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HOLOMORPHIC FUNCTIONS WITH POSITIVE REAL PART

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The main purpose of this note is to prove a special case of the following conjecture.

Conjecture. If F is holomorphic on the unit ball B_n in $\mathbf{C}^n$ and has positive real part, then F is in $H^p(B_n)$ for $0 < p < \frac{1}{2}(n + 1)$.

Here $H^p(B_n)$ $(0 < p < \infty)$ denote the usual Hardy spaces of holomorphic functions on B_n. See below for definitions. We remark that the conjecture is known for $0 < p < 1$ and that some evidence for it already exists in the literature; for example [**1**, Theorems 3.11 and 3.15] where it is shown that a particular extreme element of the convex cone of functions

$$\{F \text{ holomorphic on } B_2;\ \operatorname{Re} F > 0,\ F(0) = 1\}$$

is in $H^p(B_2)$ for $0 < p < 3/2$. The theorem below (which is stated for domains more general than balls) shows that the conjecture is true at least for functions $F = (1 + f)/(1 - f)$ where f is suitably "nice" on $\bar{B}_n$. Recall that the map $f \to (1 + f)/(1 - f)$ is a bijection from holomorphic functions of modulus less than one to holomorphic functions with positive real part. We now introduce some definitions and notation.

Let Ω be a bounded domain in $\mathbf{C}^n$. Denote by $H(\Omega)$ the collection of complex-valued functions holomorphic on Ω. If there exists an open set $W \supset \partial\Omega$ and a continuously differentiable function $\tau : W \to \mathbf{R}$ satisfying (i) the gradient of τ does not vanish on $\partial\Omega$ and (ii) $\Omega \cap W = \{w \in W; \tau(w) < 0\}$, then τ is said to be a *characterizing function* for Ω. If τ is in $C^k(W)$, i.e., is k times continuously differentiable on W, then Ω is said to have C^k *boundary*. Suppose that τ is a C^2 characterizing function for Ω. Let

$$H^p(\Omega) = \left\{ f \in H(\Omega);\ \sup_{\gamma > 0} \int_{\partial\Omega_\gamma} |f(z)|^p d\sigma_\gamma(z) < \infty \right\}$$

where $\Omega_\gamma = \{z \in \Omega; \tau(z) < -\gamma\}$ and σ_γ denotes the surface measure on $\partial\Omega_\gamma$ induced by Lebesgue measure on $\mathbf{C}^n$. The class of functions $H^p(\Omega)$ is independent of the particular characterizing function used (see [**3**];

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Section 3 of Chapter I). Let

$$P_z = \left\{ w \in \mathbf{C}^n; \sum_{k=1}^{n} w_k \frac{\partial \tau}{\partial z_k}(z) = 0 \right\}$$

and denote by $H_\tau(z)$ the hermitian form on P_z defined by

$$H_\tau(z)(u, v) = \sum_{\mu,\nu} \frac{\partial^2 \tau}{\partial z_\mu \partial \bar{z}_\nu}(z) u_\mu \bar{v}_\nu \quad u, v \text{ in } P_z$$

Thus $H_\tau(z)$ is the restriction of the Hessian of τ to the complex tangent space of $\partial\Omega$ at z. We say that Ω is *strictly pseudoconvex* if $H_\tau(z)$ is positive definite for each z in $\partial\Omega$. Denote by $R(z)$ the rank (over the complex field) of $H_\tau(z)$ and set

$$R_\Omega = \min\{R(z); z \in \partial\Omega\}.$$

The above definitions, save for $H_\tau(z)$, are independent of the characterizing function τ (see [**2**]; the proof of Theorem 2.6.12). Finally, if f is k times continuously differentiable on Ω, we say that f is in $C^k(\overline{\Omega})$ if f together with all of its partial derivatives of order at most k admit continuous extensions to $\overline{\Omega}$.

