, m + n$, has exactly n linearly independent subsets of Y.

Starting with an initial feasible point, a sequence of feasible directions is generated to find the optimal extreme point of this problem. In general, if y^{k-1} is a feasible point obtained at iteration $k - 1$ ($k = 1, 2, \ldots$), then at iteration k our procedure finds a new feasible point y^k given by

$$y^k = y^{k-1} + \gamma_{k-1}\mu^{k-1}, \tag{2.10}$$

where μ^{k-1} is the direction vector along which the point moves, and is given by

$$\mu^{k-1} = H_{k-1}p. \tag{2.11}$$

Here H_{k-1} is an $n \times n$ symmetric matrix, that is

$$H_{k-1} = \begin{cases} I & \text{if } k = 1, \\ H_{k-1}^q & \text{if } k > 1. \end{cases} \tag{2.12}$$

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For each active constraint $s = 1, 2, \ldots, q$ at the current point, H^q_{k-1} is defined as

$$H^q_{k-1} = H^{s-1}_{k-1} - \frac{H^{s-1}_{k-1} G_s G_s^T H^{s-1}_{k-1}}{G_s^T H^{s-1}_{k-1} G_s}, \tag{2.13}$$

with $H^0_{k-1} = I$. Then $H_{k-1} = H^q_{k-1}$ and the step length

$$\gamma_{k-1} = \min_{l=1,2,\ldots,m+n} \left\{ e_l \,\middle|\, e_l = \frac{g_1 - G_l^T y^{k-1}}{G_l^T \mu^{k-1}}, \text{ and } e_l > 0 \right\} \tag{2.14}$$

is always positive. Proposition 2.8 below shows that such a positive value must exist if a feasible point exists. Also due to the well-known Kuhn–Tucker condition [13, 14], for the point y^k to be an optimal solution of the linear program (2.8), there must exist $u \geq 0$ such that $G_r^T u = p$ and

$$u = (G_r G_r^T)^{-1} G_r p, \tag{2.15}$$

where G_r is a sub-matrix of the given matrix G containing only the coefficients of the set of active constraints at the current point y^k. This fact will act as a rule for stopping in our proposed algorithm.

Proposition 2.6. *For the matrix H_{k-1} defined in equation (2.12), we have $(H_{k-1})^2 = H_{k-1}$.*

Proof. This can be proved by induction. Define a matrix $Q_1 = G_1 G_1^T / G_1^T G_1$. Since $H^1_{k-1} = I - (G_1 G_1^T / G_1^T G_1)$, it follows that $H^1_{k-1} Q_1 = 0$, $Q_1^2 = Q_1$, $(H^1_{k-1})^2 = H^1_{k-1}$ and H^1_{k-1} is an orthogonal projective matrix. Also, if we define

$$Q_2 = \frac{G_2 G_2^T H^1_{k-1}}{G_2^T H^1_{k-1} G_2} \quad \text{and} \quad H^*_{k-1} = \left(I - \frac{G_2 G_2^T H^1_{k-1}}{G_2^T H^1_{k-1} G_2} \right),$$

then, since $H^2_{k-1} = H^1_{k-1}[I - (G_2 G_2^T H^1_{k-1} / G_2^T H^1_{k-1} G_2)]$, we have $H^*_{k-1} Q_2 = 0$, $Q_2^2 = Q_2$ and $(H^*_{k-1})^2 = H^*_{k-1}$. Since $H^1_{k-1}\ H^*_{k-1}$ and both matrices H^1_{k-1} and H^*_{k-1} are orthogonal projective, H^2_{k-1} is also orthogonal projective, and $(H^2_{k-1})^2 = H^2_{k-1}$. Applying the same argument, we conclude that $H_{k-1} = H^q_{k-1}$ is an orthogonal projective matrix such that $(H_{k-1})^2 = H_{k-1}$. □

