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Generalized Jordan Semiderivations in Prime Rings

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Abstract. Let R be a ring and let g be an endomorphism of R. The additive mapping $d: R \to R$ is called a Jordan semiderivation of R, associated with g, if

$$d(x^2) = d(x)x + g(x)d(x) = d(x)g(x) + xd(x) \quad \text{and} \quad d(g(x)) = g(d(x))$$

for all $x \in R$. The additive mapping $F: R \to R$ is called a generalized Jordan semiderivation of R, related to the Jordan semiderivation d and endomorphism g, if

$$F(x^2) = F(x)x + g(x)d(x) = F(x)g(x) + xd(x) \quad \text{and} \quad F(g(x)) = g(F(x))$$

for all $x \in R$. In this paper we prove that if R is a prime ring of characteristic different from 2, g an endomorphism of R, d a Jordan semiderivation associated with g, F a generalized Jordan semiderivation associated with d and g, then F is a generalized semiderivation of R and d is a semiderivation of R. Moreover, if R is commutative, then $F = d$.

1 Introduction

Throughout this paper R will be an associative prime ring of characteristic different from 2, and $Z(R)$ will denote the center of R. We will write $[x, y]$ for $xy - yx$. An additive mapping $d: R \to R$ is called a *derivation* of R, if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in R$. The additive mapping d on R is called a Jordan derivation if $d(x^2) = d(x)x + xd(x)$, for all $x \in R$. Obviously, any derivation is a Jordan derivation; the converse is not true in general. A well-known result of Herstein states that every Jordan derivation on a prime ring of characteristic different from 2 is a derivation [4]. Later, Bresar [2] gives a generalization of Herstein's result. More precisely, he proves that every Jordan derivation on a 2-torsion free semiprime ring is a derivation.

Moreover, the reader can find similar results in literature regarding other types of additive mappings. For instance, an additive map $F: R \to R$ is called a generalized derivation if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. The additive map F is called a generalized Jordan derivation if there exists a Jordan derivation d of R such that $F(x^2) = F(x)x + xd(x)$ for all $x \in R$. Of course any generalized derivation is a generalized Jordan derivation . In [5] Jing and Liu prove that any generalized Jordan derivation on a prime ring of characteristic different from 2 is a generalized derivation (Theorem 2.5).

In this paper we will extend previous results to a class of additive mappings whose concept covers the ones of derivations and generalized derivations. We first recall that in [1] Bergen introduces the following definition.

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Definition 1.1 Let g be an endomorphism of R. An additive mapping d of R into itself is called a *semiderivation* (associated with g) if, for all $x, y \in R$,

$$d(xy) = d(x)y + g(x)d(y) = d(x)g(y) + xd(y) \quad \text{and} \quad d(g(x)) = g(d(x)).$$

In [3] we introduced generalized semiderivations, defined as follows.

Definition 1.2 Let d be a semiderivation of R associated with endomorphism g. The additive map F on R is a generalized semiderivation of R if, for all $x, y \in R$,

$$F(xy) = F(x)y + g(x)d(y) = F(x)g(y) + xd(y) \quad \text{and} \quad F(g(x)) = g(F(x)).$$

Motivated by the concepts of Jordan derivations and generalized Jordan derivations, we initiate the concepts of Jordan semiderivations and generalized Jordan semiderivation as follows.

Definition 1.3 Let R be a ring, and let g be an endomorphism of R. The additive mapping $d: R \to R$ is called a *Jordan semiderivation* of R associated with g if, for $x \in R$,

$$d(x^2) = d(x)x + g(x)d(x) = d(x)g(x) + xd(x) \quad \text{and} \quad d(g(x)) = g(d(x)).$$

Definition 1.4 Let R be a ring, let g be an endomorphism of R, and let d be a Jordan semiderivation of R associated with g. The additive mapping $F: R \to R$ is called a *generalized Jordan semiderivation* of R associated with d and g if, for $x \in R$,

$$F(x^2) = F(x)x + g(x)d(x) = F(x)g(x) + xd(x) \quad \text{and} \quad F(g(x)) = g(F(x)).$$

In this paper we prove the following theorem following the line of investigation of previous cited results.

