By means of these equations it may easily be seen that an equiangular pentagon can only have its sides commensurable if it be regular.

Hexagon. $$a_1 + a_2 = a_4 + a_5 ;$$
$$a_2 + a_3 = a_5 + a_6.$$

Octagon, $$(a_2 - a_4 - a_6 + a_8) + \sqrt{2}(a_1 - a_5) = 0 ;$$
$$(a_2 + a_4 - a_6 - a_8) + \sqrt{2}(a_3 - a_7) = 0.$$

In an equiangular octagon with commensurable sides, opposite sides are equal.

Decagon, $$4(a_1 - a_6) + (a_2 - a_3 + a_4 - a_5 - a_7 + a_8 - a_9 + a_{10}) + \sqrt{5}(a_2 + a_3 - a_4 - a_5 - a_7 - a_8 + a_9 + a_{10}) = 0 ,$$
$$(a_2 + a_5 - a_7 - a_{10})^2 + (a_3 + a_4 - a_8 - a_9)^2 + \sqrt{5}\{(a_3 + a_4 - a_8 - a_9)^2 - (a_2 + a_5 - a_7 - a_{10})^2\} = 0.$$

In an equiangular decagon with commensurable sides,

$$a_1 - a_6 = a_7 - a_2 = a_3 - a_8 = a_9 - a_4 = a_5 - a_{10} ;$$

where a_1 and a_6, a_2 and a_7, &c., are opposite sides.

Dodecagon. $$2(a_1 - a_7) + (a_3 - a_5 - a_9 + a_{11}) + \sqrt{3}(a_2 - a_6 - a_8 + a_{12}) = 0 ;$$
$$2(a_4 - a_{10}) + (a_2 + a_6 - a_8 - a_{12}) + \sqrt{3}(a_3 + a_5 - a_9 - a_{11}) = 0.$$

In an equiangular dodecagon with commensurable sides,

$$a_2 - a_8 = a_{10} - a_4 = a_6 - a_{12} ; \quad a_1 - a_7 = a_9 - a_3 = a_5 - a_{11} ;$$

where a_2 and a_8, a_4 and a_{10}, &c., are opposite sides.

Third Meeting, January 14th, 1887.

W. J. MACDONALD, Esq., M.A., Vice-President, in the Chair.

On Certain Inverse Roulette Problems.

By PROFESSOR CHRYSTAL.

The problem of designing cams or centrodes to produce any given motion in one plane is one of some practical importance; and it seems worth while to illustrate by examples some simple methods by which the solution can in certain cases be arrived at. These methods are founded, for the most part, on the use of the so-called Pedal Equation (or p-r-equation), which has great advantages in the present investigation, inasmuch as it depends on the form but not on the position of the curve which it represents.

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Few of the results arrived at are absolutely new. Most of them have been found directly by Clerk Maxwell in a paper published in the Transactions of the Royal Society of Edinburgh, Vol. XVI., 1849.

If we suppose a plane Π to slide upon a fixed plane Π', it is obvious that the motion of Π is determined if the space loci of two points of Π, say P and Q, be given. There will therefore be an infinite number of ways of causing Π to move so that any point P in it may have a given locus. In other words, the problem to generate a given plane curve as a roulette is indeterminate.

I. If, however, there be given a fixed curve C′, then it is a determinate problem to find what curve C in Π must roll on C′ in order that the point P in Π may trace a given curve R.

II. Again, if there be given a curve C in Π, then it is a determinate problem to find a curve C′ such that, if C roll on C′, then P shall trace a given curve R.

Owing to the fact that the point P is fixed in Π, and thus affords an origin of reference for the curve C, or body centrode, as it is called, the first of these problems is in general easier than the second. We can at all events in general find, without much difficulty, an equation connecting the radius vector (r) from P and the perpendicular (p) from P on the tangent to C.

