

ARTICLE

# Sharp bounds for decomposing graphs into edges and triangles

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## Abstract

For a real constant  $\alpha$ , let  $\pi_3^\alpha(G)$  be the minimum of twice the number of  $K_2$ 's plus  $\alpha$  times the number of  $K_3$ 's over all edge decompositions of  $G$  into copies of  $K_2$  and  $K_3$ , where  $K_r$  denotes the complete graph on  $r$  vertices. Let  $\pi_3^\alpha(n)$  be the maximum of  $\pi_3^\alpha(G)$  over all graphs  $G$  with  $n$  vertices.

The extremal function  $\pi_3^3(n)$  was first studied by Győri and Tuza (*Studia Sci. Math. Hungar.* **22** (1987) 315–320). In recent progress on this problem, Král', Lidický, Martins and Pehova (*Combin. Probab. Comput.* **28** (2019) 465–472) proved via flag algebras that  $\pi_3^3(n) \leq (1/2 + o(1))n^2$ . We extend their result by determining the exact value of  $\pi_3^\alpha(n)$  and the set of extremal graphs for all  $\alpha$  and sufficiently large  $n$ . In particular, we show for  $\alpha = 3$  that  $K_n$  and the complete bipartite graph  $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$  are the only possible extremal examples for large  $n$ .

**2020 MSC Codes:** Primary 05C70; Secondary 05C35

## 1. Introduction

In recent progress on a problem of Győri and Tuza [27], Král', Lidický, Martins and Pehova [19] proved via flag algebras that the edges of any  $n$ -vertex graph can be decomposed into copies of  $K_2$  and  $K_3$  whose total number of vertices is at most  $(1/2 + o(1))n^2$ , where  $K_r$  denotes the clique on

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$r$  vertices. The origins of this problem can be traced back to Erdős, Goodman and Pósa [10], who considered the problem of minimizing the total number of cliques in an edge decomposition of an arbitrary  $n$ -vertex graph. They showed the following.

**Theorem 1.1** (Erdős, Goodman and Pósa [10]). *The edges of every  $n$ -vertex graph can be decomposed into at most  $\lfloor n^2/4 \rfloor$  complete graphs.*

The only extremal example for this bound is the (bipartite) Turán graph  $T_2(n) := K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ , where  $K_{a,b}$  denotes the complete bipartite graph with part sizes  $a$  and  $b$ . Moreover, this result still holds if we restrict the sizes of the cliques used in the decomposition to 2 and 3 (i.e. single edges and triangles). In a series of papers published independently by Chung [4], Györi and Kostochka [15] and Kahn [18], they proved that in fact something stronger than Theorem 1.1 is true, confirming a conjecture by Katona and Tarján.

**Theorem 1.2** (Chung [4], Györi and Kostochka [15], Kahn [18]). *Every  $n$ -vertex graph can be edge-decomposed into cliques whose total number of vertices is at most  $\lfloor n^2/2 \rfloor$ .*

For a given graph  $G$  on  $n$  vertices, let  $\pi_k(G)$  be the minimum over all decompositions of the edges of  $G$  into cliques  $C_1, \dots, C_\ell$  of size at most  $k$  of the sum  $|C_1| + |C_2| + \dots + |C_\ell|$ , where  $|G| := |V(G)|$  denotes the order of a graph  $G$ . Let  $\pi_k(n)$  be the maximum of  $\pi_k(G)$  over all graphs  $G$  with  $n$  vertices. With this notation, the conclusion of the above theorem is that  $\min_{k \in \mathbb{N}} \pi_k(n) \leq \lfloor n^2/2 \rfloor$ . In light of Theorem 1.2, Tuza [27] conjectured that  $\pi_3(n) \leq n^2/2 + o(n^2)$ , and in fact that  $\pi_3(n) \leq n^2/2 + O(1)$ . Györi and Tuza [16] showed that  $\pi_3(n) \leq 9n^2/16$ . This was the best known bound until recently, when using the celebrated flag algebra method of Razborov [24], Král', Lidický, Martins and Pehova [19] proved the asymptotic version of Tuza's conjecture.

**Theorem 1.3** (Král' et al. [19]). *We have  $\pi_3(n) \leq (1/2 + o(1))n^2$  as  $n \rightarrow \infty$ .*

In this paper we show, by building upon the proof in [19], that for all large  $n$  it holds in fact that  $\pi_3(n) \leq n^2/2 + 1$ . Moreover, if a graph  $G$  of order  $n$  attains  $\pi_3(n)$ , then  $G$  is the complete graph  $K_n$  or the Turán graph  $T_2(n)$ .

