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A FUNCTIONAL ANALYTIC PROOF OF A SELECTION LEMMA

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1. Introduction. If r is a mapping of a set X onto a set Y, then a *selection for* r is a mapping s of Y into X for which $r \circ s$ is the identity. Analogously, if F is a mapping of a set Y into the power set 2^Z of a set Z, then a *selection for* F is a mapping f of Y into Z such that $f(y)$ is an element of $F(y)$ for all y in Y. These two notions are formally equivalent: Given r mapping X onto Y, define $F(y) = r^{-1}(y)$ and $Z = X$. Conversely, given F mapping Y into 2^Z, define X to be the subset of $Y \times Z$ consisting of the pairs (y, z) for which z belongs to $F(y)$, and define r on X by $r(y, z) = y$. A third notion is also formally equivalent to these, namely the notion of a *cross-section* or *transversal* for an equivalence relation R on a set X. If Y is the set of equivalence classes and $r(x)$ is the equivalence class $R[x]$, r maps X onto Y and if s is a selection, then $s(Y)$ is a set which meets each equivalence class exactly once. Conversely, define $x_1 R x_2$ if and only if $r(x_1) = r(x_2)$. Then if T is a transversal for R, $(r|T)^{-1}$ is a selection for r. We shall restrict our attention to the first notion of selection, the other two notions having completely analogous results.

Selections of course always exist by the axiom of choice. However, one is normally interested in situations where r, X and Y possess additional structure, and then the selection is desired to have corresponding properties. Assume for example that X and Y are topological spaces and that r is continuous. A topologist would then want the selection also to be continuous. However in analysis "measurability" of the selection is often a sufficient property. Indeed, except for the nagging question of exactly what kind of measurability one wants, a measurable selection nearly always exists. The simplest example of such a selection theorem, and one which is the cornerstone of all the rest, is the selection version of the Federer-Morse Theorem about the existence of transversals [**7**, Theorem 5.1]:

LEMMA. *Let r be a continuous mapping of a compact metric space onto a compact metric space Y. Then there exists a Borel selection for r.*

The set of ideas involved in selection theorems has traditionally been considered to be a part of pointset topology. It is our aim here to show how some of these results can be obtained from a completely different viewpoint, using functional analytic methods. The standard proof to the Federer-Morse

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lemma, for example, is based on a detailed topological construction. We present here a proof which is essentially a corollary to a kind of constructive Hahn-Banach theorem.

It is well-known that the Hahn-Banach theorem depends on, and is in some sense equivalent to, a kind of choice axiom. It is surprising then that a "Constructive" proof exists. The precise functional analytic theorem we prove is the following special selection theorem:

THEOREM. *Let G be a subspace of a separable normed linear space F over the reals, and let X and Y denote the closed unit balls in the conjugate spaces F^* and G^* of F and G respectively, each equipped with its weak-star topology. Let r be the restriction mapping, taking X into Y. Then there exists a Borel mapping s of Y into X which satisfies:*

i) *$r \circ s$ is the identity. (In particular, r is onto.) and*

ii) *If φ is an extreme point of Y, then $s(\varphi)$ is an extreme point of X.*

Of course, the main difference between our result and the Hahn-Banach theorem is simply the fact that s is Borel. Our proof is easy, and it is based on the classical proof to the Hahn-Banach Theorem. We give this proof in Section 2 and show how the existence of Borel selections for continuous functions on compact metric spaces is an immediate consequence. Of course, the latter result would prove part i) of the Theorem, but not part ii). We should point out that E. Effros used the Hahn-Banach Theorem to make Borel choices of vectors in subspaces of the dual of a separable normed space [**6**, Theorem 2]. Also, P. R. Andenaes used the endpoint choice to get extensions which are maximal in the order theoretic sense [1]. We are indebted to the referee for the latter reference.