In fact, if K (Fig. 56) be the point of contact of the body and space centrodes C and C′, then, if P be the corresponding position of the point which traces out the given curve R, we know that PK is normal to R. Moreover, the tangent to C′ at K is the tangent to C at K. Hence, if PM be perpendicular to this tangent, we have (P being fixed with reference to C) $PK = r$, $PM = p$.

Now the two curves C′ and R are given in the plane Π'; and from their properties we can deduce a relation between PM, and PK, P being a variable point on R, and PK normal to R.

If this relation be f(PM, PK) = 0, then the p-r-equation to the curve C, with respect to P as origin, will be

$$f(p, r) = 0 \qquad (1).$$

Since $1/p^2 = 1/r^2 + (dr/r^2 d\theta)^2$, (1) gives us the differential equation

$$f\left\{\left(\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2\right)^{-\frac{1}{2}}, \; r\right\} = 0 \qquad (2).$$

The obtaining of the relation (1) is a comparatively simple matter in many cases; but as a rule the integration of the equation (2) presents great difficulty.

Cases where the Space Centrode is a Straight Line.

In such cases, if we take the given straight line as the x-axis, we have merely to find an equation between the normal PG and the ordinate PN to the given curve R; and this relation will furnish at once the p-r-equation to the body centrode, if we put $PN = p$, and $PG = r$.

Generation of a circle, the space centrode being any straight line. (Fig. 57).

Let a be the radius of the circle, b the distance of its centre A from the given straight line C′. We have

$$p = PN = b + a \sin\phi,$$
$$r = PG = a + b \operatorname{cosec}\phi,$$

ϕ being the angle PGD.

Hence $$(p-b)(r-a) = ab,$$

or $$\frac{a}{r} + \frac{b}{p} = 1. \qquad (1).$$

From (1) we deduce

$$\frac{a^2}{r^2} - \frac{2a}{r} + 1 = b^2\left(\frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{a\theta}\right)^2\right).$$

Hence $$d\theta = \pm \frac{bdr}{r^2\sqrt{\left(1 - \frac{2a}{r} + \frac{a^2 - b^2}{r^2}\right)}} \qquad (2).$$

If $a > b$ we have

$$d\theta = \mp \frac{b}{\sqrt{a^2 - b^2}} \frac{d\left(\frac{1}{r} - \frac{a}{a^2 - b^2}\right)}{\sqrt{\left\{\left(\frac{1}{r} - \frac{a}{a^2 - b^2}\right)^2 - \frac{b^2}{(a^2 - b^2)^2}\right\}}},$$

$$\chi = \frac{\sqrt{a^2 - b^2}}{b}\theta + \alpha = \log\left\{\frac{1}{r} - \frac{a}{a^2 - b^2} + \sqrt{\left(\frac{1}{a^2 - b^2} - \frac{2a}{(a^2 - b^2)}r + \frac{1}{r^2}\right)}\right\},$$

$$e^{\chi} - \left(\frac{1}{r} - \frac{a}{a^2 - b^2}\right) = \sqrt{(\&c.)},$$

$$e^{2\chi} - 2\left(\frac{1}{r} - \frac{a}{a^2 - b^2}\right)e^{\chi} + \left(\frac{1}{r} - \frac{a}{a^2 - b^2}\right)^2 = \frac{1}{r^2} - \frac{2a}{(a^2 - b^2)r} + \frac{1}{a^2 - b^2},$$

$$\frac{1}{2}\left(e^{\chi} + \frac{b^2}{(a^2 - b^2)^2}e^{-\chi}\right) = \frac{1}{r} - \frac{a}{a^2 - b^2}.$$

This we may write

$$\frac{1}{2}e^{\alpha}\left(e^{c\theta/b} + \frac{b^2 e^{-2\alpha}}{(a^2 - b^2)^2}e^{-c\theta/b}\right) = \frac{1}{r} - \frac{a}{c^2},$$

where $c = \sqrt{a^2 - b^2}$.

