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$(Z_2)^k$-ACTIONS FIXING A PRODUCT OF SPHERES AND A POINT

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ABSTRACT. In the paper we identify up to bordism all manifolds with $(Z_2)^k$-action whose fixed point set is $S^n \times S^m \cup$ point.

1. **Introduction.** In [6], Conner and Floyd showed that the fixed point structure of a differentiable involution on a closed manifold determines the bordism class. This fact allowed the analysis of the following question: given a smooth closed manifold F, not necessarily connected, can one identify up to bordism all manifolds and involutions (M, T) with F as the fixed point set? For instance, in [9] and in [1] this question was considered for $F = \mathrm{RP}(2n)$ and $F = \mathrm{RP}(n) \cup \mathrm{RP}(m)$ (disjoint union), respectively.

In [8], Stong showed more generally that the stationary point structure of a differentiable $(Z_2)^k$-action determines the bordism class, and this fact made possible to take into account the above question for $(Z_2)^k$-actions. In this direction, Capobianco [2] obtained this classification for $F = \mathrm{RP}(n)$, $\mathrm{CP}(n)$, or S^n.

In this paper we want to consider the case $F = S^n \times S^m \cup p$ ($p =$ point), $n, m > 0$.

We recall that in [6] Conner and Floyd exhibited involutions $\big(K_iP(2), T_i\big)$, $i = 1, 2, 4$, or 8, where $K_iP(2)$ denotes the appropriate projective plane, with $F_{T_i} = S^i \cup p$ ($F_{T_i} =$ fixed point set of T_i), and with the property that if (M^n, T) is an involution with $F_T = S^j \cup p$, then $n = 2j$, $j = 1, 2, 4$, or 8, and in each case (M^{2j}, T) is bordant to $\big(K_jP(2), T_j\big)$. The normal bundle to S^i in $K_iP(2)$, $\xi^i \longrightarrow S^i = K_iP(1)$, satisfies $w_i(\xi^i) \neq 0$.

Consider the set $L = L_1 \cup L_2$, where $L_1 = \{(1,1), (2,2), (4,4), (8,8)\}$, $L_2 = \{(1,3), (1,7), (2,6), (3,5)\}$. One has that, for each $(n, m) \in L$, there is an involution $(W^{2(n+m)}_{n,m}, \tau_{n,m})$ with $F_{\tau_{n,m}} = S^n \times S^m \cup p$. In fact, observe first that the fixed data of the involution $\big(K_iP(2) \times K_iP(2), T_i \times T_i\big)$ is bordant to $(\xi^{2i}_{i,i} = \xi^i \times \xi^i \longrightarrow S^i \times S^i) \cup (R^{4i} \longrightarrow p)$, hence there is an involution $(W^{4i}_{i,i}, \tau_{i,i})$ having precisely this fixed data (see proof of 25.2 in [5]). Next, consider the "smash product" $S^n \wedge S^m = (S^n \times S^m)/(S^n \times y_0 \cup x_0 \times S^m)$, $x_0 \in S^n$, $y_0 \in S^m$, which is homeomorphic to S^{n+m}, and the quotient map $q_{n,m}: S^n \times S^m \longrightarrow S^{n+m}$. For $(n, m) \in L_2$, consider the induced bundle $\xi^{n+m}_{n,m} = q^*_{n,m}(\xi^{n+m}) \longrightarrow S^n \times S^m$. One knows that $q^*_{n,m}: H^{n+m}(S^{n+m}, Z_2) \longrightarrow H^{n+m}(S^n \times S^m, Z_2)$ is an isomorphism, and so by the naturality of Whitney classes we get $w_{n+m}(\xi^{n+m}_{n,m}) \neq 0$ (and also that both $w_n(\xi^{n+m}_{n,m})$ and $w_m(\xi^{n+m}_{n,m})$ are zero). By computing characteristic numbers we may conclude that the line bundles