The authors feel that the existence of a Borel choice of Hahn-Banach extensions can have other uses in mathematical problems. For example, we use it in Section 3 to prove the Isomorphism Theorem for Borel functions due to Lusin and Souslin [**9**, pp. 396-398]: If r is a one-one Borel mapping of a Borel set X in a Polish space into a Polish space Y, then the range of r is a Borel set in Y and r is a Borel isomorphism between X and $r(X)$. (A space is called *Polish* if it is homeomorphic to a complete separable metric space.) This result shows that the selection and transversal versions of the Federer-Morse Theorem are equivalent. We point out that the same functional analytic idea can be used to prove both theorems, as well as another selection theorem for compact spaces which seems different from the known results (Lemma 5).

Perhaps the most general hypotheses one would want for measurable selection theorems would be for r to be a Borel mapping from a metric space X onto a metric space Y. It is known however that no Borel selection need exist, even when X and Y are both Polish and r is continuous [**3**, Theorem 4.3]. Theorem 2.2 of [**12**] shows that for separable metric spaces we can assume that a Borel function is continuous, but that doesn't help. There is a kind of measurability, however, which is always attainable in this case, which we will call *Souslin* *measurability.* A subset S of a separable metric space Y is called a *Souslin set* (or analytic) if there is a Borel set B in a Polish space and a Borel function f taking B onto S. A function g from Y into a metric space Z is *Souslin measurable* if $g^{-1}(E)$ is in the σ-algebra $\mathscr{S}$ generated by the Souslin sets whenever E is Borel in Z. The σ-algebra $\mathscr{S}$ is important, partly because if μ is a finite Borel measure on a Polish space, then its completion is defined at least on $\mathscr{S}$ [**12**, Theorem 6.1]. The famous selection theorem of von Neumann [**13**, p. 448] asserts that if r is a continuous mapping of a Souslin set into the reals, then r has a Souslin measurable selection. The proof actually covers more generality. In Section 4 we show how von Neumann's result follows from that of Federer-Morse, by a method which also leads to the transversal theorems of Bourbaki [**2**, p. 135] and Dixmier [**4**, Lemma 2].

We point out that our functional analytic result leads to a strengthening of von Neumann's theorem in this special normed linear space context. Thus if X_0 and Y_0 denote the sets of extreme points in X and Y respectively, then the continuous mapping r of X_0 onto Y_0 has a Borel selection. Since both X_0 and Y_0 are Polish spaces, von Neumann's theorem would have applied, but the resulting selection would only have been known to be Souslin measurable.

2. A Borel Hahn-Banach Theorem. Our first goal is to prove the functional analytic selection theorem stated in the introduction, using the basic inequality of the classical proof. We see that the original result for the separable case depends only on induction.

THEOREM 1. *Let G be a subspace of a separable normed linear space F over the reals, and let X and Y denote the closed unit balls in the conjugate spaces F^* and G^* of F and G respectively, each equipped with its weak-star topology. Let r be the restriction mapping, taking X into Y. Then there is a Borel mapping s of Y into X which satisfies:*

i) *$r \circ s$ is the identity. (In particular, r is onto.) and*

ii) *If φ is an extreme point of Y, then $s(\varphi)$ is an extreme point of X.*

Proof. Let $f_1, f_2, \ldots$ be such that the linear span $G_\infty = [G, f_1, f_2, \ldots]$ is dense in F and define $G_0 = G$, $G_n = [G, f_1, \ldots, f_n]$ for $1 \leqq n < \infty$. For each n, let Y_n be the closed unit ball in G_n^*, with its weak-star topology. These are all compact and metrizable, and since G_∞ is norm dense in F, X is naturally homeomorphic to Y_∞. Thus to define $s: Y \to X$ we may define $s: Y_0 \to Y_\infty$.