Proposition 2.7. *Any solution y^k given by this algorithm through equation (2.10) is feasible, and it increases the objective function value for the linear programming problem defined by equation (2.8).*

Proof. Here H^q_{k-1} is defined as

$$\begin{aligned} F(y^k) - F(y^{k-1}) &= p^T y^k - p^T y^{k-1} \\ &= \gamma_{k-1} p^T H_{k-1} p \\ &= \gamma_{k-1} p^T H^2_{k-1} p \\ &= \gamma_{k-1} \|H_{k-1} p\| > 0, \end{aligned}$$

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which proves that y^k increases the objective function. Next, to prove that y^k is a feasible point, it must satisfy all constraints of problem (2.9). Then

$$G_l^T(y^{k-1} + \gamma_{k-1}\mu^{k-1}) \le g_l,$$

for all $l \in \{1, 2, \ldots, m + n\}$, which can be written as

$$\gamma_{k-1} G_l^T \mu^{k-1} \le g_l - G_l^T y^{k-1}, \quad l = 1, 2, \ldots, m + n.$$

This is valid for any l, since if there is $w \in \{1, 2, \ldots, m + n\}$ such that $G_w^T \mu^{k-1} > 0$ and $G_w^T \mu^{k-1} > g_w - G_w^T y^{k-1}$, then $(g_w - G_w^T y^{k-1})/G_w^T \mu^{k-1} < \gamma_{k-1}$. This contradicts our definition of γ_{k-1}. □

The next result guarantees the existence of γ_{k-1} defined in equation (2.10).

Proposition 2.8. *At any iteration k, if a feasible point that increases the objective function exists, then γ_{k-1} as defined in equation (2.14) must exist.*

Proof. Here, it is enough to prove that

$$G_l^T \mu^{k-1} \le 0 \tag{2.16}$$

cannot be true for all $i \in \{1, 2, \ldots, m + n\}$. Now suppose that relation (2.16) is true for $l \in \{1, 2, \ldots, m + n\}$. Then, rewriting (2.16) in matrix form and multiplying both sides by u^T, we get

$$u^T G \mu^{k-1} \le 0,$$

that is,

$$u^T G H_{k-1} p \le 0. \tag{2.17}$$

Since the constraints of the dual problem for the linear programming problem (2.8) can be written in the form $u^T G = p^T$, $u \ge 0$. Then equation (2.17) can be written as $p^T H_{k-1}^2 p \le 0$, since $(H_{k-1})^2 = H_{k-1}$, that is, $\|H_{k-1} p\| \le 0$. This contradicts the fact that the norm must be positive, which implies that relation (2.16) cannot be true for all $l \in \{1, 2, \ldots, m + n\}$. Thus, if a feasible point y^k exists then γ_{k-1} as defined before must exist. □

Remark 2.9. If at any feasible point y^k we get the directions $\mu^k = H_k p = 0$, then the point y^k is optimal, and we cannot improve the value of the objective function. Also, we note that although the matrix H_k is singular and the vector p is nonzero, this does not cause the breakdown of this algorithm, rather it indicates that all subsequent search directions μ^{k+1} are orthogonal to p.

Based on the above results, in the next section we give a full description of our algorithm for solving the equivalent MOLP to find all efficient extreme points of MOLFP in two phases as follows.

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3. New algorithm for solving MOLFP problems

Phase I: Use first the linear programming problem (2.9) to find an initial efficient extreme point for the equivalent MOLP through the following steps.

Step 0 Set $k = 1$, $H_0 = I$, $\mu^0 = p$, let y^0 be an initial feasible point, and apply relation (2.14) to compute γ_0.
Step 1 Apply relation (2.10) to find a new solution y^k.
Step 2 Apply relation (2.15) to compute u. If $u \geq 0$ stop. The current solution y^k is the optimal solution, otherwise go to Step 3.
Step 3 Set $k = k + 1$, apply relations (2.12)–(2.14) to compute H_{k-1}, μ^k and γ_{k-1} respectively, and go to Step 1.