Which of these two graphs is extremal is a matter of divisibility of  $n$  by 6. In the case of the Turán graph, we trivially have  $\pi_3(T_2(n)) = 2 \lfloor n/2 \rfloor \lceil n/2 \rceil$ , giving  $n^2/2$  for even  $n$  and  $(n^2 - 1)/2$  for odd  $n$ . In order to determine  $\pi_3(K_n)$ , we have to determine the maximum number of edge-disjoint triangles in  $K_n$ . Clearly, the graph made of their edges is *triangle-divisible*, that is, each vertex has even degree and the total number of edges is divisible by three. It is routine to see that the minimum size of a graph  $H$  on  $n$  vertices whose complement  $\overline{H}$  is triangle-divisible is attained by taking at most one copy of the claw  $K_{1,3}$  and a perfect matching on the remaining vertices for even  $n$ , and isolated vertices plus at most one copy of the 4-cycle  $K_{2,2}$  for odd  $n$ . (Note that  $\binom{n}{2}$  is never equal to 2 modulo 3.) In fact this gives the value of  $\pi_3(K_n)$  for all large  $n$  by the following general result (which we will also use in our proof).

**Theorem 1.4** (Barber, Kuhn, Lo and Osthus [2]). *For every  $\varepsilon > 0$ , if  $G$  is a triangle-divisible graph of large order  $n$  and minimum degree at least  $(0.9 + \varepsilon)n$ , then  $G$  has a perfect triangle decomposition.*

The constant 0.9 in the minimum degree condition in Theorem 1.4 comes from the result of Dross [6] on fractional triangle decompositions, and Nash-Williams [21] conjectured that it can be replaced by  $3/4$ . Very recently, Dukes and Horsley [7] and Delcourt and Postle [5] improved the constant to 0.852 and  $(7 + \sqrt{21})/14 = 0.8273\dots$ , respectively.

**Table 1.** Values of  $\pi_3(K_n)$  and  $\pi_3(T_2(n))$  for large  $n$

$n \pmod 6$	$K_2$ 's in an optimal decomposition of $K_n$	$\pi_3(K_n)$	$\pi_3(T_2(n))$
0	perfect matching	$\frac{n^2}{2}$	$\frac{n^2}{2}$
1	none	$\binom{n}{2}$	$\frac{n^2 - 1}{2}$
2	perfect matching	$\frac{n^2}{2}$	$\frac{n^2}{2}$
3	none	$\binom{n}{2}$	$\frac{n^2 - 1}{2}$
4	$K_{1,3}$ + perfect matching	$\frac{n^2}{2} + 1$	$\frac{n^2}{2}$
5	$C_4$	$\binom{n}{2} + 4$	$\frac{n^2 - 1}{2}$

In Table 1 we list the values of  $\pi_3$  for the graphs  $K_n$  and  $T_2(n)$  for large  $n$ . Let us define

$$\mathcal{E}_n := \begin{cases} \{T_2(n), K_n\} & \text{if } n \equiv 0, 2 \pmod 6, \\ \{T_2(n)\} & \text{if } n \equiv 1, 3, 5 \pmod 6, \\ \{K_n\} & \text{if } n \equiv 4 \pmod 6, \end{cases}$$

and

$$\ell(n) := \begin{cases} n^2/2 & \text{for } n \equiv 0, 2 \pmod 6, \\ (n^2 - 1)/2 & \text{for } n \equiv 1, 3, 5 \pmod 6, \\ n^2/2 + 1 & \text{for } n \equiv 4 \pmod 6. \end{cases}$$

Thus, by the calculations of Table 1, we have for all large  $n$  that  $\mathcal{E}_n$  consists of those graphs in  $\{T_2(n), K_n\}$  which maximize  $\pi_3$  while  $\ell(n)$  is this maximum value.

Clearly,  $\ell(n)$  is a lower bound on  $\pi_3(n)$  for large  $n$ . Our main result is that this is equality.

**Theorem 1.5.** *There exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ , we have  $\pi_3(n) = \ell(n)$ , and the set of  $\pi_3(n)$ -extremal graphs up to isomorphism is exactly  $\mathcal{E}_n$ .*

A simple corollary of Theorem 1.5 is an affirmative answer to a question of Pyber [23] (see also [27, Problem 45]) for sufficiently large  $n$ . A *covering* of a graph  $G$  is a collection of subgraphs of  $G$  such that every edge of  $G$  appears in at least one subgraph. (For comparison, a decomposition requires that every edge appears in exactly one subgraph.)

