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LETTERS TO THE EDITOR

THE NUMBER OF CRITICAL CONNECTION VECTORS OF L-SUPERADDITIVE STRUCTURE FUNCTIONS

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Abstract

A conjecture due to Block et al. (1989), concerning the number of critical connection vectors to the various performance levels of a discrete L-superadditive structure function, is proved. When the components of the discrete L-superadditive structure function are further assumed to satisfy a certain relevance condition due to Griffith (1980), it is shown that there is exactly one critical connection vector to each performance level.

LEVELS OF PERFORMANCE; CARDINALITY

1. Introduction

The theory of discrete multistate structures describes situations in which a system as well as its components can perform at more than two levels of performance. A basic ingredient in this theory is a non-decreasing structure function $\phi:\{0, 1, \cdots, M\}^n \rightarrow \{0, 1, \cdots, M\}$ which relates the level of performance of the system to those of its components, where the $M+1$ levels of performance range from complete failure (0) to perfect functioning (M). The binary case is now a special case where $M=1$. Various classes of discrete multistate structures have been introduced and studied by several researchers (see El-Neweihi and Proschan (1984) for a survey).

Block et al. (1989) study the class of L-superadditive structures by imposing the following condition on the structure function ϕ:

$$\phi(\boldsymbol{x} \vee \boldsymbol{y}) + \phi(\boldsymbol{x} \wedge \boldsymbol{y}) \geqq \phi(\boldsymbol{x}) + \phi(\boldsymbol{y}) \text{ for all } \boldsymbol{x} \text{ and } \boldsymbol{y}, \tag{1.1}$$

where $\boldsymbol{x} \vee \boldsymbol{y}$ ($\boldsymbol{x} \wedge \boldsymbol{y}$) is the vector of componentwise maxima (minima).

Given an L-superadditive (LSP) structure function ϕ, its dual $\phi^{\mathrm{D}}(\boldsymbol{x}) = M - \phi(\boldsymbol{M} - \boldsymbol{x})$ is an L-subadditive (LSB) structure function for which the reverse inequality in (1.1) holds. Such functions have the interesting interpretation of describing whether a system is more series-like than parallel-like or vice versa (see Block et al. (1989)).

A vector $\boldsymbol{x}$ is said to be a critical connection vector to level $k>0$ for a multistate structure ϕ if $\phi(\boldsymbol{x})=k$ and $\boldsymbol{y}<\boldsymbol{x}$ implies $\phi(\boldsymbol{y})<k$, where $\boldsymbol{y}<\boldsymbol{x}$ means $y_i \leqq x_i$ for each i and strict inequality holds for at least one i. Critical connection vectors to the various performance levels of a multistate structure ϕ play a central role similar to the one played by minimal path vectors in the binary case. So naturally, Block et al. (1989) examine critical connection

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vectors to levels of performance of an LSP structure function. They conjecture an upper bound for the number of such vectors and prove it for some special cases.

In the following section we prove this conjecture. We also show that imposing some coherence conditions on an LSP structure can severely limit the number of its critical connection vectors to its various levels of performance.

2. The number of critical connection vectors

In the following main theorem of this paper we establish an upper bound on the number of critical connection vectors to the various performance levels of an LSP structure function. This upper bound is conjectured by Block et al. (1989).

Theorem 2.1. Let ϕ be an n-component discrete LSP structure function, then the number of critical connection vectors to level k is at most $\binom{M-k+n-1}{M-k}$, $k = 1, \cdots, M$.

Proof. First we prove the theorem when $n = 2$. Let $(x_1, y_1), \cdots, (x_r, y_r)$ be the critical connection vectors to level $k, 1 \leqq k \leqq M$. Without loss of generality we may assume $x_1 < \cdots < x_r$ and $y_1 > \cdots y_r$. By L-superadditivity we have

$$\phi(x_i, y_1) - \phi(x_{i-1}, y_1) \geqq \phi(x_i, y_i) - \phi(x_{i-1}, y_i) \geqq 1, \qquad 2 \leqq i \leqq r.$$

It now follows that $\phi(x_r, y_1) \geqq \phi(x_1, y_1) + r - 1$. Since the level of ϕ is at most M, we must have $r \leqq M - k + 1$.

Now suppose the theorem is true for n and all k, $1 \leqq k \leqq M$. Let ϕ be an $(n + 1)$-component discrete LSP structure function and let k be a performance level of ϕ, $1 \leqq k \leqq M$. Let $C_{k,\phi} = \boldsymbol{x} \in \{0, 1, \cdots, M\}^{n+1} : \boldsymbol{x}$ is a critical connection vector of ϕ to level $k\}$. Consider the projection map $f : \{0, 1, \cdots, M\}^{n+1} \to \{0, 1, \cdots, M\}^n$ defined by $f(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_n)$. Since two distinct critical connection vectors of ϕ to level k must differ in at least two coordinates, $C_{k,\phi}$ and its image $f(C_{k,\phi})$ have the same cardinality.