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over $\mathrm{RP}(\xi_{n,m}^{n+m})$ and $\mathrm{RP}(2n+2m-1)$ are bordant. By using [5; 25.2] we obtain then an involution $(W_{n,m}^{2(n+m)}, \tau_{n,m})$ with fixed data $(\xi_{n,m}^{n+m} \longrightarrow S^n \times S^m) \cup (R^{2(n+m)} \longrightarrow p)$. Note that, excluding the $(n,m) = (8,8)$ case, this latter approach may be employed also to obtain $(W_{n,m}^{2(n+m)}, \tau_{n,m})$ for $(n,m) \in L_1$, and the resulting involutions are bordant to those already considered. Actually, $(W_{n,m}^{2(n+m)}, \tau_{n,m})$ is bordant to $(W_{n',m'}^{2(n'+m')}, \tau_{n',m'})$ if $n+m = n'+m'$ and $n+m = 4$ or 8, and is bordant to $\big(K_{n+m}P(2), T_{n+m}\big)$ if $n+m = 2, 4$ or 8.

Now let (M,Φ) be a smooth $(Z_2)^k$-action; here, and throughout this paper, $(Z_2)^k$ will be considered as the group generated by k commuting involutions $T_1, T_2, \ldots, T_k$. The normal bundle η of $F = F_\Phi$ in M decomposes as Whitney sum of subbundles on which $(Z_2)^k$ acts as one of the irreducible (nontrivial) real representations. To describe this decomposition one may use sequences $a = (a_1, a_2, \ldots, a_k)$ where each a_j is either 0 or 1. Let $\varepsilon_a \subset \eta$ be the subbundle on which each T_j acts as multiplication by $(-1)^{a_j}$ for each j; then

$$\eta = \bigoplus_{a \neq (0)} \varepsilon_a$$

where $(0) = (0,0,\ldots,0)$ (trivial sequence). In this way, choosing an order for $\{a : a \neq (0)\}$, F and the ordered set of the $2^k - 1$ vector bundles ε_a $(a \neq (0))$ constitute the fixed data of the action on M. We will always consider the standard order given inductively by $(c_1,0), (c_2,0), \ldots, (c_{2^{k-1}-1}, 0), (c_1,1), \ldots, (c_{2^{k-1}-1}, 1), (0,0,\ldots,0,1)$, where $c_1, c_2, \ldots, c_{2^{k-1}-1}$ denote the (ordered) irreducible nontrivial representations of $(Z_2)^{k-1}$ (the case $k = 1$ is trivial).

Consider now an involution (M,T) with fixed data $\eta \longrightarrow F$. For $1 \leq t \leq k$ let $(Z_2)^k$ act on $M^{2^{t-1}}$, the cartesian product of 2^{t-1} copies of M, by $T_1(x_1, \ldots, x_{2^{t-1}}) = \big(T(x_1), \ldots, T(x_{2^{t-1}})\big)$, letting $T_2, \ldots, T_t$ act by permuting factors so that the points fixed by $T_2, \ldots, T_t$ form the diagonal copy of M, and letting $T_{t+1}, \ldots, T_k$ act trivially; we denote this action by $\Gamma_t^k(M,T)$. The fixed data of $\Gamma_t^k(M,T)$ may be described using induction on t: it is $\bigoplus_{i=1}^{2^k-1} \varepsilon_{a_i} \longrightarrow F$ where both $\bigoplus_{i=1}^{2^{t-1}-1} \varepsilon_{a_i}$, and $\bigoplus_{i=2^{t-1}}^{2^t-2} \varepsilon_{a_i}$ are equal to the fixed data of $\Gamma_{t-1}^{t-1}(M,T)$, and where $\varepsilon_{a_1} = \eta$, $\varepsilon_{a_{2^t-1}} = T(F)$ and $\varepsilon_{a_j} = 0$ for $2^t \leq j \leq 2^k - 1$, here $T(F)$ and 0 denoting, respectively, the tangent bundle and the 0-dimensional bundle over F.