Since a depends merely on the choice of the prime radius vector through P, we may so select the latter that $b^2e^{-2a}/(a^2-b^2)^2=1$, we then have

$$\frac{1}{2}\frac{b}{c^2}\left(e^{c\theta/b}+e^{-c\theta/b}\right)=\frac{1}{r}-\frac{a}{c^2},$$

$$r=\frac{c^2}{a+b\cosh(c\theta/b)}.$$

If $a<b$, (2) may be written

$$d\theta=+\frac{b}{\sqrt{b^2-a^2}}\frac{d\left(\frac{1}{r}+\frac{a}{b^2-a^2}\right)}{\sqrt{\left\{\frac{b^2}{(b^2-a^2)^2}-\left(\frac{1}{r}+\frac{a}{b^2-a^2}\right)^2\right\}}}.$$

Hence

$$a+\frac{c\theta}{b}=\cos^{-1}\left(\frac{\frac{c^2}{r}+a}{b}\right);$$

where $c=\sqrt{b^2-a^2}$.

We may choose the prime radius so as to annul a; hence we have

$$b\cos\frac{c\theta}{b}=\frac{c^2}{r}+a,$$

or

$$r=\frac{c^2}{b\cos(c\theta/b)-a}.$$

If $a=b$, (2) becomes

$$d\theta=+\frac{ad\left(\frac{1}{r}\right)}{\sqrt{\left(1-\frac{2a}{r}\right)}},$$

$$\theta+a=\sqrt{\left(1-\frac{2a}{r}\right)};$$

which, by properly choosing the prime radius, we may write

$$r=\frac{2a}{1-\theta^2}.$$

Generation of a straight line, the space centrode being a straight line inclined at an angle a to the given straight line. (Fig. 58).

Here we have

$$\text{PG}\cos a=\text{PN};$$

$$r\cos a=p.$$

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Hence the body centrode is an equiangular spiral whose pole is and whose angle is $\frac{1}{2}\pi - a$.

Generation of an ellipse, the space centrode being the major axis. (Fig. 59).

If the co-ordinates of P be (x, y), so that $y^2 = (1 - e^2)(a^2 - x^2)$,

we have $$p^2 = y^2 = (1 - e^2)(a^2 - x^2);$$

and $$\begin{aligned} r^2 = \text{PG}^2 &= \text{NG}^2 + \text{PN}^2, \\ &= (1 - e^2)^2 x^2 + y^2, \\ &= (1 - e^2)(a^2 - e^2 x^2). \end{aligned}$$

Hence $$r^2 - e^2 p^2 = (1 - e^2)^2 a^2;$$

that is $$p^2 = \frac{a^2}{a^2 - b^2} r^2 - \frac{b^4}{a^2 - b^2} \qquad (1).$$

Now (see Williamson Diff. Calc. p. 346) if α be the radius of the fixed, β the radius of the rolling circle of an epicycloid, and if the centre of the fixed circle be the origin, the p-r-equation is

$$p^2 = \frac{(\alpha + 2\beta)^2}{4\beta(\alpha + \beta)} r^2 - \frac{\alpha^2(\alpha + 2\beta)^2}{4\beta(\alpha + \beta)} \qquad (2)$$

Comparing this with (A) we see that they will agree provided

$$\frac{(\alpha + 2\beta)^2}{4\beta(\alpha + \beta)} = \frac{a^2}{a^2 - b^2}, \quad \frac{\alpha^2(\alpha + 2\beta)^2}{4\beta(\alpha + \beta)} = \frac{b^4}{a^2 - b^2}.$$

This gives

$$\alpha = \frac{b^2}{a}, \ \beta = \frac{1}{2}\frac{b}{a}(a - b) \text{ or } = -\frac{1}{2}\frac{b}{a}(a + b).$$

Hence any ellipse can be generated by the rolling of an epicycloid, or of a hypocycloid, upon its major axis, the tracing point being the centre of the fixed circle. The radius of the fixed circle is the radius of curvature at the end of the major axis; and the radius of the generating circle is half the difference, or half the sum, of this radius of curvature and of the semi-axis minor.