Now let $a = \max \{(\boldsymbol{x})_{n+1} : \boldsymbol{x} \in C_{k,\phi}\}$, where $(\boldsymbol{x})_{n+1}$ denotes the $(n + 1)$th coordinate of the vector $\boldsymbol{x}$. Let ψ be an n-component discrete structure function defined by $\psi(\boldsymbol{x}) = \phi(\boldsymbol{x}; a)$ for all $\boldsymbol{x} \in \{0, 1, \cdots, M\}^n$. It is easy to see that ψ is an LSP structure function. Let $C_{t,\psi} = \{\boldsymbol{x} \in \{0, 1, \cdots, M\}^n : \boldsymbol{x}$ is a critical connection vector of ψ to level $t\}$, $1 \leqq t \leqq M$.

We are now ready to show that $f(C_{k,\phi}) \subset \bigcup_{t=k}^{M} C_{t,\psi}$. To see this let $\boldsymbol{x} \in f(C_{k,\phi})$ and $\boldsymbol{y} < \boldsymbol{x}$. There exists $x_{n+1} \leqq a$ such that $(\boldsymbol{x}; x_{n+1}) \in C_{k,\phi}$. Now $\psi(\boldsymbol{x}) = \phi(\boldsymbol{x}; a) \geqq \phi(\boldsymbol{x}; x_{n+1}) = k$. Moreover, we have by L-superadditivity,

$$\psi(\boldsymbol{x}) - \psi(\boldsymbol{y}) = \phi(\boldsymbol{x}; a) - \phi(\boldsymbol{y}; a) \geqq \phi(\boldsymbol{x}; x_{n+1}) - \phi(\boldsymbol{y}; x_{n+1}) \geqq 1.$$

Now by the induction hypothesis, the number of elements in $C_{t,\psi}$ is at most $\binom{M-t+n-1}{M-t}$. Therefore the number of elements in $f(C_{k,\phi})$ is at most

$$\sum_{t=k}^{M} \binom{M-t+n-1}{M-t} = \binom{M-k+n}{M-k}.$$

Remark 2.2. Using a standard duality argument a similar result can be formulated for LSB structure functions.

Usually relevance and other assumptions are imposing on discrete structure functions. Such assumptions generalize familiar ones in the binary case. However, no such conditions have been assumed for LSP (LSB) structure functions by Block et al. (1989). The following, unexpected, result shows that the upper bound in Theorem 2.1 (which is sharp in general) is quite crude when such additional relevance and other conditions are imposed on LSP structure functions.

Theorem 2.3. Let $\phi:\{0, 1, \cdots, M\}^n \to \{0, 1, \cdots, M\}$ be a discrete LSP structure function. Assume further that ϕ satisfies the following conditions: (i) $\phi(k, k, \cdots, k) \geqq k$, (ii) for every component i and level $j \geqq 1$, there exists $(\cdot i, \boldsymbol{x})$ such that $\phi(j_i, \boldsymbol{x}) > \phi((j-1)_i, \boldsymbol{x})$, where $(l_i, \boldsymbol{x})$ denotes a vector whose ith component is l, $0 \leqq l < M$. Then there is exactly one critical connection vector to each performance level of ϕ.

Proof. By L-superadditivity and condition (ii) above we have $\phi(j_i, \boldsymbol{M}) - \phi((j-1)_i\boldsymbol{M}) \geqq 1$, for $1 \leqq j \leqq M$, and $1 \leqq i \leqq n$, where $(l_i, \boldsymbol{M})$ is the vector whose ith component is l and all the others are equal to M. Hence $\phi(0_i, \boldsymbol{M}) < \phi(1_i, \boldsymbol{M}) < \cdots < \phi(M_i, \boldsymbol{M})$, and we must have $\phi(j_i, \boldsymbol{M}) = j$ for all $1 \leqq i \leqq n$ and all $0 \leqq j \leqq M$. This together with condition (i) above implies that ϕ is the series structure, i.e., $\phi(\boldsymbol{x}) = \min_{1 \leqq i \leqq n} x_i$. Therefore the vector $(k, k, \cdots, k)$ is the only critical connection vector to level k, $1 \leqq k \leqq M$.

Remark 2.4. Condition (i) in Theorem 2.3 is equivalent to requiring that $\phi(\boldsymbol{x}) \geqq \min_{1 \leqq i \leqq n} x_i$ which generalizes the familiar fact in the binary case (series system is the weakest coherent system). Condition (ii) of Theorem 2.3 is a mild relevance condition due to Griffith (1980).

Remark 2.5. In view of the proof of Theorem 2.3, conditions (i) and (ii) of that theorem characterize the series structure among the discrete LSP structures. Using a standard duality argument a similar result can be proved for LSB structures. Namely, condition (i)′ $\phi(\boldsymbol{x}) \leqq \max_{1 \leqq i \leqq n} x_i$ and condition (ii) of Theorem 2.3 characterize the parallel structures among LSB structure functions.

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