A NOTE ON THE HADAMARD PRODUCT

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Let $A = (a_{ij})$, $B = (b_{ij})$, be two n-square matrices over the complex numbers. Then the n-square matrix $H = (h_{ij}) = (a_{ij}b_{ij})$ is called the Hadamard product of A and B, $H = A \circ B$, [1; p. 174]. Let the n^2-square matrix $K = A \otimes B$ denote the Kronecker product of A and B. The element k_{rs} of K is given by $k_{rs} = a_{i_1 j_1} b_{i_2 j_2}$, where

$$(1) \qquad r = i_2 + n(i_1 - 1) \qquad 1 \leqslant i_1, j_1, i_2, j_2 \leqslant n.$$

$$s = j_2 + n(j_1 - 1).$$

We show that H and K are related and from this obtain some new information regarding the eigenvalues of H. We first recall that a k-square matrix C is said to be a principal submatrix of an n-square matrix D, if there exists a sequence of integers $1 \leqslant i_1 < \ldots < i_k \leqslant n$ such that $d_{i_s i_t} = c_{st}$, $s, t = 1, \ldots, k$.

THEOREM. H is a principal submatrix of K.

Proof. Let $r_\alpha = n(\alpha - 1) + \alpha$, $1 \leqslant \alpha \leqslant n$. We show that the principal submatrix $(k_{r_\alpha r_\beta})$ of K is actually H.

From (1)

$$k_{r_\alpha r_\beta} = a_{i_1 j_1} b_{i_2 j_2}, \text{ where}$$

$$n(\alpha - 1) + \alpha = r_\alpha = i_2 + n(i_1 - 1)$$

$$n(\beta - 1) + \beta = r_\beta = j_2 + n(j_1 - 1).$$

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Then

$$n(i_1 - \alpha) + (i_2 - \alpha) = 0$$

(2)

$$n(j_1 - \beta) + (j_2 - \beta) = 0 .$$

If $i_1 - \alpha \neq 0$ then from (2) $(i_2 - \alpha) \equiv 0(n)$; but $|i_2 - \alpha| < n$ and hence $i_2 - \alpha = 0$. Thus $i_1 = \alpha = i_2$ and similarly $j_1 = \beta = j_2$. Hence $k_{r_\alpha r_\beta} = a_{\alpha\beta} b_{\alpha\beta}$ and the proof is complete.

The eigenvalues of $A \otimes B$ are $\alpha_i \beta_j$, $i,j = 1, \ldots, n$. Assume A and B are non-negative hermitian and order the eigenvalues of $A \otimes B$ as follows:

(3) $$\lambda_1 = \alpha_1 \beta_1 \geqslant \cdots \geqslant \lambda_{n^2} = \alpha_n \beta_n .$$

COROLLARY 1. If A and B are non-negative (positive definite) hermitian then AoB is non-negative (positive definite) hermitian and

$$\alpha_n \beta_n \leqslant \lambda_{s+n^2-n} \leqslant \mu_s \leqslant \lambda_s \leqslant \alpha_1 \beta_1, \quad s = 1, \ldots, n.$$

where $\mu_1 \geqslant \ldots \geqslant \mu_n$ are the eigenvalues of AoB and the λ_i are as in (3).

Proof. By the theorem, AoB is a principal submatrix of $A \otimes B$ and hence we may apply* the Cauchy inequalities [2; p. 75]. These inequalities state the following: If D is any n-square hermitian matrix with eigenvalues $d_1 \geqslant \ldots \geqslant d_n$ and C is any k-square principal submatrix of D with eigenvalues $d_1' \geqslant \ldots \geqslant d_k'$ then $d_{s+n-k} \leqslant d_s' \leqslant d_s$, $s = 1, \ldots, k$. The fact that C is hermitian when D is hermitian is obvious.

Another proof of the fact that AoB is non-negative (positive definite) is given in [1; p. 173].

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* Referee's footnote: The fact that AoB is a principal submatrix of $A \otimes B$ implies that $AoB = P_{\mathfrak{M}} A \otimes B P_{\mathfrak{M}}$, where $P_{\mathfrak{M}}$ is an orthogonal projector of B_{n^2}, an n^2-dimensional vector space, onto $\mathfrak{M}$, a linear manifold of rank n. Thus 14.2 of [2] applies.

We write $A \geqslant 0$, $(A > 0)$, to indicate that for the n-square matrix A, $a_{ij} \geqslant 0$, $(a_{ij} > 0)$. It is clear that if $A \geqslant 0$, (> 0), and $B \geqslant 0$, (> 0), then $AoB \geqslant 0$, (> 0). It is known that if $A \geqslant 0$ then there exists an eigenvalue $\lambda_M(A) \geqslant 0$ such that if λ is any other eigenvalue of A then $|\lambda| \leqslant \lambda_M(A)$.

COROLLARY 2. If $A \geqslant 0$, (> 0), and $B \geqslant 0$, (> 0), then

$$\lambda_M(AoB) \leqslant, (<), \lambda_M(A)\lambda_M(B). \tag{4}$$

Proof. In [3] it is proved that if $X \geqslant 0$, (> 0), and Y is any principal submatrix of X then $\lambda_M(Y) \leqslant$, $(<)$, $\lambda_M(X)$. We apply this to AoB as a principal submatrix of $A \ B$.

As an application of Corollary 2, let $A^{(1/n)}$ denote the matrix whose (i,j) element is ${a_{ij}}^{1/n}$. Then from (4)

$$\lambda_M(A) = \lambda_M(A^{(1/n)}o \ldots oA^{(1/n)}) \leqslant \lambda_M{}^n(A^{(1/n)}),$$

$$\lambda_M{}^{1/n}(A) \leqslant \lambda_M(A^{(1/n)}).$$

Let A' denote the transpose of A. Suppose A is non-negative hermitian. Then A' is also non-negative hermitian and hence by Corollary 1, with B = A',

$$\lambda_M(AoA') \leqslant \lambda_M(A)\,\lambda_M(A') = \lambda_M{}^2(A).$$

Thus if $h_{ij} = |a_{ij}|^2$ then

$$\lambda_M{}^{1/2}(H) \leqslant \lambda_M(A).$$

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