Cor. 1.—A parabola can be described as a roulette by the rolling of the involute of a circle upon its axis, the tracing point being the centre of the circle, and the radius of the circle being half the latus rectum.

Cor. 2.—The p-r-equation of the body centrode for a hyperbola in case above considered is

$$p^2 = \frac{a^2 + b^2}{a^2} r^2 - \frac{b^4}{a^2 + b^2}.$$

Cor. 3.—If the space centrode be the minor axis of the ellipse, the body centrode is one or other of the hypocycloids given by

$$\alpha = a^2/b, \ \beta = \tfrac{1}{2}a(a \pm b)/b.$$

Generation of a parabola, the space centrode being the directrix. (Fig. 60).

We have here, $4a$ denoting the latus rectum of the given parabola,

$$\frac{NG}{p} = \frac{PM}{MH}$$

$$= \frac{\sqrt{4a(p-a)}}{2a}$$

$$NG = p\sqrt{\left(\frac{p}{a} - 1\right)}.$$

Hence
$$r^2 = p^2 + p^2\left(\frac{p}{a} - 1\right);$$

that is,
$$ar^2 = p^3 \qquad (1).$$

This leads to the differential equation

$$d\theta = \pm \frac{a^{\frac{1}{3}}dr}{r\sqrt{\left(r^{\frac{2}{3}} - a^{\frac{2}{3}}\right)}}$$

Hence, by properly choosing the prime radius, we have

$$\theta = 3\sin^{-1}\left(\frac{a}{r}\right)^{\frac{1}{3}},$$

that is
$$r = a\operatorname{cosec}^3\frac{\theta}{3} \qquad (2);$$

or, in Cartesian coordinates,

$$27a(x^2 + y^2) = (4a + y)^3 \qquad (2^1).$$

The form of this cubic is given in Fig. 61.

Cor.—If we write R in place of p, and R^2/P in place of r, in equation (1), we obtain the equation to the pedal of the generating cubic. The result is

$$P^2 = a\,R,$$

which represents a parabola identical with the given parabola.

Hence the body centrode for a parabola, when the space centrode is the directrix, is the first negative pedal of the given parabola, with respect to its focus, the tracing point being the focus

Generation of a straight line, the space centrode being a circle. (Fig. 62).

Let $CA = d$ be the perpendicular from the centre of the circle on the given line; and let $KCA = \theta$.

Then, if a be the radius of the circle, we have

$$r = PK = a\cos\theta - d;$$

$$p = r\cos\theta.$$

Hence $\quad r^2 = ap - dr,$

that is, $\quad ap = r^2 + dr \qquad (1).$

This equation can be integrated; but we confine ourselves here to the case where $d = 0$. We then have

$$ap = r^2 \qquad (2).$$

This is the p-r-equation to a circle where diameter is a, the origin being a point on the circumference. Hence the straight line is generated by a circle whose diameter is equal to the radius of the given circle, the tracing point being on the circumference, a very familiar result (Fig. 63). It is to be observed, however, that it is only the part DE of the line that can be thus generated, and that it will be impossible to generate the rest of the line with the given circle as space centrode. Similar limitations will obviously occur in the more general case where d is not zero.

Generation of the cardioid, the space centrode being a circle passing through the cusp, whose centre lies on the axis of the curve, and whose diameter is half the length of the axis. (Fig. 64).

The polar equations to the cardioid and to the circle are

$$r = a(1 + \cos\theta) \text{ and } r = a\cos\theta$$

respectively.

Since for the cardioid $rd\theta/dr = -\cot\frac{1}{2}\theta$, if PK be the normal, we have $OPK = \frac{1}{2}\theta$.