Given now a $(Z_2)^k$-action (M,Φ), $\Phi = (T_1, T_2, \ldots, T_k)$, we observe that each automorphism $\sigma\colon (Z_2)^k \longrightarrow (Z_2)^k$ yields a new action given by $\big(M; \sigma(T_1), \sigma(T_2), \ldots, \sigma(T_k)\big)$; we denote this action by $\sigma(M,\Phi)$. The fixed data of $\sigma(M,\Phi)$ is obtained from the fixed data of (M,Φ) by a permutation of subbundles.

The desired classification will be obtained with the

THEOREM. *If (M^r,Φ), $\Phi = (T_1, T_2, \ldots, T_k)$, is a $(Z_2)^k$-action whose fixed point set is $S^n \times S^m \cup p$, then $(n,m) \in L$ and there is an integer t, $1 \leq t \leq k$, and an automorphism $\sigma\colon (Z_2)^k \longrightarrow (Z_2)^k$ such that $r = (n+m)2^t$ and (M^r,Φ) is bordant to $\sigma\Gamma_t^k(W_{n,m}^{2(n+m)}, \tau_{n,m})$.*

We shall prove first this result for involutions. The general case will be obtained then by induction.

2. **Involutions fixing products of spheres and a point.** We start with an involution (M^r, T) with $S^n \times S^m \cup p$ as the fixed point set. Let $\eta^s \longrightarrow S^n \times S^m$ denote the normal bundle to $S^n \times S^m$ in M^r, where $s = \dim(\eta)$. We write

$$W(\eta^s) = 1 + w_n + w_m + w_{n+m}.$$

Since $\chi(M^r) \equiv \chi(S^n \times S^m \cup p) \equiv 1 \bmod 2$, where χ denotes the Euler characteristic [6; 27.2], one has from [6; 27.5] that some Whitney number of η involving $w_s(\eta)$ is non-zero.

Assuming first $n < m$, we have therefore three possibilities:

i) $s = n$, $m = in$ for some $i > 1$ and $w_n^{i+1}[S^n \times S^m] \neq 0$;

ii) $s = m$ and $w_n w_m[S^n \times S^m] \neq 0$;

iii) $s = n + m$ and $w_{n+m}[S^n \times S^m] \neq 0$.

The structure of $H^*(S^n \times S^m, Z_2)$ excludes validity of i). Assuming ii) valid and using the Wu formula [10], we get

$$Sq^n(w_m) = \binom{m-n-1}{0} w_n w_m + \binom{m-1}{n} w_{m+n} = w_n w_m.$$

But $Sq^i = 0$ in $H^*(S^n \times S^m, Z_2)$ if $i > 0$, so one has a contradiction. It remains only the possibility $s = n + m$ and $w_{n+m} \neq 0$.

Suppose first $w_n \neq 0$ and $w_m \neq 0$. As is well know, $H^*(\mathrm{BO}, Z_2)$ is generated over the Steenrod algebra by the classes w_{2^j}. Since both w_n and w_m cannot be obtained over the Steenrod algebra from lower classes, both n and m must be a power of 2, say $n = 2^p$, $m = 2q, p < q$.

Now let c be the characteristic class of the line bundle over $\mathrm{RP}(\eta)$. One has

$$W\big(\mathrm{RP}(\eta)\big) = 1 + c^{2^p} + w_{2^p} + c^{2^q} + w_{2^q}$$

so the Whitney number

$$w_{2^{q+1}}\big(\mathrm{RP}(\eta)\big) c^{2^{p+1}-1}[\mathrm{RP}(\eta)]$$

of this line bundle is zero. Since

$$W\big(\mathrm{RP}(2n + 2m - 1)\big) = 1 + \alpha^{2^{p+1}} + \alpha^{2^{q+1}},$$

where $\alpha \in H^1\big(\mathrm{RP}(2n+2m-1), Z_2\big)$ is the generator, the corresponding Whitney number of the line bundle over $\mathrm{RP}(2n + 2m - 1)$ is non-zero. By [5; 25.2], $(\eta \longrightarrow S^n \times S^m) \cup (R^{2(n+m)} \longrightarrow p)$ cannot be the fixed data of an involution; hence either $w_n = 0$ or $w_m = 0$ (and so $n + m$ must be a power of 2).