Theorem *Let R be a prime ring of characteristic different from 2, let g be an endomorphism of R, let d be a Jordan semiderivation associated with g, and let F be a generalized Jordan semiderivation associated with d and g. Then F is a generalized semiderivation of R and d is a semiderivation of R. Moreover, if R is commutative, then $F = d$.*

2 Proof of Theorem

In all that follows we will assume R has characteristic different from 2.

Remark 2.1 In order to prove our result we must show the following

$$F(xy) = F(x)y + g(x)d(y), \quad \forall x, y \in R, \tag{2.1}$$

$$F(xy) = F(x)g(y) + xd(y), \quad \forall x, y \in R. \tag{2.2}$$

Notice that proofs of (2.1) and (2.2) are analogous to each other. Thus, without loss of generality, we will show only that (2.1) holds.

Remark 2.2 We notice that if g is the identity map on R, then F is a Jordan generalized derivation. In this case, by [5, Theorem 2.5], F is an ordinary generalized derivation of R, and a fortiori F is a generalized semiderivation of R.

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Lemma 2.3 $(F(x)y + g(x)d(y) - F(xy))[x, y] = 0$ *for all* $x, y \in R$.

Proof Let $x, y \in R$; then by the definition of F we have

$$F\big((x+y)^2\big) = F(x+y)(x+y) + g(x+y)d(x+y) = F(x^2) + F(y^2) + F(x)y + g(x)d(y) + F(y)x + g(y)d(x). \tag{2.3}$$

On the other hand,

$$F\big((x+y)^2\big) = F(x^2) + F(y^2) + F(xy+yx). \tag{2.4}$$

Equations (2.3) and (2.4) imply

$$F(xy+yx) = F(x)y + g(x)d(y) + F(y)x + g(y)d(x). \tag{2.5}$$

If we replace y with $xy + yx$ in (2.5), we have

$$G(x,y) = F\big(x(xy+yx) + (xy+yx)x\big) = F(x)(xy+yx) + g(x)d(xy+yx) + F(xy+yx)x + g(xy+yx)d(x)$$

and using (2.5),

$$G(x,y) = F(x)(xy+yx) + g(x)d(x)y + g(x)g(x)d(y) + g(x)d(y)x + g(x)g(y)d(x) + F(x)yx + g(x)d(y)x + F(y)x^2 + g(y)d(x)x + g(xy+yx)d(x). \tag{2.6}$$

Moreover, we can also write

$$G(x,y) = F\big(x^2y + yx^2\big) + 2F(xyx),$$

and again using (2.5),

$$G(x,y) = F(x)xy + g(x)d(x)y + g(x)^2d(y) + F(y)x^2 + g(y)d(x)x + g(y)g(x)d(x) + 2F(xyx). \tag{2.7}$$

Comparing (2.6) with (2.7) and since $char(R) \neq 2$, it follows that

$$F(xyx) = F(x)yx + g(x)d(y)x + g(x)g(y)d(x). \tag{2.8}$$

Now replace x with $x + z$ in (2.8), for any $z \in R$, so that

$$F(xyz+zyx) = F(x)yz + g(x)d(y)z + g(x)g(y)d(z) + F(z)yx + g(z)d(y)x + g(z)g(y)d(x). \tag{2.9}$$

In particular, for $z = xy$,

$$H(x,y) = F\big((xy)(xy) + (xy)(yx)\big),$$

and using (2.9) we get

$$H(x,y) = F(x)yxy + g(x)d(y)xy + g(x)g(y)d(xy) + F(xy)yx + g(xy)d(y)x + g(xy)g(y)d(x). \tag{2.10}$$

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On the other hand

$$H(x,y) = F\big((xy)^2\big) + F(xy^2x) = F(xy)xy + g(xy)d(xy) + F(x)y^2x + g(x)d(y)yx + g(x)g(y)d(y)x + g(x)g(y^2)d(x). \tag{2.11}$$

Comparing (2.10) with (2.11), one has

$$\big(F(x)y + g(x)d(y) - F(xy)\big)(xy - yx) = 0. \tag{2.12}$$

■

Lemma 2.4 *Assume that R is not commutative and let $x, y \in R$ be such that $[x,y] = 0$. Then $F(xy) = F(x)y + g(x)d(y)$.*