Let OK bisect the angle POB. The $OK = KP$, and we have $2OK\cos\frac{1}{2}\theta = OP = 2a\cos^2\frac{1}{2}\theta$.

Hence $OK = a\cos\frac{1}{2}\theta$; and it follows that K lies on the circle.

If C be the centre of the circle $CKO = COK = \frac{1}{2}\theta$. It follows that CK is parallel to OP. Hence the tangent to the circle at K is perpendicular to OP.

We have therefore

$$p = \mathrm{PM} = a\cos^2\tfrac{1}{2}\theta\,;$$
$$r = \mathrm{PK} = a\cos\tfrac{1}{2}\theta.$$

Hence $$ap = r^2 \qquad (1).$$

Now (1) is the p-r-equation to a circle of diameter a, the origin being a point on the circumference. Hence the body centrode in the present case is a circle whose diameter is the same as that of the fixed circle, and the tracing point is a point on its circumference.

Cor.—We have, incidentally, a proof that the pedal of a circle with respect to a point on its circumference is the cardioid generated by a pair of circles whose diameters are each equal to the radius of the given circle.

For M is a point on the pedal of the circle OKB; and OM = MP.

Hence the locus of M is a cardioid similar to OPA, the ratio of similarity being 1 : 2.

Generation of an ellipse, the space centrode being the major auxiliary circle. (Fig. 65).

Here PK $= r$, PM $= p$. Denote PH, the perpendicular from P on the conjugate diameter by P. Then, with the usual notation, we have

$$\mathrm{KH}^2 - \mathrm{HP}^2 = \mathrm{CK}^2 - \mathrm{CP}^2,$$

that is $$(r - \mathrm{P})^2 - \mathrm{P}^2 = a^2 - \mathrm{CP}^2$$
$$= \mathrm{CD}^2 - b^2$$
$$= a^2b^2/\mathrm{P}^2 - b^2 \qquad (1).$$

Also, since $$\mathrm{KCH} \approx \mathrm{PKM},$$
$$p/r = (r - \mathrm{P})/a.$$

Hence $$\mathrm{P} = r - ap/r = (r^2 - ap)/r \qquad (2).$$

We have therefore, from (1) and (2)

$$r^2 - 2(r^2 - ap) = a^2b^2r^2/(r^2 - ap) - b^2\,;$$

that is $$2(r^2 - ap)^3 - (r^2 + b^2)(r^2 - ap)^2 + a^2b^2r^2 = 0 \qquad (3).$$

This is the p-r-equation to the body centrode. The derivation of the polar equation would obviously be difficult.

Generation of a given Curve as a Roulette when the Body Centrode is given.

Referring back to figure 56, we see that we have now the curve C given in the plane Π, and R given in the plane Π. P being a point fixed with respect to C, we may suppose the p-r-equation of C given with respect to P as origin say

$$f(\mathrm{P},\ \mathrm{R})=0 \qquad (1).$$

The problem now is, given a curve (Fig. 66) to determine the locus of a point K on the normal at P which is such that, if PM be perpendicular to the tangent of this locus at K and PM = P, PK = R, then the relation (1) shall be satisfied.

Let the co-ordinates of P and K, with reference to any pair of rectangular axes be (X, Y) and (x, y) respectively; and let the equation to (R) be

$$\phi(\mathrm{X},\ \mathrm{Y})=0 \qquad (2),$$

Then, since PK is normal to (R), we have

$$(\mathrm{X}-x)/\phi_{\mathrm{X}}=(\mathrm{Y}-y)/\phi_{\mathrm{Y}} \qquad (3),$$

Also, if p denote dy/dx, we have

$$\mathrm{P}=\{(\mathrm{X}-x)\,p-(\mathrm{Y}-y)\}/\sqrt{(1+p^2)} \qquad (4),$$

And, finally,

$$\mathrm{R}^2=(\mathrm{X}-x)^2+(\mathrm{Y}-y)^2 \qquad (5).$$

Between these five equations we can eliminate P, R, X, Y. The resultant is a differential equation of the first order connecting x and y, the integration of which will give the equation in Cartesian co-ordinates to the required space centrode.