Actually, if $n + m = 2^t$ and $\eta \longrightarrow S^n \times S^m$ is a $n + m$-dimensional bundle satisfying the above conditions, then we can show that $(\eta \longrightarrow S^n \times S^m) \cup (R^{2(n+m)} \longrightarrow p)$ is the fixed data of an involution. Therefore, the next step will consist in restricting the occurrence of these bundles.

2.1 *The above bundles can occur only for* $(n,m) \in L_2$. Let (X,x_0), (Y,y_0) be compact CW-complexes with basepoint, and let $\widetilde{KO}(X)$ denote the (real) reduced Grothendieck ring of X. According to [4; 2.4.8], one has

$$\widetilde{KO}(X \times Y) \cong \widetilde{KO}(X) \oplus \widetilde{KO}(Y) \oplus \widetilde{KO}(X \wedge Y).$$

The proof of this fact, which is based upon arguments involving split exact sequences, yields precisely the following: if $p_1: X \times Y \longrightarrow X$, $p_2: X \times Y \longrightarrow Y$ are the projections, $i_1: X \longrightarrow X \times Y$ is the inclusion $x \longmapsto (x, y_0)$ and $q: X \times Y \longrightarrow X \wedge Y$ is the quotient map, then given $a \in \widetilde{KO}(X \times Y)$, there are elements $b \in \widetilde{KO}(X \wedge Y)$, $c \in \widetilde{KO}(Y)$ such that $a = p_1^* i_1^*(a) + q^*(b) + p_2^*(c)$.

We then have, in particular, that there are bundles $P \longrightarrow S^{n+m}$, $Q \longrightarrow S^m$, so that the bundles η and $p_1^* i_1^*(\eta) \oplus q^*(P) \oplus p_2^*(Q)$ are stably equivalent. Letting $W(P) = 1 + W_{n+m}$, $W(Q) = 1 + v_m$, one then has

$$\begin{aligned} 1 + w_n + w_m + w_{n+m} &= (1 + w_n)\big(1 + q^*(W_{n+m})\big)\big(1 + p_2^*(v_m)\big) \\ &= 1 + w_n + p_2^*(v_m) + w_n p_2^*(v_m) + q^*(W_{n+m}). \end{aligned}$$

Hence $p_2^*(v_m) = w_m$, and so $w_n p_2^*(v_m) = w_n w_m = 0$. It follows that $q^*(W_{n+m}) = w_{n+m}$, that is, $W_{n+m} \neq 0$. But Milnor [3] shows that $n + m = 1, 2, 4$, or 8 in that case. Since $n < m$, $n + m = 4$ or 8. This completes 2.1.

Since in each case η has the same characteristic numbers as $\xi_{n,m}^{n+m}$, (M^r, T) is bordant to $(W_{n,m}^{2(n+m)}, \tau_{n,m})$.

Finally, suppose $n = m$, with $W(\eta) = 1 + w_n + w_{2n}$. As before, we then have $s = 2n$ and $w_{2n} \neq 0$. Assuming first $w_n \neq 0$, we may obtain the bundle $p^*(\eta)$ over S^n with non-zero Whitney class $w_n(\eta)$ by choosing suitable inclusion $p: S^n \longrightarrow S^n \times S^n$. Hence $n = 1$, 2, 4 or 8 in that case. Otherwise, if $w_n = 0$, we can use the arguments and terminology outlined before to obtain

$$1 + w_{2n} = \big(1 + q^*(W_{2n})\big)\big(1 + p_2^*(v_n)\big) = 1 + p_2^*(v_n) + q^*(W_{2n}).$$

Thus $W_{2n} \neq 0$ and so $n = 1, 2$, or 4. In any case, $(n,m) \in L$ and (M^r, T) is bordant to $(W_{n,m}^{2(n+m)}, \tau_{n,m})$.