First define $s_n: Y_n \to Y_{n+1}$ for $0 \leqq n < \infty$ as follows; for $\varphi \in Y_n$, let

$$a_n(\varphi) = \sup \{\varphi(f) - \|f - f_{n+1}\| : f \in G_n\} \text{ and}$$
$$b_n(\varphi) = \inf \{\varphi(f) + \|f - f_{n+1}\| : f \in G_n\}.$$

The classical proof shows that $a_n(\varphi) \leqq b_n(\varphi)$ and that $\psi \to \psi(f_{n+1})$ takes $S(\varphi) = \{\psi \in G_{n+1}^* : \psi|G_n = \varphi \text{ and } \|\psi\| = \|\varphi\|\}$ one to one onto $[a_n(\varphi), b_n(\varphi)]$. We define $s_n(\varphi)$ to be the extension of φ with $s_n(\varphi)(f_{n+1}) = b_n(\varphi)$.

Notice that b_n is an inf of continuous functions, so it is upper semi-continuous. Thus for $f \in G_{n+1}$ $s_n(\varphi)(f)$ depends in a Borel way on φ. Since the evaluation functionals induce a compact metrizable topology on Y_{n+1}, s_n is Borel. Suppose φ is extreme in Y_n and $s_n(\varphi) = t\psi_1 + (1-t)\psi_2$ where $0 < t < 1$ and $\psi_1, \psi_2 \in Y_{n+1}$. Then $\varphi = t(\psi_1|G_n) + (1-t)(\psi_2|G_n)$, so we must have $\psi_1 \in S(\varphi)$ and $\psi_2 \in S(\varphi)$. Combining this with the fact that $b_n(\varphi) = s_n(\varphi)(f_{n+1}) = r\psi_1(f_{n+1}) + (1-t)\psi_2(f_{n+1})$, it follows that $\psi_1(f_{n+1}) = \psi_2(f_{n+1}) = b_n(\varphi)$. Thus $\psi_1 = \psi_2 = s_n(\varphi)$, and $s_n(\varphi)$ is an extreme in Y_{n+1}.

Now define $s: Y_0 \to Y_\infty$ by defining $s(\varphi)(f_n) = ((s_{n-1} \circ \ldots \circ s_0)(\varphi))(f_n)$ for $n = 1, 2 \ldots$. Then $s(\varphi)(f_n)$ is Borel in φ for each n, so $s(\varphi)(f)$ is Borel in φ for $f \in G_\infty$. Hence s is Borel. If φ is extreme in Y_0, an easy induction shows that $s(\varphi)|G_n$ is extreme in Y_n for $n = 1, 2, \ldots$. Hence $s(\varphi)$ is extreme in Y_∞.

LEMMA 2. (Federer and Morse [**7**]) *If X and Y are compact metric spaces and $r: X \to Y$ is continuous and onto, then r has a Borel selection.*

Proof. Let $F = C(X, \mathbf{R})$, $G = \{g \circ r : g \in C(Y, \mathbf{R})\}$. Then X can be identified with the set of those extreme points in the unit ball of F^* whose values at 1 are 1, and likewise Y for G^*. Then taking s from Theorem 1, $s|Y$ is a Borel selection.

Remark. Federer and Morse actually prove there is a Borel transversal: a Borel set B such that $r|B$ is one to one and onto. Using the result of our next section, Borel selections and transversals are equivalent.

The next lemma is proved by a standard technique, which can be found in the proof of Lemma 1.1 of [**11**].

LEMMA 3. *If X is a σ-compact metric space, Y is a metric space and $r: X \to Y$ is continuous, then r has a Borel selection.*

Proof. If $K_1, K_2 \ldots$ are compact in X and have union X, or even if only $r(X) = \bigcup_{n \geq 1} r(K_n)$, we can let s_n be a Borel selection for r/K_n, $n = 1, 2, \ldots$, and define s to agree with s_n on $r(K_n) \setminus \bigcup_{j<n} r(K_j)$. For each n, $r(K_n)$ is compact and hence Borel, so $r(X)$ is Borel and s is Borel.

3. The isomorphism theorem. This fundamental result can be stated in several ways, using facts about Polish spaces. We shall use the fact that a Borel set in a Polish space is Borel isomorphic to a compact metric space, indeed any one with the same cardinality [**9**, p. 358, Remark 1]. Thus the lemma below implies that a one to one Borel image of a Borel set in a Polish space, under a mapping into a Polish space, is a Borel set. Since this applies to all Borel sets, a one to one Borel mapping of a Borel set is an isomorphism onto a Borel set. We will use the separation theorem for Souslin sets [**9**, p. 393], as do other proofs.