**Corollary 1.1.** *There exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ , the edge set of every  $n$ -vertex graph can be covered with triangles and edges so that the sum of their orders is at most  $\lfloor n^2/2 \rfloor$ .*

**Proof.** Theorem 1.5 directly implies the corollary unless  $n \equiv 4 \pmod 6$  and the graph under consideration is  $K_n$ . So assume that  $n \equiv 4 \pmod 6$ . Denote the vertices of  $K_n$  by  $v_1, \dots, v_n$ . Recall that an optimal decomposition for  $K_n$  is obtained by taking edges  $v_1v_2, v_1v_3, v_1v_4$  and  $v_iv_{i+1}$  for all odd  $i$  with  $5 \leq i \leq n - 1$ . The rest of the graph becomes triangle-divisible and Theorem 1.4 can be applied. This gives a decomposition of cost  $n^2/2 + 1$ . A covering of cost at most  $n^2/2$  can be

obtained from this decomposition by replacing edges  $v_1 v_2$  and  $v_1 v_3$  with a triangle  $v_1 v_2 v_3$ . (Note that the pair  $v_2 v_3$  is covered by two triangles in the resulting covering.)  $\square$

We also study an extension of Theorem 1.5, where we consider decompositions into  $K_2$ 's and  $K_3$ 's but we modify the cost of  $K_3$ 's to be  $\alpha$  (with the cost of  $K_2$  still being 2). The minimum over all costs of such decompositions of a graph  $G$  is denoted by  $\pi_3^\alpha(G)$ . The maximum value of  $\pi_3^\alpha(G)$  over all  $n$ -vertex graphs  $G$  is denoted by  $\pi_3^\alpha(n)$ . Note that  $\pi_3^3(G) = \pi_3(G)$  and  $\pi_3^3(n) = \pi_3(n)$ . Denote  $K_n$  without one edge by  $K_n^-$  and  $K_n$  without a matching of size two by  $K_n^=$ . Then the following result holds.

**Theorem 1.6.** *For every real  $\alpha$  there exists  $n_0 \in \mathbb{N}$  such that every  $\pi_3^\alpha$ -extremal graph  $G$  with  $n \geq n_0$  vertices satisfies the following (up to isomorphism).*

- If  $\alpha < 3$ , then  $G = T_2(n)$ .
- If  $\alpha = 3$ , then Theorem 1.5 applies.
- If  $3 < \alpha < 4$  and  $n \equiv 0, 2, 4, 5 \pmod{6}$ , then  $G = K_n$ .
- If  $3 < \alpha < 4$  and  $n \equiv 1, 3 \pmod{6}$ , then  $G = K_n^-$ .
- If  $\alpha = 4$  and  $n \equiv 1, 3 \pmod{6}$ , then  $G \in \{K_n, K_n^-, K_n^=\}$  and, moreover, the three listed graphs are all  $\pi_3^\alpha$ -extremal.
- If  $\alpha = 4$  and  $n \equiv 0, 2, 4, 5 \pmod{6}$ , then  $G = K_n$ .
- If  $4 < \alpha$ , then  $G = K_n$ .

This paper is organized as follows. In Section 2 we give an outline of the proof of Theorem 1.3 from [19] that we build on. Theorem 1.5 is proved in Section 3. An extension for other weights of triangles is in Section 4. Some related results are mentioned in Section 5.

**Notation.** We follow standard graph theory notation (see e.g. [3]).

For a graph  $G$ , we denote the set neighbours of  $x \in V(G)$  by  $\Gamma_G(x)$  (or just  $\Gamma(x)$  when  $G$  is understood) and the number of edges in a set  $B \subseteq E(G)$  incident with  $x$  by  $d_B(x)$ . We let  $K[V_1, V_2]$  denote the complete bipartite graph with vertex partition  $(V_1, V_2)$ . The term  $[X, Y]$ -edges refers to edges  $xy \in E(G)$  such that  $x \in X$  and  $y \in Y$ . We write  $[x, Y]$ -edges as shorthand for  $\{x\}, Y$ -edges.

Let  $t_2(n) := |E(T_2(n))|$  be the number of edges in the Turán graph  $T_2(n)$ . Recall that  $t_2(n) = \lfloor n^2/4 \rfloor$ . By a *cherry* we mean a path with two edges.

We consider graphs up to isomorphism; in particular, we write  $G = H$  to denote that  $G$  and  $H$  are isomorphic graphs.