3. **The general case.** Let (M^r, Φ), $\Phi = (T_1, T_2, \ldots, T_k)$, be a $(Z_2)^k$-action with fixed data $\eta = \bigoplus_{a \neq (0)} \varepsilon_a \longrightarrow F$. For each $a \neq (0)$, let $f_a: (Z_2)^k \longrightarrow Z_2 = \{+1, -1\}$ denote the homomorphism given by $f_a(T_i) = (-1)^{a_i}$. We note that $\ker(f_a)$ is isomorphic to $(Z_2)^{k-1}$. Let F^j be a j-dimensional component of F and let N_a represent the fixed point set of the action $\big(M^r, \ker(f_a)\big)$. If $T \notin \ker(f_a)$, then $\varepsilon_a \longrightarrow F^j$ is a component of the normal bundle to the fixed point set of the manifold with involution $(N_a^j, T|_{N_a^j})$, where N_a^j represents the component of N_a containing F^j.

Suppose now that (M^r, Φ) has $F_\Phi = S^n \times S^m \cup p$ as the fixed point set, and denote by

$$\Big(\bigoplus_{a \neq (0)} \varepsilon_a \longrightarrow S^n \times S^m\Big) \cup \Big(\bigoplus_{a \neq (0)} \mu_a \longrightarrow p\Big)$$

the fixed data of Φ.

For each a, denote by S_a the component of $F_{\ker(f_a)}$ containing $S^n \times S^m$ and by P_a the component containing p. Let a be chosen so that $\dim(P_a) > 0$, and take $T \notin \ker(f_a)$. Since an involution cannot have precisely one fixed point [6; 25.1], the fixed set of the involution (P_a, T) is $S^n \times S^m \cup p$, that is, $P_a = S_a$. But then, by the previous section, $\dim(P_a) = 2(n+m)$ and (P_a, T) is bordant to $(W^{2(n+m)}_{n,m}, \tau_{n,m})$ for some $(n,m) \in L$, and the normal bundle ε_a in that case has $w_{n+m}(\varepsilon_a)[S^n \times S^m]$ as the only non-zero Whitney number; throughout, we will denote these bundles by η. Evidently, $\dim(\mu_a) = 2(n+m)$ in that case.

Suppose next that $\dim(P_a) = 0$; taking again $T \notin \ker(f_a)$, one has that (S_a, T) is an involution fixing $S^n \times S^m$. Since $W(S^n \times S^m) = 1$, it follows by [6; 25.3] that $\varepsilon_a \longrightarrow S^n \times S^m$ bounds. In that case, if $\dim(\varepsilon_a) = n+m$, we have that ε_a is bordant to the tangent bundle over $S^n \times S^m$; throughout, we will use the notation $\Upsilon \longrightarrow S^n \times S^m$ for these bundles.

There is at least one a_0 with $\dim(P_{a_0}) > 0$; otherwise $r = 0$. Denote by

$$\bigoplus_c \vartheta_c \longrightarrow P_{a_0}$$

the fixed data of the action $\big(M^r, \ker(f_{a_0})\big)$ restricted to P_{a_0}, where c runs through the $2^{k-1} - 1$ nontrivial representations of $\ker(f_{a_0})$.