THEOREM. *Let Ω be a bounded domain in $\mathbf{C}^n$ with C^3 boundary. Suppose that f is in $H(\Omega) \cap C^3(\overline{\Omega})$ and that $|f| < 1$ on Ω. Then* $(1 + f)/(1 - f)$ *is in $H^p(\Omega)$ for $0 < p < 1 + R_\Omega/2$.*

COROLLARY. *Suppose that $\Omega \subset C^n$ is strictly pseudoconvex with C^3 boundary (in particular Ω could be B_n). If f is in $H(\Omega) \cap C^3(\overline{\Omega})$ and $|f| < 1$ on Ω, then $(1 + f)/(1 - f)$ is in $H^p(\Omega)$ for $0 < p < (n + 1)/2$.*

The theorem will be proved by means of the following lemma.

LEMMA. *Suppose that F and G are in $C^3(\omega)$ where ω is an open neighbourhood of* 0 *(the origin) in $\mathbf{R}^n$ and suppose that $F(0) = G(0) = 0$, $F \geqq 0$, and*

$$\nabla G(0) = \left(\frac{\partial G}{\partial x_1}(0), \ldots \frac{\partial G}{\partial x_n}(0) \right) \neq 0.$$

Let r be the rank of the quadratic form

$$\sum_{i,j} \frac{\partial^2 F}{\partial x_i \partial x_j}(0) x_i x_j$$

restricted to the subspace $\{x \in \mathbf{R}^n; \langle x, \nabla G(0) \rangle = 0\}$.

Then $(F^2 + G^2)^{-p/2}$ $(0 < p < \infty)$ is integrable in some neighbourhood of the origin if and only if $p < 1 + r/2$.

The proof of the relevant half of the lemma is given at the end of the paper.

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Proof of theorem. Fix p such that $0 < p < 1 + R_\Omega/2$ and fix τ, a C^3 characterizing function for Ω. We claim that the conclusion of the theorem will follow once we have shown that $\int_{\partial\Omega} |1 - f(z)|^{-p}\, d\sigma(z) < \infty$. Indeed, it is easy to see that there exists $\delta > 0$ such that for each $0 < \gamma \leqq \delta$ and z in $\partial\Omega$ there is a unique z_γ in $\partial\Omega_\gamma$ satisfying

(a) The vector $z - z_\gamma$ is perpendicular to the tangent space of $\partial\Omega$ at z.

(b) The open ball with centre z_γ and radius $|z - z_\gamma|$ is contained in Ω.

If $0 < \gamma < \delta$, then using the pluriharmonicity of $1 - \operatorname{Re} f$ together with the Poisson kernel inequality $P_r(\theta) \geqq (1 - r)/2$, we have

$$(1)\qquad 1 - \operatorname{Re} f(z_\gamma) = \frac{1}{2\pi}\int_0^{2\pi} [1 - \operatorname{Re} f(z_\delta + e^{i\theta}(z - z_\delta))]P_r(\theta)d\theta$$

$$\left(r = \frac{|z_\gamma - z_\delta|}{|z - z_\delta|}\right)\qquad \geqq \frac{1 - r}{2}(1 - \operatorname{Re} f(z_\delta))$$

$$= \frac{|z - z_\gamma|}{2|z - z_\delta|}(1 - \operatorname{Re} f(z_\delta)) \geqq \lambda |z - z_\gamma|$$

where

$$\lambda = \inf\left\{\frac{1 - \operatorname{Re} f(z)}{2d(z, \partial\Omega)};\quad z \in \partial\Omega_\delta\right\} > 0.$$

If $A = \sup\{|\nabla f(z)|; z \in \Omega\}$, then by (1)

$$\left|\frac{1 - f(z)}{1 - f(z_\gamma)}\right| \leqq 1 + \left|\frac{f(z) - f(z_\gamma)}{1 - f(z_\gamma)}\right| \leqq 1 + A/\lambda$$

and hence

$$(2)\qquad |1 - f(z_\gamma)|^{-p} \leqq (1 + A/\lambda)^p |1 - f(z)|^{-p} \quad z \text{ in } \partial\Omega,\ 0 < \gamma < \delta.$$

The claim made at the beginning of the proof now follows from (2) and the fact that for small γ, the map $z \to z_\gamma$ is "close" to being an isomorphism of the measure spaces $(\partial\Omega, \sigma)$ and $(\partial\Omega_\gamma, \sigma_\gamma)$.