LEMMA 4. *Let X and Y be compact metric spaces and let $r: X \to Y$ be one to one and Borel. Then $r(X)$ is Borel.*

Proof. Let $\mathscr{U}$ be a countable base for the topology of X. For $U \in \mathscr{U}$, $r(U)$ and $r(X - U)$ are disjoint analytic sets, so there is a Borel set $B(U)$ in Y containing $r(U)$ and disjoint from $r(X - U)$. Let G_1 be the algebra of real functions on Y generated by $C(Y, \mathbf{R})$ and $\{\varphi_{B(U)} : U \in \mathscr{U}\}$. Then G_1 is separable relative to the sup norm, so the set

$$Y_1 = \{y \in Y : g \in G_1 \text{ implies } |g(y)| \leqq \sup\{|g(r(x))| : x \in X\}\}$$

is determined by a countable dense set of g's in G_1 and thus is Borel.

Let G_2 be the uniform closure of $\{g|Y_1 : g \in G_1\}$. Then $G_2 + iG_2$ is uniformly closed and closed under complex conjugation, so it is isomorphic to the complex $C(Z)$ for some compact metric space Z. If $y \in Y_1$, it follows that $\delta_y(g) = g(y)$ defines an extreme point of the unit ball of G_2^*. Since G_2 is separable and each $g \in G_2$ is Borel, $y \to \delta_y$ is Borel.

Now $G = \{g \circ r : g \in G_2\}$ is a closed algebra of functions on X isometric to G_2. Let F be the algebra generated by G and $C(X, \mathbf{R})$. Then F is separable under the sup norm, so if X^* and Y^* are the closed unit balls in F^* and G^* we have a Borel map $s^* : Y^* \to X^*$ given by Theorem 1. For $y \in Y_1$, let $\lambda_y = s^*(\delta_y)$ and $\mu_y = \lambda_y|C(X)$. Then μ_y is of norm 1 and $\mu_y(1) = 1$, so μ_y is a probability measure on X. Thus μ_y has a non-empty support S_y, which we claim has only one element. If not, let x_1, x_2 be distinct elements, and choose a continuous $f_0 : X \to [0, 1]$ which is 1 in a neighborhood of x_1 and 0 in a neighborhood of x_2. Then the number $t = \mu_y(f_0)$ satisfies $0 < t < 1$. Define λ_1, λ_2 on F by

$$\lambda_1(f) = t^{-1}\lambda_y(f_0 f), \quad \lambda_2(f) = (1 - t)^{-1}\lambda_y((1 - f_0)f).$$

Then $\lambda_y = t\lambda_1 + (1 - t)\lambda_2$ and if $f_1(x_2) > 0$ but $f_1 = 0$ on supp (f_0) then $\lambda_1(f_1) \neq \lambda_2(f_1)$. Also $\|\lambda_1\| = \|\lambda_2\| = 1$, so λ_y is not extreme. Hence S_y is a singleton $\{s(y)\}$. Now $\lambda_y(f) = f(s(y))$ for $f \in C(X)$, so $f \circ s$ is Borel. Hence s is Borel from Y_1 to X.

Suppose $s(r(x)) \neq x$ for some $x \in X$. Then there is a continuous $f : X \to [0, 1]$ such that $f(s(r(x))) > 0$ and $f = 0$ on a $U \in \mathscr{U}$ which contains x. Then the characteristic function $g = 1 - \varphi_{B(U)}$ in G_2 is such that $g \circ r = 0$ on U and $g \circ r = 1$ on $X - U$. Then $f \leqq g \circ r$, so $\lambda_{r(x)}(f) \leqq \lambda_{r(x)}(g) = 0$, but $f > 0$ on supp $(\mu_{r(x)})$, so $\lambda_{r(x)}(f) = \mu_{r(x)}(f) > 0$. Hence $s(r(x)) = x$. Now it is easy to see that $r(X) = \{y \in Y_1 : r(s(y)) = y\}$, so $r(X)$ is Borel.