Proof We start from (2.12) and replace x with $x + z$, for any $z \in R$; then

$$\big(F(x)y+g(x)d(y)-F(xy)\big)[z,y]+\big(F(z)y+g(z)d(y)-F(zy)\big)[x,y] = 0. \tag{2.13}$$

Analogously, replacing y with $y + z$ in (2.12), it follows that

$$\big(F(x)y+g(x)d(y)-F(xy)\big)[x,z]+\big(F(x)z+g(x)d(z)-F(xz)\big)[x,y] = 0 \tag{2.14}$$

for any $x, y, z \in R$. Now let x, y be such that $[x,y] = 0$; therefore, by (2.13) we have

$$\big(F(x)y + g(x)d(y) - F(xy)\big)[z,y] = 0, \quad \forall z \in R.$$

The primeness of R implies easily that if $y \notin Z(R)$, then $F(x)y + g(x)d(y) - F(xy) = 0$, as required by the conclusion Lemma 2.4.

Similarly, by (2.14) and $[x,y] = 0$, one has

$$\big(F(x)y + g(x)d(y) - F(xy)\big)[x,z] = 0, \quad \forall z \in R,$$

and if $x \notin Z(R)$, then $F(x)y + g(x)d(y) - F(xy) = 0$ follows again.

Thus, we consider the case both $x \in Z(R)$ and $y \in Z(R)$. Since R is not commutative, there exists $r \in R$ such that $r \notin Z(R)$. Hence $x + r \notin Z(R)$ and $[y, x+r] = [y,r] = 0$. By the previous argument, we have that

$$F(x+r)y + g(x+r)d(y) - F((x+r)y) = 0$$

and

$$F(r)y + g(r)d(y) - F(ry) = 0,$$

implying that $F(x)y + g(x)d(y) - F(xy) = 0$. Therefore, in any case

$$[x,y] = 0 \Longrightarrow F(xy) = F(x)y + g(x)d(y).$$

■

Lemma 2.5 *Assume that R is a non-commutative domain. Then $F(xy) = F(x)y + g(x)d(y)$ for all $x, y \in R$.*

Proof By Lemma 2.3, we have that $(F(x)y + g(x)d(y) - F(xy))[x,y] = 0$ for all $x, y \in R$. Since R is a domain, for all $x, y \in R$, either $F(xy) = F(x)y + g(x)d(y)$ or $[x,y] = 0$. But in this last case, $F(xy) = F(x)y + g(x)d(y)$ follows from Lemma 2.4, and we are done. ■

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Convention 2.6 In all that follows, if R is not commutative, then we always assume that R is not a domain.

Remark 2.7 Assume that d is a Jordan semiderivation of R. Then $d(xyx) = d(x)yx + g(x)d(y)x + g(x)g(y)d(x)$ for all $x, y \in R$.

Proof This follows by (2.8), with $F = d$. ∎

Lemma 2.8 *Assume that R is not commutative and let $x, y \in R$ be such that $xy = 0$. Then $0 = F(xy) = F(x)y + g(x)d(y)$.*

Proof In the case where $yx = 0$, $[x, y] = 0$, and we conclude by Lemma 2.4. Let $yx \neq 0$. Right multiplying (2.14) by y, since $xy = 0$, we have

$$\big(F(x)y + g(x)d(y)\big)xzy = 0 \quad \forall z \in R,$$

and by the primeness of R we have

$$\big(F(x)y + g(x)d(y)\big)x = 0.$$

Replace y with yry, for any $r \in R$, so that

$$\big(F(x)yry + g(x)d(yry)\big)x = 0,$$

and by Remark 2.7 we have

$$\big(F(x)y + g(x)d(y)\big)ryx = 0 \quad \forall r \in R.$$

Once again by the primeness of R we get $F(x)y + g(x)d(y) = 0 = F(xy)$. ∎

Corollary 2.9 *Assume that R is not commutative and let $x, y \in R$ be such that $xy = 0$. Then $F(yx) = F(y)x + g(y)d(x)$.*

Proof By Lemma 2.8, $F(xy) = F(x)y + g(x)d(y) = 0$. On the other hand, by using equation (2.5),

$$F(yx) = F(xy + yx) = F(y)x + g(y)d(x). \qquad ∎$$

Remark 2.10 Assume that R is not commutative, let d be a Jordan semiderivation of R, and let $x, y \in R$ be such that $xy = 0$. Then $0 = d(xy) = d(y)x + g(y)d(x)$.