Fix z in $\partial\Omega$. We shall now use the lemma to show that $|1 - f|^{-p}$ is σ-integrable in some $\partial\Omega$-neighbourhood of z. This, together with the compactness of $\partial\Omega$ and the claim just established, will complete the proof of the theorem. Without loss of generality we suppose $z = e = (1, 0, \ldots, 0)$, $\nabla\tau(e) = e$, and $f(e) = 1$. Set $z_k = x_{2k-1} + ix_{2k}$ for $1 \leqq k \leqq n$. For z in an appropriate (small) neighbourhood N of e, define $\Sigma: N \to N$ by

$$\Sigma(z) = (\beta(x_2, z_2, \ldots z_n) + ix_2, z_2, \ldots z_n)$$

where the function β (defined on the tangent space of $\partial\Omega$ at e) is chosen so that $\Sigma(z)$ is in $\partial\Omega$ for z in N. Set $u(z) = 1 - \operatorname{Re} f(z)$ and $v(z) = \operatorname{Im} f(z)$. Let $F = u \circ \Sigma$ and $G = v \circ \Sigma$. If we can show that $(F^2 + G^2)^{-p/2} = |1 - f \circ \Sigma|^{-p}$ is Lebesgue integrable in some $\mathbf{C}^n$-neighbourhood of e,

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then it will follow that $|1 - f|^{-p}$ is σ-integrable in some $\partial\Omega$-neighbourhood of e and we will be done.

Clearly the function β is C^3. Applying the chain rule to the equation $\tau \circ \Sigma \equiv 0$ and recalling that $\partial\tau/\partial x_1(e) = 1$ we obtain

$$(3) \qquad \frac{\partial \beta}{\partial \bar{z}_\nu}(e) = -\frac{\partial \tau}{\partial \bar{z}_\nu}(e) = 0 \quad 2 \leqq \nu \leqq n$$

$$(4) \qquad \frac{\partial^2 \beta}{\partial z_\mu \partial \bar{z}_\nu}(e) = -\frac{\partial^2 \tau}{\partial z_\mu \partial \bar{z}_\nu}(e) \quad 2 \leqq \mu, \nu \leqq n.$$

Let $a = -\partial u/\partial x_1(e)$. Setting $z = e$ in (1) we see that

$$a = -\frac{\partial u}{\partial x_1}(e) = \lim_{t \to 1} \frac{u(te) - u(e)}{1 - t} = \lim_{t \to 1} \frac{1 - \operatorname{Re} f(te)}{1 - t} \geqq \lambda > 0.$$

A simple calculation yields

$$(5) \qquad \frac{\partial G}{\partial \bar{z}_1}(e) = \frac{i}{2}\frac{\partial G}{\partial x_2}(e) = \frac{i}{2}\frac{\partial v}{\partial x_2}(e) = -\frac{i}{2}\frac{\partial u}{\partial x_1}(e) = \frac{ia}{2} \neq 0$$

$$\frac{\partial G}{\partial \bar{z}_\nu}(e) = \frac{\partial v}{\partial \bar{z}_\nu}(e) = -i\frac{\partial u}{\partial \bar{z}_\nu}(e) = -i\frac{\partial F}{\partial \bar{z}_\nu}(e) = 0 \quad 2 \leqq \nu \leqq n$$

since F achieves a relative minimum at e. Applying the chain rule to F and using (3) and (5) yields

$$\frac{\partial^2 F}{\partial z_\mu \partial \bar{z}_\nu}(e) = -a\frac{\partial^2 \beta}{\partial z_\mu \partial \bar{z}_\nu}(e) + \frac{\partial^2 u}{\partial z_\mu \partial \bar{z}_\nu}(e) \quad 2 \leqq \mu, \nu \leqq n$$

and now using (4) and the pluriharmonicity of u we obtain

$$(6) \qquad \frac{\partial^2 F}{\partial z_\mu \partial \bar{z}_\nu}(e) = a\frac{\partial^2 \tau}{\partial z_\mu \partial \bar{z}_\nu}(e) \quad 2 \leqq \mu, \nu \leqq n.$$