Given an initial feasible point y^0 and a vector p, Step 0 computes γ_0 in $O(m + n)$. Computing y^k in Step 1 requires $O(n)$, while testing the optimality of the current solution y^k in Step 2 requires $O(n^3)$. Step 3 of the algorithm requires $O(n^3)$ to compute H_{k-1} while computing μ^{k-1} and the feasible direction that increases the value of the objective function requires $O(n^2)$. Finally, computing of γ_{k-1} requires $O(m + n)$. Hence the application of each iteration of our algorithm requires $O(\max\{m + n, n^3\})$.

Remark 3.1. Assuming that q is the number of active constraints at point y^k, if $q < n$ and relation (2.15) is satisfied, this indicates that y^k is an optimal nonextreme point. In this case, the objective function cannot be improved through any feasible direction.

Remark 3.2. If y^{k-1} is an extreme but nonoptimal point (that is, there are n active constrains at point y^{k-1} and relation (2.15) is not satisfied), then a move is made through a direction μ^{k-1} lying in the nullity of a subset of the set the active constrains at y^{k-1}, where each constraint in this subset satisfies relation (2.14).

Next, we prove that the number of iterations that our algorithm requires to solve the linear programming problem defined by equation (2.9) is at most n.

Proposition 3.3. *Our algorithm solves the linear programming problem* (2.9) *in at most n iterations.*

Proof. Since our allowed directions (2.11) which improve the value of the objective function lie in the nullity of a subset of the given matrix G, in all iterations we are moving in a direction parallel to a certain subset of the $(m + n)$ constraints. Also, since at least one constraint is added at a time starting with $H_0^0 = I$, an optimal extreme point may be reached in at most n steps. □

In our analysis to find all efficient extreme points in multiple objective linear programming problems, we proceed from a given efficient point defined by Phase I to its adjacent efficient extremes. This is done by defining a frame for a Cone (H) denoted by F, called a minimal spanning system. For an $n \times n$ matrix H, denote the set of indices of the columns of H by Id_H. Hence if $H = (h^1, \ldots, h^n)$, then $\mathrm{Id}_H = \{1, 2, \ldots, n\}$.

For matrix H, we define the positive cone spanned by the columns of H (called a conical or positive hull [19]) as

$$\begin{aligned}\text{Cone}(H) &= \text{Cone}(h^i, i \in \text{Id}_H)\\ &= \left\{h \in R^n \;\middle|\; h = \sum_{i \in \text{Id}_H} \tau_i h^i, \tau_i \geq 0\right\}.\end{aligned}$$

A frame F of Cone(H) is defined as a collection of columns of H such that Cone $(h^i, i \in \text{Id}_H) = \text{Cone}(H)$ and, for each $i \in \text{Id}_H$,

$$\text{Cone}(h^i \mid i \in \text{Id}_H) \neq \text{Cone}(H).$$

Based on the above definitions, we start Phase II to find all efficient extreme points for the equivalent MOLP problem through a finite number of steps as follows.

Phase II:

Step 1 Let y^k be an efficient point. Compute H_k corresponding to this point y^k.
Step 2 Construct a frame F of Cone (H_k) using the method of Wets and Witzgall [22].
Step 3 For each $h^i \in F$, determine γ^* obtained by solving the system of linear inequalities of the form $\gamma A h^i \leq g - G y^k$ (the boundary points of this interval give γ^*).
Step 4 Compute $y^* = y^k + \gamma^* h^i$ as an efficient extreme point for this MOLP problem, and go to Step 1.