It follows that if s is a Borel selection for $r : X \to Y$ and Y is Polish, then $s(Y)$ is a Borel set B meeting each level set of r just once, i.e., a Borel transversal. Conversely, $(r|B)^{-1}$ is a Borel selection if B is a Borel transversal.

The same proof applies to give the following result. It seems natural in our functional analytic approach, but it is not obvious that it is either stronger or weaker than the more familiar selection lemmas.

LEMMA 5. *Let X and Y be compact metric spaces and let $r : Y \to Y$ be Borel. Suppose there are Borel sets $E_1, E_2, \ldots$ in Y such that whenever $x \notin r^{-1}(y)$ there* *is an n such that $y \notin E_n$ and x is interior to $r^{-1}(E_n)$. Then there is a Borel set $Y_1 \supseteq r(X)$ and a Borel $s: Y_1 \to X$ such that $s(y) \in r^{-1}(y)$ for $y \in r(X)$.*

4. More general selection lemmas. Here we are going to use the Federer-Morse Lemma to prove two selection lemmas with weaker hypotheses. The first one contains the essence of von Neumann's selection lemma. After the proof we will briefly recall some known methods for passing to still weaker hypotheses keeping the same conclusion. Then we show how the same proof applies to get a Borel selection under slightly stricter hypotheses. The proof uses Lemma 3, and a slight extension of the method of proof of Lemma 3.

LEMMA 6. (von Neumann [**13**, p. 448]) *Let X and Y be Polish spaces and let $r: X \to Y$ be continuous. Then there is a Souslin measurable selection $s: r(X) \to X$ for r.*

Proof. First we reduce to a special case. Recall that any separable metric space can be homeomorphically imbedded in a compact metric space [**9**, p. 119]. Suppose X_0, Y_0 are compact metric spaces containing X, Y respectively. Since r is continuous its graph Γ is a subset of $X_0 \times Y_0$ which is homeomorphic to X. Thus Γ is a G_δ in $X_0 \times Y_0$ [**9**, p. 337]. A selection for the projection mapping of Γ into Y gives a selection for r by composition with the projection into X. Since the latter projection is continuous, any measurability is preserved. Thus we reduce to the following case: Z and Y are compact, X is a G_δ in Z, $r: Z \to Y$ is continuous and we seek a selection for $r|X$.

Let $X_1 \supseteq X_2 \supseteq \ldots$ be open sets in Z whose intersection is X. For each n, there is a sequence of open sets $U_1^n, U_2^n, \ldots$ whose union is X_n, such that each has diameter at most $1/n$ and each has closure contained in X_n. For each n, order N^n lexicographically and for $i \in N^n$ let $V_i = U_{i_1}^1 \cap \ldots \cap U_{i_n}^n$.

For $y \in r(X)$, let $i_1(y)$ be the first element of $\{j: y \in r(X \cap U_j^1)\}$, let $i_2(y)$ be the first element of $\{j: y \in r(X \cap V_{i_1(y),j})\}$, etc. Notice that for $j \in N^n$, $n \geqq 2$, we have

$$V_j \subseteq V_{j_1,\ldots,j_{n-1}}.$$

Thus for any n, $(i_1(y), \ldots, i_n(y))$ is the first element of $\{j \in N^n: y \in r(X \cap V_j)\}$, and

$$V_{i_1(y),\ldots,i_n(y)} \supseteq V_{i_1(y),\ldots,i_{n+1}(y)}.$$

Now each V_j is open in Z and hence σ-compact, so there is a Borel selection s_j for $r|V_j$, by Lemma 3. We define $s^{(n)}$ to agree with s_j on $r(V_j \cap X) \setminus \bigcup_{i<j} r(V_i \cap X)$. Then $s^{(n)}$ is Souslin measurable because each $r(V_j \cap X)$ is a Souslin set. We have

$$s^{(n)}(y) = s_{i_1(y),\ldots,i_n(y)}(y), \text{ so}$$
$$s^{(n)}(y) \in V_{i_1(y),\ldots,i_n(y)} \cap r^{-1}(y).$$