**2. Outline of the proof of Theorem 1.3 from [19]**

In this section we give a short outline of the proof of [19, Lemma 5], which was a key step in proving  $\pi_3(n) \leq n^2/2 + o(n^2)$  and is a starting point of our argument towards Theorem 1.5. For an  $n$ -vertex graph  $G$  and each  $i \in \mathbb{N}$ , let  $K_i(G)$  be the set of all  $i$ -cliques in  $G$ . Let  $\pi_{3,f}(G)$  be the minimum of

$$2 \sum_{xy \in K_2(G)} c(xy) + 3 \sum_{xyz \in K_3(G)} c(xyz)$$

over fractional  $\{K_2, K_3\}$ -decompositions  $c$  of  $E(G)$ , that is, over maps  $c: K_2(G) \cup K_3(G) \rightarrow [0, 1]$  such that for every edge  $xy \in E(G)$  we have  $c(xy) + \sum_{z: xyz \in K_3(G)} c(xyz) \geq 1$ . Of course,  $\pi_{3,f}(G) \leq \pi_3(G)$ . By a result of Haxell and Rödl [17] or a more general version by Yuster [28], it also holds that  $\pi_3(G) \leq \pi_{3,f}(G) + o(n^2)$ . So, to show that  $\pi_3(G) \leq n^2/2 + o(n^2)$ , it suffices to consider the fractional equivalent  $\pi_{3,f}(G)$ .

**Lemma 2.1.** *Let  $G$  be an  $n$ -vertex graph. Then*

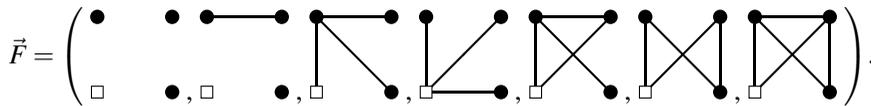
$$\binom{n}{7}^{-1} \sum_{W \in \binom{V(G)}{7}} \pi_{3,f}(G[W]) \leq 21 + o(1),$$

where the sum is taken over 7-vertex subsets  $W$  of  $V(G)$ .

**Outline of proof.** Let  $M$  be the positive semidefinite matrix

$$M := \frac{1}{12 \cdot 10^9} \begin{pmatrix} 1800000000 & 2444365956 & 640188285 & -1524146769 & 1386815580 & -732139362 & -129387078 \\ 2444365956 & 4759879134 & 1177441152 & -1783771230 & 2546923788 & -1397639394 & -143552208 \\ 640188285 & 1177441152 & 484273772 & -317303211 & 1038156300 & -591902130 & -6783162 \\ -1524146769 & -1783771230 & -317303211 & 1558870290 & -651906630 & 305728704 & 154602378 \\ 1386815580 & 2546923788 & 1038156300 & -651906630 & 2285399634 & -1283125950 & -10755036 \\ -732139362 & -1397639394 & -591902130 & 305728704 & -1283125950 & 734039016 & -1621938 \\ -129387078 & -143552208 & -6783162 & 154602378 & -10755036 & -1621938 & 23860164 \end{pmatrix} \geq 0,$$

and let  $\vec{F} := (F_1, \dots, F_7)$  be the following vector of rooted graphs, each having four vertices with the root denoted by the white square:



Take any graph  $G$  of order  $n \rightarrow \infty$ . For  $w \in V(G)$ , let  $\mathbf{v}_{G,w} \in \mathbb{R}^7$  denote the column vector whose  $i$ th component is  $p(F_i, (G, w))$ , the density of the 1-flag  $F_i$  in the rooted graph  $(G, w)$ , which is  $G$  with the vertex  $w$  designated as the root.

It was shown in [19] that

$$\frac{1}{\binom{n}{7}} \sum_{W \in \binom{V(G)}{7}} \pi_{3,f}(G[W]) + \frac{1}{n} \sum_{w \in V(G)} \mathbf{v}_{G,w}^T M \mathbf{v}_{G,w} \leq 21 + o(1). \tag{2.1}$$

Namely, if we rewrite the left-hand side as a linear combination  $\sum_H c_H p(H, G)$ , where  $H$  ranges over all 7-vertex unlabelled graphs and  $p(H, G)$  is the density of  $H$  in  $G$ , then each coefficient  $c_H$  is at most 21. Since  $\sum_H p(H, G) = 1$ , the claimed inequality (2.1) follows.

In particular, since  $M$  is positive semidefinite, the quantity

$$\frac{1}{n} \sum_{w \in V(G)} \mathbf{v}_{G,w}^T M \mathbf{v}_{G,w}$$

is always non-negative, yielding the result. □

The main result of [19], that  $\pi_3(n) \leq n^2/2 + o(n^2)$ , now follows directly from Lemma 2.1.