3.1. Application to a production example Consider a company that manufactures three kinds of products A_1, A_2 and A_3, with a profit of \$4, \$5 and \$3 per unit, respectively. The production cost for each unit of the above products is \$1, \$2 and \$1, respectively. It is assumed that a fixed amount of \$10 is added to the profit function, and also a fixed cost of \$5 is added to the cost function as an effect of expected duration through the process of production. The first product takes 5 employment hours to produce, while the second product takes 3 h, and the third product takes 7 h. Also, it is assumed that a fixed employment demands amount 5 h is added to the employment function. Furthermore, one of the objectives of this company is to maximize the ratio of the profit to the total cost, and also the company wishes to maximize the ratio of the employment to total cost in order to receive state aid for business development, provided that the company has the raw materials for manufacturing and supposing the materials needed for production is per ton 1, 2 and 1, respectively, with supply for this raw material restricted to 10 tons. If we consider x_1, x_2 and x_3 as the amount of units of A_1, A_2 and A_3, respectively, to be produced then the above problem can be formulated as

$$\begin{aligned}&\text{maximize} \quad z_1(x) = \frac{4x_1 + 5x_2 + 3x_3 + 10}{x_1 + 2x_2 + x_3 + 5},\\ &\text{maximize} \quad z_2(x) = \frac{5x_1 + 3x_2 + 7x_3 + 5}{x_1 + 2x_2 + x_3 + 5}\\ &\text{subject to} \quad x_1 + 2x_2 + x_3 \leq 10, \quad x_1 \geq 0,\ x_2 \geq 0 \text{ and } x_3 \geq 0.\end{aligned}$$

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For this MOLFP, $C_1^T = (4 \quad 5 \quad 3)$, $C_2^T = (5 \quad 3 \quad 7)$, $d^T = (1 \quad 2 \quad 1)$, $\alpha_1 = 10$, $\alpha_2 = 5$ and $\beta = 5$. Then, the equivalent MOLP takes the form

$$\begin{aligned} &\text{maximize} \quad z_1(y) = 2y_1 + y_2 + y_3 + 2, \\ &\text{maximize} \quad z_2(y) = 4y_1 + y_2 + 6y_3 + 1 \\ &\text{subject to} \quad 3y_1 + 6y_2 + 3y_3 \le 2, \quad y_1 \ge 0,\ y_2 \ge 0 \text{ and } y_3 \ge 0. \end{aligned}$$

For this MOLP, the first efficient extreme point is obtained by solving the linear programming problem

$$\begin{aligned} &\text{maximize} \quad z = 6y_1 + 2y_2 + 7y_3 \\ &\text{subject to} \quad 3y_1 + 6y_2 + 3y_3 \le 2, \quad -y_1 \le 0,\ -y_2 \le 0, \text{ and } -y_3 \le 0. \end{aligned}$$

Phase I:

Step 0 $k = 1$, $H_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mu^0 = \begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix}$, and let $y^0 = \begin{bmatrix} 1/8 \\ 1/8 \\ 1/8 \end{bmatrix}$ be an initial feasible point. Then equation (2.14) gives $\gamma_0 = 1/102$, and we go to Step 1.

Step 1 Apply equation (2.10) to get $y^1 = \begin{bmatrix} 1/8 \\ 1/8 \\ 1/8 \end{bmatrix} + (1/102)\begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 150/816 \\ 118/816 \\ 158/816 \end{bmatrix}$, and we go to Step 2.

Step 2 For this point, y^1, the first constraint is the only active constrain. Relation (2.15) is not satisfied, which indicates that this point is not optimal, and we go to Step 3.