Proof This follows by Lemma 2.8, with $F = d$. ∎

Lemma 2.11 *Assume R is not commutative and let $x, y \in R$ be such that $xy = 0$. Then $F(yxr) = F(yx)r + g(yx)d(r)$, for all $r \in R$.*

Proof By using equation (2.9), for $xy = 0$ and for all $r \in R$,

$$\begin{aligned} F(rxy + yxr) = F(yxr) = g(r)d(x)y + g(r)g(x)d(y) \\ + F(y)xr + g(y)d(x)r + g(y)g(x)d(r), \end{aligned}$$

and by Corollary 2.9

$$F(yxr) = g(r)\big(d(x)y + g(x)d(y)\big) + g(y)g(x)d(r) + F(yx)r.$$

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Hence, applying Remark 2.10, $d(x)y + g(x)d(y) = 0$, and we conclude that

$$F(yxr) = g(y)g(x)d(r) + F(yx)r. \qquad \blacksquare$$

Remark 2.12 Define the following subset of R:

$$S = \{a \in R : F(ax) = F(a)x + g(a)d(x), \quad \forall x \in R\}.$$

We remark that by Lemma 2.6 one has that $ab = 0$, which implies $ba \in S$.

Here we fix an element $b \in R$, and introduce the following map $\phi_b\colon R \to R$ such that $\phi_b(x) = F(xb) - F(x)b - g(x)d(b)$ for all $x \in R$. We notice that the following hold:

$$\phi_{b+c}(x) = \phi_b(x) + \phi_c(x) \qquad \forall b, c, x \in R;$$
$$\phi_b(c) = -\phi_c(b) \qquad \forall b, c \in R.$$

We need a few lemmas to prove the main theorem. These results are contained in the classical paper of Herstein [4], but we prefer to state them for sake of completeness.

Lemma 2.13 *Let $t \in S$, $t \notin Z(R)$. If $y \in R$ such that $[t, y] = 0$, then $y \in S$.*

Proof The proof is contained in [4, Lemma 3.8]. ■

Lemma 2.14 *Let $x \in R$ such that $x^2 = 0$. Then $x \in S$.*

Proof Of course we assume $x \neq 0$, if not we are done, in particular $x \notin Z(R)$ Since $x(xr) = 0$ for any $r \in R$, then by Lemma 2.11, $F(xrx) = F(xr)x + g(xr)d(x)$. Moreover by Remark 2.12 we also have $xrx \in S$. Finally, since $x \notin Z(R)$, there exists $r \in R$ such that $xrx \notin Z(R)$. Hence by $[xrx, x] = 0$ and Lemma 2.13, it follows $x \in S$. ■

Lemma 2.15 *Let $x, y \in S$; then $\phi_b(a)[x, y] = 0$, for all $a, b \in R$.*

Proof This is [4, Lemma 3.10]. ■

We are now ready to prove our result.

Theorem *Let R be a prime ring of characteristic different from 2, let g be an endomorphism of R, let d be a Jordan semiderivation associated with g, and let F be a generalized Jordan semiderivation associated with d and g. Then F is a generalized semiderivation of R and d is a semiderivation of R. Moreover, if R is commutative, then $F = d$.*

Proof Our target is to show that $\phi_r(s) = 0$ for all $r\,s \in R$.

First, we consider the case where R is not commutative. In light of Lemma 2.5 we also assume R is not a domain. Let $z \in R$ be such that $z^2 = 0$. By Lemma 2.14 it follows that $z \in S$. Therefore, for any $t \in R$ such that $t^2 = 0$, Lemma 2.15 implies $\phi_a(b)[z, t] = 0$ for all $a, b \in R$. Right multiplying by z, we get

$$\phi_a(b)ztz = 0 \tag{2.15}$$

for all $a, b \in R$ and for all square-zero elements $z, t \in R$.