**Proof of Theorem 1.3.** Let  $G$  be any graph of order  $n \rightarrow \infty$ . As mentioned before,  $\pi_3(G) \leq \pi_{3,f}(G) + o(n^2)$ . Also, we have

$$\binom{n}{2}^{-1} \pi_{3,f}(G) \leq \binom{7}{2}^{-1} \binom{n}{7}^{-1} \sum_{W \in \binom{V(G)}{7}} \pi_{3,f}(G[W]),$$

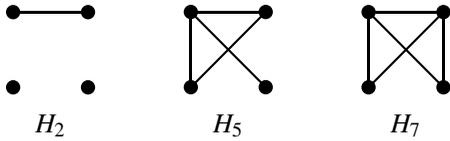


Figure 1. Graphs  $H_2, H_5$  and  $H_7$ .

by averaging optimal fractional decompositions of all 7-vertex induced subgraphs. Combining this inequality with Lemma 2.1 immediately gives that  $\pi_3(G) \leq (1/2 + o(1))n^2$ .  $\square$

### 3. Proof of Theorem 1.5

We use the so-called *stability approach*, where the first step is to describe the approximate structure of all almost  $\pi_3$ -extremal graphs of order  $n \rightarrow \infty$  within  $o(n^2)$  adjacencies. Namely, our Corollary 3.2 will show that every such graph is close to  $K_n$  or  $T_2(n)$ .

For this purpose, we start by showing that all almost  $\pi_3$ -extremal graphs contain almost no copies of the three graphs in Figure 1 (which are obtained by taking the unlabelled versions of the corresponding graphs in  $\vec{F}$ ). This is achieved by the following lemma, which builds on the results from [19].

**Lemma 3.1.** *For every  $c > 0$  there exist  $\varepsilon > 0$  and  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ , if  $G$  is a graph of order  $n$  with  $\pi_3(G) \geq (1/2 - \varepsilon)n^2$ , then  $G$  has at most  $c\binom{n}{4}$  copies of each of the graphs*

$$\begin{aligned} H_2 &:= (\{a, b, c, d\}, \{ab\}), \\ H_5 &:= (\{a, b, c, d\}, \{ab, bc, ac, ad\}), \\ H_7 &:= (\{a, b, c, d\}, \{ab, bc, ac, bd, ad\}) \end{aligned}$$

from Figure 1.

**Proof.** Given  $c > 0$ , let  $\varepsilon \gg 1/n_0 > 0$  be sufficiently small. Let  $G$  be a graph as in the lemma. Let  $M$  and  $\vec{F}$  be as in the proof of Lemma 2.1.

First, the rank of the matrix  $M$  is 6 with  $\mathbf{v} = (1, 0, 3, 1, 0, 3, 0)$  being the only zero eigenvector. (Thus all other eigenvalues of  $M$  are strictly positive by  $M \succeq 0$ .)

Second, by the almost optimality of  $G$  and the fact that each term on the left-hand side of (2.1) is non-negative, we have

$$\sum_{w \in V(G)} \mathbf{v}_{G,w}^T M \mathbf{v}_{G,w} = o_\varepsilon(n). \tag{3.1}$$

We now show that  $G$  must contain few copies of the graphs  $H_2, H_5$  and  $H_7$ . Suppose, for contradiction, that  $G$  contains at least  $c\binom{n}{4}$  copies of  $H_2$ . Then, by a simple double-counting argument, we have that at least  $cn/4$  vertices in  $G$  contain at least  $c\binom{n}{3}/4$  copies of the rooted flag  $F_2$ . In particular, the second coordinate of at least  $cn/4$  of the vectors  $\mathbf{v}_{G,w}$  is at least  $c/4$ . For each such vector  $\mathbf{u}$ , let  $\mathbf{u}' := \mathbf{u}/\|\mathbf{u}\|_2$  be the scalar multiple of  $\mathbf{u}$  of  $\ell^2$ -norm 1. Since  $\|\mathbf{u}\|_2 \leq \sqrt{7}$ , we have that its second coordinate  $u'_2$  is at least  $c/4\sqrt{7}$ . The scalar product of  $\mathbf{u}'$  and the  $\ell^2$ -normalized zero eigenvector  $\mathbf{v}/\sqrt{20}$  (whose second coordinate is 0) is at most

$$\sqrt{1 - (c/4\sqrt{7})^2}.$$

Thus the projection of  $\mathbf{u}$  on the orthogonal complement  $L = \mathbf{v}^\perp$  of the zero eigenspace of  $M$  has  $\ell^2$ -norm at least  $c/4\sqrt{7}$ . Thus  $\mathbf{u}^T M \mathbf{u} \geq \lambda_2 (c/4\sqrt{7})^2$ , where  $\lambda_2 > 0$  is the smallest positive eigenvalue of  $M$  (in fact one can check with the computer that  $\lambda_2 = 0.0005228 \dots$ ). Thus the left-hand

side of (3.1), in which each term is non-negative by  $M \geq 0$ , is at least  $(cn/4) \times \lambda_2(c/4\sqrt{7})^2 = \Omega(n)$ , a contradiction.