Step 3 Set $k = 2$, then

$$H_1 = \begin{bmatrix} 45/54 & -18/54 & -9/54 \\ -18/54 & 18/54 & -18/54 \\ -9/54 & -18/54 & 45/54 \end{bmatrix}, \quad \mu^1 = \begin{bmatrix} 171/54 \\ -198/54 \\ 225/54 \end{bmatrix}$$

and $\gamma_1 = 6372/161\,568$. Then we go to Step 1, to get

$$y^2 = \begin{bmatrix} 150/816 \\ 118/816 \\ 158/816 \end{bmatrix} + (632/161\,568)\begin{bmatrix} 171/54 \\ -198/54 \\ 225/54 \end{bmatrix} = \begin{bmatrix} 49\,878/161\,568 \\ 0 \\ 57\,834/161\,568 \end{bmatrix}.$$

For this point, the first and the second constraints are the only active constraints, and since equation (2.15) is not satisfied, we go to Step 3 to get

$$H_2 = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}, \quad \mu^2 = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} \quad \text{and} \quad \gamma_2 = 99\,756/161\,568$$

and again we go to Step 1, to get

$$y^3 = \begin{bmatrix} 49\,878/161\,568 \\ 0 \\ 57\,834/161\,568 \end{bmatrix} + (99\,756/161\,568)\begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2/3 \end{bmatrix}.$$

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Note that for the point y^3, equation (2.15) is satisfied with $u^T = [8 \quad 39]$, indicating that this point is optimal for this linear programming, and consequently it is the first efficient extreme point generated for this MOLP. Then we start Phase II.

Phase II: For the above MOLP problem, we compute the matrices

$$T_1 = \begin{bmatrix} 46/93 & -6/93 \\ 38/93 & -9/93 \\ -37/93 & 21/93 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -5/2 \\ 4 \\ 1 \end{bmatrix} \quad \text{and}$$

$$P = \begin{bmatrix} 8/47 & 0 & 39/47 & 0 \\ 2/41 & 0 & 0 & 39/41 \\ 0 & 8/13 & 5/13 & 0 \\ 0 & 2/7 & 0 & 5/7 \end{bmatrix}$$

such that $PGT_2 = 0$. This matrix P is also used to compute

$$PGT_1 = \begin{bmatrix} 6/47 & 3/47 \\ 21/41 & -9/41 \\ -6/13 & 1/13 \\ 1/7 & -1/7 \end{bmatrix}.$$

We note that the first row in PGT_1 has only strictly positive values, so $\lambda^T = [6 \quad 3]$ as the only positive weight defined for this problem. Hence, at the point y^3, a subset of the set of active constraints such that

$$\begin{bmatrix} 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 4 & 1 & 6 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 3 & 6 & 3 \\ 0 & -1 & 0 \end{bmatrix}$$

has a solution $u_1 = 8, u_2 = 39$ which is selected to compute

$$H_3 = \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}.$$

A frame of the columns of H_3 is used as feasible direction to find the adjacent extreme point for x^3 by solving the system of linear inequalities

$$\begin{bmatrix} 3 & 6 & 3 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} \gamma \leq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2/3 \end{bmatrix}.$$

Then with $\gamma^* = 4/3$, we have an efficient extreme point of the form

$$y_1^* = \begin{bmatrix} 0 \\ 0 \\ 2/3 \end{bmatrix} + (4/3) \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 0 \\ 0 \end{bmatrix}.$$

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For this new point, if we repeat the above steps we get the adjacent point

$$y^3 = \begin{bmatrix} 2/3 \\ 0 \\ 0 \end{bmatrix} - (4/3) \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2/3 \end{bmatrix}.$$

We conclude that the two efficient extreme points y^3 and y_1^* are the only efficient extreme points for this MOLP problem. Finally, using equation (2.4), we get the points $x_1^{T^*} = [0 \quad 0 \quad 10]$ and $x_2^{T^*} = [10 \quad 0 \quad 0]$ as the only efficient extreme points for this MOLFP.

4. Conclusion

In this paper a new method is presented to find all efficient extreme points for MOLFPs with equal denominator, based on a feasible direction of movement from an efficient extreme point to its adjacent one. This method starts with an initial efficient extreme point found by solving a single linear programming problem using the conjugate gradient projection method [11, 12]. Since most of the proposed methods for solving MOLFPs depend on the simplex tableau, we expect our method to be less sensitive to problem size.

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