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Moreover, by Lemma 2.3, $\phi_y(x)[x,y]=0$ holds for all $x,y\in R$. This means that $([x,y]r\phi_y(x))^2=0$, so that $[x,y]r\phi_y(x)\in S$, for all $x,y,r\in R$. Applying equation (2.15) yields that, for all $a,b,x,y,r,s,t,z\in R$,

$$\phi_a(b)\big([x,y]r\phi_y(x)\big)\big([z,t]s\phi_t(z)\big)\big([x,y]r\phi_y(x)\big)=0;$$

that is,

$$\phi_t(z)[x,y]r\phi_y(x)[z,t]R\phi_t(z)[x,y]r\phi_y(x)=(0).$$

By the primeness of R, either $\phi_t(z)[x,y]=0$ or $\phi_y(x)[z,t]=0$. In particular, for $z=y$ one has either $0=\phi_t(y)[x,y]=-\phi_y(t)[x,y]$ or $\phi_y(x)[y,t]=0$. On the other hand, by (2.13), $\phi_y(t)[x,y]+\phi_y(x)[t,y]=0$, and this implies both $\phi_y(t)[x,y]=0$ and $\phi_y(x)[t,y]=0$. Therefore, in any case for all $x,y,t\in R$, $\phi_y(x)[t,y]=0$. Replacing t with rx, for any $r\in R$, we have $\phi_y(x)r[x,y]=0$. We recall that, if $[x,y]=0$, then $\phi_y(x)=0$ follows from Lemma 2.4. Thus $\phi_y(x)r[x,y]=0$ and the primeness of R imply $\phi_y(x)=0$ for all $x,y\in R$.

Finally we consider the case where R is commutative. We recall that, by Remark 2.2, if g is the identity map on R, then we are done. Therefore here we assume again g is not the identity map on R.

Since d is a generalized Jordan semiderivation associated with d and g, (2.5) yields

$$2d(xy)=d(x)y+g(x)d(y)+d(y)x+g(y)d(x)\quad \text{for all } x,y\in R.$$

Replacing y by yz, we get

$$2d(xyz)=d(x)yz+g(x)d(yz)+d(yz)x+g(yz)d(x)\quad \text{for all } x,y,z\in R. \tag{2.16}$$

On the other hand, (2.9) yields

$$\begin{aligned}2d(xyz)=d(x)yz+g(x)d(y)z+g(x)g(y)d(z)\\ +d(x)g(y)g(z)+xd(y)g(z)+xyd(z).\end{aligned} \tag{2.17}$$

Comparing (2.16) with (2.17) we obtain

$$g(x)d(y)z+g(x)g(y)d(z)+xd(y)g(z)+xyd(z)=g(x)d(yz)+xd(yz)$$

for all $x,y,z\in R$, so that

$$\big(g(x)-x\big)\big(d(yz)-d(y)z-g(y)d(z)\big)=0\quad \text{for all } x,y,z\in R.$$

Since R is a domain and g is not the identity map on R, we conclude that $d(yz)=d(y)z+g(y)d(z)$ for all $y,z\in R$.

Now, to prove that $F=d$, rewriting equation (2.5), we get

$$2F(xy)=F(x)\big(y+g(y)\big)+\big(x+g(x)\big)d(y),$$

and in particular

$$\begin{aligned}2F(x^2y)&=F(x^2)\big(y+g(y)\big)+\big(x^2+g(x^2)\big)d(y)\\ &=\big(F(x)x+g(x)d(x)\big)\big(y+g(y)\big)+\big(x^2+g(x^2)\big)d(y).\end{aligned} \tag{2.18}$$

Moreover, by equation (2.8),

$$2F(x^2y)=2F(x)yx+2g(x)d(y)x+2g(x)g(y)d(x). \tag{2.19}$$

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Comparing (2.18) with (2.19) it follows that

$$F(x)x\big(g(y)-y\big)+d(x)g(x)\big(y-g(y)\big)+d(y)\big(x-g(x)\big)^2=0, \tag{2.20}$$

and for $x=y$,

$$\big(F(x)-d(x)\big)x\big(g(x)-x\big)=0 \quad \forall x\in R.$$

Therefore, for any $x\in R$, either $F(x)=d(x)$ or $g(x)=x$. Assume that $g(x)=x$; moreover, since g is not the identity map, there exists $y\in R$ such that $g(y)\neq y$. Thus by (2.20) we get $(F(x)-d(x))x=0$; that is, $F(x)=d(x)$ holds in any case. ∎

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