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Hence $s^{(1)}, s^{(2)}, \ldots$ converges uniformly on $r(X)$ to a function s such that for each n,

$$s(y) \in \bar{V}_{i_1(y),\ldots,i_n(y)} \cap r^{-1}(y).$$

Hence $f \circ s$ is Souslin measurable for each $f \in C(Z)$, so s is Souslin measurable. Also, $s(y) \in X_n$ for each n, so $s(y) \in X$, and s is a selection for $r|X$.

The original proof of von Neumann produced a Souslin measurable selection for a continuous map r of a Souslin set into the reals. If S is Souslin and $X = N^N$ with the product topology, he used the fact that there is a continuous f taking X onto S[**9**, p. 386]. If s_1 is a selection for $r \circ f$, measurable relative to a σ-algebra $\mathscr{A}$, he observed that $f \circ s_1$ is a selection for r, measurable relative to $\mathscr{A}$. If r is only Borel and S is Souslin, the graph of r is a Souslin set, and Borel isomorphic to S by [**9**, p. 398]. The projection is continuous, so we reduce to the case in which r is continuous. These techniques have been used more widely, for instance in [**8**].

The proof of Lemma 6 also proves the following special case of a result of K. Kuratowski and A. Maitra [**10**, Corollary 2].

LEMMA 7. *Let X and Y be Polish and let $r:X \to Y$ be continuous. If $r(U)$ is Borel whenever U is open in X, or if $r(C)$ is Borel whenever C is closed in X, then r has a Borel selection.*

Proof. If r takes closed sets to Borel sets, it must do the same for all F_σ's and in particular for open sets. Thus we may concern ourselves only with the case that r takes open sets to Borel sets. Then in the proof of Lemma 6 the sets $r(V_j \cap X)$ are Borel in Y, so the function s turns out to be Borel, and $r(X)$ is Borel.

There is an important application of Lemma 7 which is useful for selections of set valued mappings and also for producing transversals of equivalence relations. Let X be a Polish space, a dense G_δ in a compact metric space Z. Let $\mathscr{C}_0(X)$ denote the space of non-empty closed sets in X. Under the mapping $C \to \bar{C}$, $\mathscr{C}_0(X)$ is mapped one to one to a subspace of the space $\mathscr{C}(Z)$ of all closed sets in Z under the Hausdorff metric. $\mathscr{C}(Z)$ is compact, and E. Effros shows in [**5**] that $\mathscr{C}_0(X)$ corresponds to a G_δ, so that $\mathscr{C}_0(X)$ is Polish. Let $X^* = \{(x, F) \in X \times \mathscr{C}_0(X): x \in F\}$ and let r be the projection of X^* onto $\mathscr{C}_0(X)$. Then it is easy to show that X^* is closed and that r is open. Hence r has a Borel selection. Projecting into X gives a Borel $s:\mathscr{C}_0(X) \to X$ such that for each C, $s(C) \in C$ (c.f. [**3**, Theorem 4.2]).

Now suppose R is an equivalence relation on X with closed equivalence classes $R[x]$ and define $F(x) = R[x]$ for $x \in X$. Then $f = s \circ F$ maps X to X, and is as measurable as F, because s is Borel. Suppose $R[C]$ is Borel for each closed $C \subseteq X$. Since each open set is an F_σ, it follows that $R[U]$ is Borel if U is open. Now $F:X \to \mathscr{C}(X)$ is Borel in the latter case [**4**, Proof of Theorem],

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so f is Borel. Thus $\{x:f(x) = x\}$, which is a transversal for R, is a Borel set. This proves the Bourbaki and Dixmier transversal lemmas [**2**, p. 135; **4**, Lemma 2].

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