The analogous argument shows that the densities of  $H_5$  and  $H_7$  in  $G$  are also at most  $c$ .  $\square$

Let us say that two graphs  $G_1$  and  $G_2$  of the same order are  $k$ -close in the edit distance (or simply  $k$ -close) if there is a relabelling of the vertices of  $G_2$  so that  $|E(G_1) \Delta E(G_2)| \leq k$ . In other words we can make  $G_1$  and  $G_2$  isomorphic by changing at most  $k$  adjacencies.

**Corollary 3.2.** *For every  $\delta > 0$  there exists  $n_1 \in \mathbb{N}$  such that if  $G$  is a graph of order  $n \geq n_1$  with  $\pi_3(G) \geq \ell(n) - n^2/n_1$ , then  $G$  is  $\delta n^2$ -close in edit distance to  $K_n$  or to  $T_2(n)$ .*

**Proof.** Given any  $\delta > 0$ , choose sufficiently small constants  $\delta \gg c \gg 1/n_1 > 0$ . Take any graph  $G$  on  $n \geq n_1$  vertices such that  $\pi_3(G) \geq \ell(n) - n^2/n_1$ .

By Lemma 3.1 and the Induced Removal Lemma [1],  $G$  can be made  $\{H_2, H_5, H_7\}$ -free by changing at most  $cn^2$  adjacencies. Denote this new graph by  $G'$  and note that  $\pi_3(G') \geq \pi_3(G) - 2cn^2$ . By  $c \ll \delta$ , it is enough to show that  $G'$  is  $\delta n^2/2$ -close to  $K_n$  or  $T_2(n)$ .

Let us show that  $G'$  is either triangle-free or the disjoint union of at most two cliques. Indeed, if some vertices  $a, b, c$  span a triangle in  $G'$  then, by the  $\{H_5, H_7\}$ -freeness of  $G$ , all the remaining vertices of  $G'$  have either no or three neighbours among  $\{a, b, c\}$ . Let  $A_0$  be the set of vertices in  $G' \setminus \{a, b, c\}$  which see none of  $\{a, b, c\}$ , and let  $A_3$  be the set of vertices which see all of  $\{a, b, c\}$ . Then  $A_3$  is a clique because  $G'$  is  $H_7$ -free. The set  $A_0$  is also a clique because  $G'$  is  $H_2$ -free. Also, no pair  $xy$  in  $A_3 \times A_0$  can be an edge, as otherwise, for example, the 4-set  $\{a, b, x, y\}$  spans a copy of  $H_5$  in  $G$ . It follows that  $G$  is the disjoint union of the cliques on  $A_0$  and  $A_3 \cup \{a, b, c\}$ , as required.

Now, if  $G'$  is triangle-free, then

$$e(G') = \pi_3(G')/2 \geq \ell(n)/2 - n^2/n_1 - 2cn^2 \geq t_2(n) - 3cn^2.$$

Thus, by the stability result for Mantel's theorem by Erdős [8] and Simonovits [26], the graph  $G'$  must indeed be  $\delta n^2/2$ -close in edit distance to  $T_2(n)$ .

Otherwise  $G'$  is the disjoint union of two cliques. Let us show that one of them has size at most  $\delta n/2$ . Indeed, otherwise  $G'$  has a triangle packing covering all but at most  $n/2 + 2$  edges by Theorem 1.4, meaning that  $\pi_3(G') \leq e(G') + n/2 + 2$ . Also,  $e(G')$  is maximum when clique sizes are as far apart as possible. Thus, by the lower bound on  $\pi_3(G) \leq \pi_3(G') + 2cn^2$ , we conclude that, for example,

$$\ell(n) - 3cn^2 \leq \binom{\delta n/2}{2} + \binom{(1 - \delta/2)n}{2},$$

leading to a contradiction to our choice of constants. Therefore  $G'$  is at most  $n \cdot \delta n/2$  adjacency edits away from  $K_n$ , as desired.  $\square$

The key steps in proving Theorem 1.5 are Lemmas 3.3–3.5.