Equation (6) shows that up to multiplication by a positive constant, the restriction of the Hessians of F and τ to the complex tangent space of $\partial\Omega$ at e are identical. This is the main step of the proof.

Now let D denote the complex $(n - 1) \times (n - 1)$ matrix

$$\left(\frac{\partial^2 F}{\partial z_\mu \partial \bar{z}_\nu}(e)\right) \quad 2 \leqq \mu, \nu \leqq n$$

and let M denote the real $(2n - 2) \times (2n - 2)$ matrix

$$\left(\frac{\partial^2 F}{\partial x_i \partial x_j}(e)\right) \quad 3 \leqq i, j \leqq 2n.$$

For any $x = (x_3, \ldots x_{2n})$ in $\mathbf{R}^{2n-2}$ let $\tilde{x} = (x_3 + ix_4, \ldots, x_{2n-1} + ix_{2n})$. Clearly the map $x \to \tilde{x}$ is a (real) linear isomorphism between $\mathbf{R}^{2n-2}$ and $\mathbf{C}^{n-1}$. If D_k (respectively M_k) denotes the kth row of D (respectively M), then $4D_k = \widetilde{M}_{2k-1} - i\widetilde{M}_{2k}$. Thus the rank of the real matrix M is at least as large as the rank (over $\mathbf{C}$) of the complex matrix D which by (6) and our hypothesis is at least R_Ω. Taking into account (5), this shows that the rank of the quadratic form $\sum_{i,j} \partial^2 F/\partial x_i \partial x_j(e) x_i x_j$ restricted to the subspace $\{x \in \mathbf{R}^{2n}; \langle x, \nabla G(e)\rangle = 0\}$ is at least R_Ω. The lemma now shows that $(F^2 + G^2)^{-p/2}$ is Lebesgue integrable in some neighbourhood of e and by remarks made earlier, this completes the proof of the theorem.

Remark. Let $h_n(z) = \sum_{k=1}^n (z_k)^2$ for z in B_n. Clearly h_n is in $H(B_n) \cap C^3(\overline{B_n})$ and $|h_n| < 1$ on B_n. A simple calculation shows that $(1 + h_n)/(1 - h_n)$ is not in $H^{(n+1)/2}(B_n)$ and thus the range of p given in the conclusion of the above theorem cannot in general be extended.

Proof of Lemma. We shall only prove the "if" half of the lemma. Clearly the hypotheses and conclusion of the lemma are invariant under non-singular linear changes of variable. Thus we may assume that the matrix $(\partial^2 F/\partial x_i \partial x_j(0))_{i,j}$ is diagonal and, since F achieves a relative minimum at 0, that the entries are either 0 or 1. Furthermore, by renumbering the co-ordinate functions we may assume that

$$\text{(i)}\ \frac{\partial^2 F}{\partial x_i \partial x_j}(0) = \begin{cases} 1 & 1 \leqq i = j \leqq r \\ 0 & i \neq j \end{cases}$$

and

$$\text{(ii)}\ \frac{\partial G}{\partial x_n}(0) \neq 0.$$

By the implicit function theorem, the equations $\nabla F(0) = 0$ and (i) show that there are neighbourhoods U and V of the origins in $\mathbf{R}^r$ and $\mathbf{R}^{n-r}$ respectively $(\overline{U \times V} \subset \omega)$ and a function $\alpha: V \to U$ such that

$$\text{(7)} \qquad \frac{\partial F}{\partial x_i}(\alpha(w), w) = 0 \quad w \text{ in } V,\ 1 \leqq i \leqq r.$$

