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A CONNECTED METRIC SPACE WITHOUT AN EQUALLY SPACED CHAIN OF POINTS

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We construct a connected subspace M of the euclidean plane $\mathbb{R}^2$ containing two points A and B such that, for every pair of points $\{P, Q\}$ of $M\backslash\{A, B\}$, the three real numbers $d(A, P)$, $d(P, Q)$ and $d(Q, B)$ are not the same. This solves a question posed by Väisälä.

In [3], Väisälä proved the following result:

"Let (X, d) be an arcwise connected metric space and let $a, b \in X$. Then for every positive integer n there is a sequence of distinct points $a = x_0, \ldots, x_n = b$ in X such that $d(x_{j-1}, x_j)$ is independent of j ".

He also asked if this result is true when arcwise connectedness is replaced by connectedness.

We solve completely Väisälä's question by means of the construction of a connected metric space (M, d) with two points A and B such that, for every $P, Q \in M\backslash\{A, B\}$, the real numbers $d(A, P)$, $d(P, Q)$ and $d(Q, B)$ are not the same. The metric space (M, d) is a subspace of the plane $\mathbb{R}^2$ endowed with the euclidean distance.

Let $A = (0, 0)$ and $B = (1, 0)$. If $P, Q \in \mathbb{R}^2$ we shall say that

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P and Q are related (and we will write $r(P, Q)$), if $P, Q \subset \mathbb{R}^2 \setminus \{A, B\}$ and either $d(A, P) = d(P, Q) = d(Q, B)$ or $d(A, Q) = d(Q, P) = d(P, B)$. For every $P \in \mathbb{R}^2$ the set $\{Q \in \mathbb{R}^2 : r(P, Q)\}$ contains, at most, four points.

Let Ω be the first ordinal number equipotent to the continuum. Select a well ordering $\langle P_\alpha : \alpha < \Omega \rangle$ of the points of $\mathbb{R}^2$ such that $P_0 = A$ and $P_1 = B$. Select a well ordering $\langle F_\alpha : \alpha < \Omega \rangle$ of the perfect subsets of $\mathbb{R}^2$ such that $A \in F_0$ and $B \in F_1$.

Now we define $\langle Q_\alpha : \alpha < \Omega \rangle$ inductively in the following way. Suppose we have chosen, for every $\beta < \alpha$, Q_β . Then let Q_α be the first point P in the well ordering $\langle P_\alpha : \alpha < \Omega \rangle$, satisfying

(a) $P \neq Q_\beta$ for every $\beta < \alpha$,

(b) $P \in F_\alpha$,

(c) if $\beta < \alpha$, P is not related to Q_β .

There exist points that satisfy (a), (b) and (c) because the cardinal of F_α is the cardinal of the continuum [Levy, 2.15], (c) is equivalent to the assertion $P \notin \{Q \in \mathbb{R}^2 : \text{there exists } \beta < \alpha \text{ with } r(Q, Q_\beta)\} = H_\alpha$ and the cardinal of $H_\alpha \cup \{Q_\beta : \beta < \alpha\}$ is smaller than the cardinal of the continuum.

We define $M = \{Q_\alpha : \alpha < \Omega\} \subset \mathbb{R}^2$ endowed with the induced metric. It is clear that $A = Q_0$ and $B = Q_1$, hence $A, B \in M$.

There exist no pair of points $\{P, Q\}$ in $M \setminus \{A, B\}$ such that $d(A, P) = d(P, Q) = d(Q, B)$ because otherwise P and Q would be related points and the definition of M would imply either $P \notin M$ or $Q \notin M$.

It is clear that $Q_\alpha \in F_\alpha \cap M$. It follows that the set M has nonempty intersection with every perfect subset of $\mathbb{R}^2$. The set $M \subset \mathbb{R}^2$ is connected according to Sierpinski [2], since its complement does not

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contain any perfect set.

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