**Lemma 3.3.** *There exist constants  $\delta > 0$  and  $n_1 \in \mathbb{N}$  such that, among all graphs on  $n \geq n_1$  vertices which are  $\delta n^2$ -close to  $T_2(n)$ , the maximizer of  $\pi_3$  is  $T_2(n)$ .*

**Proof.** Choose sufficiently small  $\varepsilon \gg \delta \gg 1/n_1 > 0$ . Let  $G$  be an arbitrary graph with  $n \geq n_1$  vertices which is  $\delta n^2$ -close to  $T_2(n)$ . We will show that  $\pi_3(G) \leq \pi_3(T_2(n))$  with equality if and only if  $G = T_2(n)$ . In fact this claim can be directly derived from the result of Győri [11, Theorem 1] that a graph with  $n$  vertices and  $t_2(n) + k$  edges, where  $n \rightarrow \infty$  and  $k = o(n^2)$ , has at least  $k - O(k^2/n^2)$  edge-disjoint triangles. More specifically, for each  $\varepsilon > 0$  there exists  $\delta > 0$  and  $n_0 \in \mathbb{N}$  such that

every graph with  $n \geq n_0$  vertices and  $t_2(n) + k$  edges, where  $k \leq \delta n^2$ , has at least  $k - \varepsilon k^2/n^2$  edge-disjoint triangles. (See also [12, Theorem 1] for a generalization of this to  $r$ -cliques for any fixed  $r \geq 3$ .) Since  $G$  is  $\delta n^2$ -close to  $T_2(n)$ , it must have at most  $t_2(n) + \delta n^2$  edges. From this and  $1/n \ll \delta \ll \varepsilon \ll 1$ , we have that, for  $k := e(G) - t_2(n)$ ,

$$\pi_3(G) \leq 2(t_2(n) + k) - 3(k - \varepsilon k^2/n^2) = 2t_2(n) - k(1 - 3\varepsilon k/n^2) \leq 2t_2(n).$$

Clearly, if equality is achieved then  $k = 0$ , i.e.  $e(G) = t_2(n)$ ; furthermore,  $G$  must be triangle-free and thus  $G = T_2(n)$ , as required.  $\square$

Next we need to analyse graphs that are close to  $K_n$ . If  $n \equiv 1, 3 \pmod{6}$ , then let  $\mathcal{E}'_n$  consist of those graphs which are obtained from  $K_n$  by removing a matching of size  $m \equiv 2 \pmod{3}$ ; otherwise let  $\mathcal{E}'_n := \{K_n\}$ . Also, define

$$w(n) := \begin{cases} n/2 & n \equiv 0, 2 \pmod{6}, \\ 2 & n \equiv 1, 3 \pmod{6}, \\ n/2 + 1 & n \equiv 4 \pmod{6}, \\ 4 & n \equiv 5 \pmod{6}. \end{cases}$$

Using Theorem 1.4 and the calculation for  $K_n$  described in Table 1, one can show that  $\pi_3(G) = \binom{n}{2} + w(n)$  for all large  $n$  and every  $G \in \mathcal{E}'_n$ . We are going to show that these are exactly the extremal graphs among those close to  $K_n$ . It is more convenient to do first the case when we have some bound on the minimum degree of a graph and then derive the general case (in a separate Lemma 3.5).

**Lemma 3.4.** *There exist constants  $\delta > 0$  and  $n_0 \in \mathbb{N}$  such that the following holds. Let  $G$  be a graph on  $n \geq n_0$  vertices with minimum degree at least  $n/8$  such that  $G$  is  $\delta n^2$ -close to  $K_n$  and  $\pi_3(G) \geq \binom{n}{2} + w(n)$ . Then  $G \in \mathcal{E}'_n$ .*

**Proof.** Choose small constants in the following order:  $c \gg \delta \gg 1/n_0 > 0$ . Suppose that  $G$  is a graph of order  $n \geq n_0$  as in the statement of the lemma. Let  $w := w(n)$ .