Generation of a straight line, the body centrode being a straight line.

In this case the equations of the general theory reduce as follows:—

$$\mathrm{P}=c \qquad (1),$$

where c is the distance of the tracing point from the rolling straight line.

$$\mathrm{Y}=0 \qquad (2),$$

if we take the generated straight line as axis of X.

$$\mathrm{X}-x=0 \qquad (3).$$

$$\mathrm{P}=\{(\mathrm{X}-x)p-(\mathrm{Y}-y)\}/\sqrt{(1+p^2)} \qquad (4).$$

$$\mathrm{R}^2=(\mathrm{X}-x)^2+(\mathrm{Y}-y)^2 \qquad (5)$$

The result of the elimination is

$$c\sqrt{(1+p^2)}=y \qquad (6).$$

This gives

$$c\,dy/\sqrt{(y^2-c^2)}=dx.$$

Hence

$$\log(y+\sqrt{y^2-c^2})=x/c+\mathrm{A},$$

or, if we so choose the origin that $x=o$ when $y=c$,

$$\log\frac{y+\sqrt{y^2-c^2}}{c}=x/c.$$

This gives

$$y=\frac{c}{2}(e^{x/c}+e^{-x/c}).$$

Hence the space centrode is a catenary, whose directrix is the axis of x.

In point of fact if K M be the tangent at any point K of the catenary (Fig. 67) KP the ordinate, and PM perpendicular to KM, then we have, as is well known, PM $=c$ and KM $=$ arc AK, this verifies the result we have just arrived at.

Generation of a circle, the body centrode being a straight line.

Let the distance of the tracing point from the line be b; and the radius of the circle a; then the equations required to determine the space centrode are as follows :—

$$P=b \qquad (1).$$

$$X^2+Y^2=a^2 \qquad (2).$$

$$(X-x)/X=(Y-y)/Y \qquad (3).$$

$$P=\{(X-x)p-(Y-y)\}/\sqrt{(1+p^2)} \qquad (4).$$

These lead to

$$(xp-y)\{a/\sqrt{(x^2+y^2)}-1\}=b\sqrt{(1+p^2)} \qquad (6).$$

or

$$x^2d\left(\frac{y}{x}\right)\{a/\sqrt{(x^2+y^2)}-1\}=b\sqrt{(dx^2+dy^2)}.$$

Changing to polar co-ordinates this equation becomes

$$r^2d\theta\left(\frac{a}{r}-1\right)=b\sqrt{(dr^2+r^2d\theta^2)}$$

which gives

$$d\theta=\frac{\pm bdr}{r^2\sqrt{\left\{1-\frac{2a}{r}+\frac{a^2-b^2}{r^2}\right\}}}$$

This is identical with the equation obtained above for the body centrode in the generation of a circle when the space centrode is a straight line.

We have, therefore, a remarkable reciprocity between the two cases, viz. :—If C be the body centrode for generating a circle when the space centrode is a straight line at a distance b from the centre, then C will be the space centrode for generating the same circle when the body centrode is a straight line, and the tracing point is at a distance b from A.

Generation of any Curve by Means of Identical Centrodes.

It is here understood that the body and space centrodes are to be congruent curves, and that the point of contact is always to be an identically corresponding point of the two curves.

When the position of a certain point (Q) in the plane of the given curve (R) is assumed, the problem of describing R by means of identical centrodes is determinate. In fact, either centrode is similar to the first negative pedal of R, the ratio of similarity being 1 : 2.*

This appears at once (Fig. 68) if we reflect that in the present case the space centrode C′ is the image of the body centrode C in the common tangent K. For if Q be the image of P in the tangent KM, then since P is a fixed point relative to C, Q will be a fixed point relative to C′. In other words, Q is a fixed point relative to R. Moreover, since $QM = \frac{1}{2}QP$, the locus of M is similar to R, and the locus of M is the pedal of C′, with reference to Q; or, what comes evidently to the same thing, the locus of P is the pedal of a curve similar to C′, the ratio of similarity being 2 : 1.