Similarly $G(0) = 0$ and (ii) imply that there are neighbourhoods M and N of the origins in $\mathbf{R}^{n-1}$ and $\mathbf{R}$ respectively $(\overline{M \times N} \subset \omega)$ and a function $\beta: M \to N$ such that $G(w, \beta(w)) = 0$ for w in M. For $x = (x_1, \ldots, x_n)$ in $\mathbf{R}^n$, set $x' = (x_{r+1}, \ldots x_n)$ and $x'' = (x_1, \ldots x_{n-1})$. By Taylor's formula with x in $(U \times V) \cap (M \times N)$ we thus have

$$\text{(8)} \qquad F(x) = F(\alpha(x'), x') + \tfrac{1}{2} \sum_{1 \leqq i,j \leqq r} \frac{\partial^2 F}{\partial x_i \partial x_j}(\alpha(x'), x')(x_i - \alpha_i(x'))(x_j - \alpha_j(x')) + \delta_1(x)$$

$$G(x) = \frac{\partial G}{\partial x_n}(x'', \beta(x''))(x_n - \beta(x'')) + \delta_2(x)$$

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where $\alpha = (\alpha_1, \dots \alpha_r)$ and

$$\sup \left\{ \frac{|\delta_1(x)|}{\left(\sum_{i=1}^{r} |x_i - \alpha_i(x')|^2\right)^{3/2}}, \frac{|\delta_2(x)|}{|x_n - \beta(x'')|^2};\right.$$

$$\left. x \in (U \times V) \cap (M \times N) \right\} < \infty.$$

By further shrinking the neighbourhoods U, V, M, and N (if necessary) and using $F \geqq 0$ together with (i), (ii), (8), and continuity, we can obtain

$$(9) \qquad F(x) \geqq \tfrac{1}{4} \sum_{i=1}^{r} (x_i - \alpha_i(x'))^2$$

$$|G(x)| \geqq \tfrac{1}{4} a |x_n - \beta(x'')|, \quad a = \left| \frac{\partial G}{\partial x_n}(0) \right| > 0$$

for x in $(U \times V) \cap (M \times N)$.

Now make the change of variables $y = Tx$ defined by

$$(10) \qquad y_i = \begin{cases} x_i - \alpha_i(x') & 1 \leqq i \leqq r \\ x_i & r < i < n \\ x_n - \beta(x'') & i = n. \end{cases}$$

Applying $\partial/\partial x_j$ to (7) and using (i) shows that

$$\frac{\partial \alpha_i}{\partial x_j}(0) = 0 \quad \text{for } 1 \leqq i \leqq r < j \leqq n$$

and hence that

$$\frac{\partial y_i}{\partial x_j}(0) = \begin{cases} 1 & i = j \\ 0 & i < j \end{cases}.$$

Thus $\det J(0) = 1$ where $J(x)$ denotes the Jacobian matrix of T at x. Let P be a neighbourhood of the origin in $\mathbf{R}^n$ such that $\bar{P} \subset (U \times V) \cap (M \times N)$ and $\det J(x) > \frac{1}{2}$ for x in P. Then if m denotes Lebesque measure on $\mathbf{R}^n$, we have from (9) and (10) that

$$\int_P (F(x)^2 + G(x)^2)^{-p/2} dm(x)$$

$$= \int_{T(P)} (F(T^{-1}y)^2 + G(T^{-1}y)^2)^{-p/2} |\det J(T^{-1}y)|^{-1} dm(y)$$

$$\leqq 2^{2p+1} \int_{T(P)} \left[\left(\sum_{i=1}^{r} y_i^2 \right)^2 + a^2 y_n^2 \right]^{-p/2} dm(y)$$

and this last integral is easily seen to be finite for $a > 0$ and $p < 1 + r/2$.

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