As described before, for each c there is a corresponding subgroup $(Z_2^{k-2})_c$ of $\ker(f_{a_0})$; if we take $T \notin \ker(f_{a_0})$ and $S \in \ker(f_{a_0})$, but $S \notin (Z_2^{k-2})_c$, then the subgroups $(Z_2^{k-2})_c \oplus \langle T\rangle$ and $(Z_2^{k-2})_c \oplus \langle ST\rangle$ of $(Z_2)^k$ determine representations a_{c_1}, a_{c_2} so that $\vartheta_c|_{S^n \times S^m} = \varepsilon_{a_{c_1}} \oplus \varepsilon_{a_{c_2}}$ and $\vartheta_c|_p = \mu_{a_{c_1}} \oplus \mu_{a_{c_2}}$. We assert that $\varepsilon_{a_{c_1}} \oplus \varepsilon_{a_{c_2}}$ is either $\eta \oplus \Upsilon$ or $0 \oplus 0$. In fact, suppose that $\dim(\varepsilon_{a_{c_2}}) > 0$ and $\varepsilon_{a_{c_2}}$ bounds (that is, $\dim(\mu_{a_{c_2}}) = 0$). Then $\dim(\mu_{a_{c_1}}) = \dim(\mu_{a_{c_1}}) + \dim(\mu_{a_{c_2}}) = \dim(\vartheta_c) \geq \dim(\varepsilon_{a_{c_2}}) > 0$. As we have seem, $\varepsilon_{a_{c_1}} = \eta$ and $\dim(\mu_{a_{c_1}}) = 2(n+m)$; hence $\dim(\varepsilon_{a_{c_2}}) = 2(n+m) - \dim(\eta) = n+m$. That is, $\varepsilon_{a_{c_2}} = \Upsilon$. In particular, we have proved that, for each a, $\dim(\varepsilon_a)$ is either $n+m$ or zero. So, if we suppose on the other hand that $\varepsilon_{a_{c_1}} = \eta$, then $\dim(\mu_{a_{c_1}}) = 2(n+m)$ and so $n + m + \dim(\varepsilon_{a_{c_2}}) = 2(n+m) + \dim(\mu_{a_{c_2}})$. It remains $\dim(\varepsilon_{a_{c_2}}) = n+m$ and $\dim(\mu_{a_{c_2}}) = 0$, that is, $\varepsilon_{a_{c_2}} = \Upsilon$.

Since $\varepsilon_{a_0} = \eta$, one has thus that the number of bundles $\varepsilon_a = \eta$ is equal to one plus the number of bundles $\varepsilon_a = \Upsilon$; we assert that this number is 2^t for some $0 \leq t \leq k-1$ and that the bundles η, Υ and 0 are settled in the fixed data of (M^r, Φ) in the same manner as the bundles $\xi^{n+m}_{n,m}$, $T(S^n \times S^m)$ and 0 are settled in the fixed data of $\sigma\Gamma^k_{t+1}(W^{2(n+m)}_{n,m}, \tau_{n,m})$ for some $\sigma \in \mathrm{Aut}\big((Z_2)^k\big)$. Indeed, suppose inductively the fact true for $(Z_2)^{k-1}$-actions. Considering $k > 1$, there is at least one $T \in (Z_2)^k$ such that the component of F_T containing p, which we call $(F_T)_p$, has positive dimension. To simplify notation we may suppose $T = T_k$ (it suffices to take an automorphism $(Z_2)^k \longrightarrow (Z_2)^k$ carrying T_k into T). On F_{T_k} one has an induced action Ψ of $(Z_2)^{k-1}$, the group generated by $T_1, T_2, \ldots, T_{k-1}$. Since a $(Z_2)^j$-action cannot fix precisely one point [6; 31.3], the fixed set of $\big((F_{T_k})_p, \Psi\big)$ is $S^n \times S^m \cup p$. Let $\Theta \longrightarrow (F_{T_k})_p$ denote the normal bundle to $(F_{T_k})_p$ in M^r. Since T_k acts as -1 in the fibers of Θ, one has

$$\Theta|_{S^n \times S^m} = \Big(\bigoplus_c \varepsilon_{(c,1)}\Big) \oplus \varepsilon_{(0,1)}.$$