Hence, when the point Q is selected, the determination of the centrodes is complete.

We may, of course, regard C′ as the envelope of the perpendicular bisector of QP.

In many cases this is the simplest way of treating the problem.

We [illegible], however, use the p-r-equation of the given curve (R).

If this equation be

$$f(p, r) = 0, \qquad (1).$$

the p-r-equation to the locus of M is

$$f(2P, 2R) = 0.$$

Hence, if (p, r) be the p-r-co-ordinates of K, we have

$$P/R = p/r, \text{ and } R = P;$$

that is $P = p^2/r$, and $R = P$.

The p-r-equation of the centrode is therefore

$$f(2p^2/r, 2p) = 0 \qquad (2).$$

* This theorem seems to have been first pointed out by Steiner.

Generation of a straight line by means of identical centrodes.

The equation to R in this case is

$$p = 2a \qquad (1),$$

Where $2a$ is the distance of any chosen fixed point Q from the line.

The equation to either centrode is therefore

$$2p^2/r = 2a\,;$$

that is,

$$p^2 = ar \qquad (2).$$

This represents a parabola, of which Q is the focus.

Hence we conclude that a straight line may be traced by any parabola rolling on an equal parabola, the tracing point being the focus, a well-known result.

Generation of a circle by means of identical centrodes (Fig 69).

If c be the radius of the circle, and Q the chosen position of Q at a distance d from the centre O, then the p-r-equation to the circle is

$$p = (r^2 + c^2 - d^2)/2c \qquad (1).$$

Hence the p-r-equation to the centrode is

$$2p^2/r = (4p^2 + c^2 - d^2)/2c.$$

That is

$$\frac{1}{p^2} = \frac{4}{c^2 - d^2}\left(\frac{c}{r} - 1\right) \qquad (2).$$

Comparing this with the focal p-r-equation of an ellipse, viz.—

$$\frac{1}{p^2} = \frac{1}{b^2}\left(\frac{2a}{r} - 1\right),$$

we see that either centrode is an ellipse whose major axis is c, and whose minor axis is $\sqrt{c^2 - d^2}$.

It is easy to see that the major axis of the space centrode must be along QO ; and since we have $\sqrt{(a^2 - b^2)} = d/2$, we see that the centre of the ellipse bisects QO.

Generation of an ellipse by identical centrodes.

Take Q at the focus.

$$\frac{1}{p^2} = \frac{1}{b^2}\left(\frac{2a}{r} - 1\right)$$

$$\frac{r^2}{4p^4} = \frac{1}{b^2}\left(\frac{a}{p} - 1\right)$$

This is the polar reciprocal of a limaçon with respect to its double point.

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Generation of a parabola by identical centrodes.

Take Q at the focus.

$$p^2 = ar,$$

the

$$\frac{4p^4}{r^2} = 2ap$$

$$p^3 = ar^2.$$

This is the polar reciprocal of a cardioid with respect to its cusp.

Generation of the lemniscate by identical centrodes.

Take Q at the double point.

$$a^2p = r^3$$

$$a^2\frac{2p^2}{r} = 8p^3$$

$$\tfrac{1}{4}a^2 = pr$$

This is an equilateral hyperbola.

Generation of an equilateral hyperbola by identical centrodes.

Take Q at the centre.

$$pr = a^2$$

$$\frac{4p^3}{r} = a^2$$

$$p^3 = \tfrac{1}{4}a^2r.$$

This is the polar reciprocal of the lemniscate.

Trigonometrical Mnemonics.

By William Renton.