The fixed data of $\big((F_{T_k})_p, \Psi\big)$ restricted to $S^n \times S^m$ is

$$\bigoplus_c \varepsilon_{(c,0)}.$$

But the induction hypothesis guarantees that this latter fixed data contains 2^{t-1} bundles $\varepsilon_{(c,0)} = \eta$, $2^{t-1} - 1$ bundles $\varepsilon_{(c,0)} = \Upsilon$ and $2^{k-1} - 2^t$ bundles $\varepsilon_{(c,0)} = 0$ for some $1 \le t \le k-1$; moreover, these bundles are settled in that fixed data as the corresponding bundles are settled in the fixed data of $\rho\Gamma_t^{k-1}(W_{n,m}^{2(n+m)}, \tau_{n,m})$, for some $\rho \in \mathrm{Aut}\big((Z_2)^{k-1}\big)$. Hence, if we assume $\varepsilon_{(0,1)} = 0$ and $\varepsilon_{(c,1)} = 0$ for all c, the fact is proved. Otherwise, the preceding comments imply that at least one of these bundles, say $\varepsilon_{(h,1)}$, must be η, which is the normal bundle to $S^n \times S^m$ in $P_{(h,1)}$. One has, as before, the fixed data $\oplus_c \vartheta_c \longrightarrow P_{(h,1)}$, and for each c the decomposition $\vartheta_c|_{S^n \times S^m} = \varepsilon_{a_{c_1}} \oplus \varepsilon_{a_{c_2}}$ with the bundles $\varepsilon_{a_{c_1}}$, $\varepsilon_{a_{c_2}}$ corresponding, respectively, to the subgroups $(Z_2^{k-2})_c \oplus \langle T_k \rangle$ and $(Z_2^{k-2})_c \oplus \langle ST_k \rangle$ (observe that $T_k \notin \ker(f_{(h,1)})$). But it can be seen that T_k acts trivially in the fibers of $\varepsilon_{a_{c_1}}$, and as multiplication by -1 in the fibers of $\varepsilon_{a_{c_2}}$. It follows that $\varepsilon_{a_{c_1}}$ is of the form $\varepsilon_{(b,0)}$ while $\varepsilon_{a_{c_2}}$ is of the form $\varepsilon_{(v,1)}$, $v \neq h$. In this manner, the occurrence of the bundles η in the fixed data of Φ is given by $\varepsilon_{(h,1)}$, by the 2^{t-1} bundles $\varepsilon_{(c,0)}$ given by induction hypothesis and by the $2^{t-1} - 1$ bundles $\varepsilon_{(v,1)}$, $v \neq h$, corresponding to the $2^{t-1} - 1$ bundles $\varepsilon_{(b,0)} = \Upsilon$ of the induction hypothesis. So, this number is 2^t. To analyse the order of the bundles we note first that the condition $\ker(f_{(b,0)}) = (Z_2)_c^{k-2} \oplus \langle T_k \rangle$ implies that $\ker(f_{(b+h,1)}) = (Z_2)_c^{k-2} \oplus \langle ST_k \rangle$, where the sum $b+h$ is taken modulo 2. This means that $(v,1) = (b+h,1)$, and hence the fixed data of (M^r, Φ) obey the following rule: if $\varepsilon_{(b,0)} = \eta$, Υ or 0, then $\varepsilon_{(b+h,1)} = \Upsilon$, η or 0, respectively. By observing (by direct inspection) that for each representation $(h,1)$ there is automorphism $\sigma\colon (Z_2)^k \longrightarrow (Z_2)^k$ such that the fixed data of $\sigma\Gamma_{t+1}^k(W_{n,m}^{2(n+m)}, \tau_{n,m})$ has $\varepsilon_{(h,1)} = \xi_{n,m}^{n+m}$ and also the part $\oplus_c \varepsilon_{(c,0)}$ equal to the fixed data of $\Gamma_t^{k-1}(W_{n,m}^{2(n+m)}, \tau_{n,m})$, and that the actions $\sigma\Gamma_{t+1}^k(W_{n,m}^{2(n+m)}, \tau_{n,m})$ obey the same above rule, the fact is proved (observe in the above proof that we may suppose that the part $\oplus_c \varepsilon_{(c,0)}$ of the fixed data of (M^r, Φ) behaves as the fixed data of $\Gamma_t^{k-1}(W_{n,m}^{2(n+m)}, \tau_{n,m})$; indeed, it suffices to take the automorphism $(Z_2)^k \longrightarrow (Z_2)^k$ which restricted to $(Z_2)^{k-1}$ is ρ^{-1} and which carries T_k into T_k).

Next consider the homomorphism

$$S\colon \bigoplus \mathcal{N}_{p',p'',n_1',n_1'',\dots,n_{2^{k-1}-1}',n_{2^{k-1}-1}''}(Z_2^0, 2^k - 1) \longrightarrow \hat{\mathcal{N}}_{p-1,n_1,\dots,n_{2^{k-1}-1}}(Z_2, 2^{k-1} - 1)$$

of the Stong's sequence [8, 4.3]. Rewriting the fixed data of (M^r, Φ) as $(\oplus_j \varepsilon_j \longrightarrow S^n \times S^m) \cup (\oplus_j \mu_j \longrightarrow p)$, $j = 1, 2, \dots, 2^k - 1$, and according to [7, 8.7], one has $S[\oplus_j \varepsilon_j \longrightarrow S^n \times S^m] = \big[\lambda \oplus \big(\varepsilon_2 \oplus (\varepsilon_3 \otimes \lambda)\big) \oplus \big(\varepsilon_4 \oplus (\varepsilon_5 \otimes \lambda)\big) \oplus \cdots \oplus \big(\varepsilon_{2^k-2} \oplus (\varepsilon_{2^k-1} \otimes \lambda)\big) \longrightarrow \mathrm{RP}(\varepsilon_1)\big] \in \mathcal{N}_{p-1}\big(\mathrm{BO}(1) \times \mathrm{BO}(n_1) \times \cdots \times \mathrm{BO}(n_{2^{k-1}-1})\big)$, where $\lambda \longrightarrow \mathrm{RP}(\varepsilon_1)$ is the line bundle. Now, for $i = 1, 2, 4$, or 8, suppose that there are ε_{j_1}, ε_{j_2} with $j_1 \neq j_2$, $v_i(\varepsilon_{j_1}) = \gamma_1$ and $v_i(\varepsilon_{j_2}) = \gamma_2$, where $\gamma_l = p_l^*(\gamma)$, $p_l\colon S^i \times S^i \longrightarrow S^i$ are the projections (for $l = 1, 2$) and $\gamma \in H^i(S^i, Z_2)$ is the generator. By computing characteristic numbers one may see then that $S[\oplus_j \varepsilon_j \longrightarrow S^i \times S^i]$ is nonzero; in the same way, one may see that $S[\oplus_j \mu_j \longrightarrow p]$ is zero.

On the other hand, if $(n, m) \in L_2$, the Milnor's result [3] implies that $v_m(\varepsilon_j) = 0$ for any j.

The above facts imply that $(\bigoplus_j \varepsilon_j \longrightarrow S^n \times S^m) \cup (\bigoplus_j \mu_j \longrightarrow p)$ is bordant as an element of $\mathcal{N}_{n+m}\big(\mathrm{BO}(n_1) \times \cdots \times \mathrm{BO}(n_{2^k-1})\big)$ to the fixed data of (replacing $t+1$ by t) $\sigma\Gamma_t^k(W_{n,m}^{2(n+m)}, \tau_{n,m})$ for some $\sigma \in \mathrm{Aut}\big((Z_2)^k\big)$, and so the proof is complete.

It is interesting observing that the above proof serves also to extend, for $(Z_2)^k$-actions, the previously mentioned Conner-Floyd's result; it is the following

THEOREM. *If (M^r, Φ) is a $(Z_2)^k$-action with fixed point set $S^n \cup p$, then $n = 1, 2, 4$, or 8 and there is an integer t, $1 \leq t \leq k$, and an automorphism $\sigma\colon (Z_2)^k \longrightarrow (Z_2)^k$ such that $r = n2^t$ and (M^r, Φ) is bordant to $\sigma\Gamma_t^k\big(K_nP(2), T_n\big)$.*

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