Let

$$U := \{v \in V(G) : d_G(v) \leq (1 - c)n\}.$$

Then

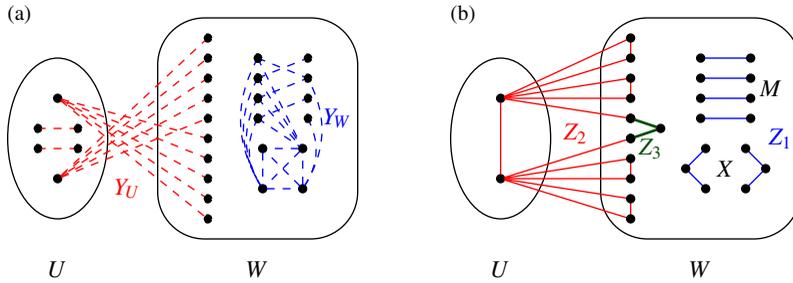
$$\frac{|U|cn}{2} \leq e(\overline{G}) \leq \delta n^2,$$

and so  $|U| \leq (2\delta/c)n$ . Denote  $W := V(G) \setminus U$ , and let  $S := \{v \in W : d_G(v) \text{ is odd}\}$ . Let  $M$  be a set of edges forming a maximum matching in  $G[S]$ , and denote  $X := S \setminus V(M)$ . Then  $X$  is an independent set and thus  $\binom{|X|}{2} \leq \delta n^2$ , which implies that rather roughly

$$|X| < cn. \tag{3.2}$$

Moreover, for every edge  $yz \in M$  and any two distinct vertices  $y', z' \in X$ , at most one of  $yy'$  and  $zz'$  can be an edge of  $G$  (otherwise  $y'yz z'$  is an augmenting path contradicting the maximality of  $M$ ). It follows that if  $|X| \neq 1$ , then for every edge  $yz \in M$  there are at least  $|X|$  edges missing between  $yz$  and  $X$ . Let  $Y_W$  denote the set of missing edges in  $G[W]$ . Thus

$$|Y_W| \geq \binom{|X|}{2} + |M|(|X| - \mathbb{1}_{|X|=1}), \tag{3.3}$$



**Figure 2.** (a) Missing edges in  $Y_W$  are coloured blue and edges in  $Y_U$  are red. (b) Edges in  $Z_1$  are coloured blue, edges in  $Z_2$  are red and in  $Z_3$  green. The same vertices are in (a), where some of the missing edges are dashed. Note that this is a sketch and vertices in  $W$  can be incident to both blue and red (dashed) edges.

where the indicator function  $\mathbb{1}_{|X|=1}$  is 1 if  $|X| = 1$  and is 0 otherwise. Moreover, the set  $Y_U$  of missing edges in  $G$  with at least one endpoint in  $U$  satisfies

$$|Y_U| \geq cn|U| - \binom{|U|}{2} \tag{3.4}$$

by the definition of  $U$ . Note that  $e(G) = \binom{n}{2} - |Y_W| - |Y_U|$ . See Figure 2 for a sketch of  $Y_W$  and  $Y_U$ .

We now build a decomposition  $\mathcal{D}$  of  $G$  into edges and triangles, starting with  $\mathcal{D} = \emptyset$ . If we add edges/triangles to  $\mathcal{D}$ , we regard them as removed from  $E(G)$ . It is convenient to split our argument into the following two cases.

*Case 1.*  $U \neq \emptyset$  or  $S = \emptyset$ .

In this case, our procedure for constructing  $\mathcal{D}$  is as follows.

- Step 1. Add the following to  $\mathcal{D}$  as  $K_2$ 's: the edges of the matching  $M$  and the edges of some  $\lfloor |X|/2 \rfloor$  cherries with distinct endpoints in  $X$  such that their middle points are pairwise distinct.
- Step 2. For each  $u \in U$ , one at a time, add to  $\mathcal{D}$  a maximum set of edge-disjoint  $K_3$ 's containing  $u$  and two vertices from  $W$ . Add all remaining edges incident to vertices in  $U$  as  $K_2$ 's to  $\mathcal{D}$ .
- Step 3. (a) Let  $S' \subseteq V(G)$  be the set of vertices with odd degree after Step 2. Add to  $\mathcal{D}$  the edges of some  $\lfloor |S'|/2 \rfloor$  cherries with distinct endpoints in  $S'$  such that their middle points are pairwise distinct.  
 (b) If the number of remaining edges is not divisible by 3, then fix this by adding to  $\mathcal{D}$  (as single edges) the edge set of some cycle of length 4 or 5.
- Step 4. Add a perfect triangle decomposition of the remaining edges to  $\mathcal{D}$ .

For  $i \in \{1, 2, 3\}$ , let  $Z_i$  be the set of edges that are added to  $\mathcal{D}$  in Step  $i$  as copies of  $K_2$ . See Figure 2 for some illustrations of the above steps.

**Claim.** *The above steps can be carried out as stated. Moreover, the obtained decomposition  $\mathcal{D}$  of  $G$  has at most  $|M| + |X| + \binom{|U|}{2} + 2|U| + 6$  copies of  $K_2$ .*

**Proof of Claim.** In order to carry out Step 1 as stated, we can iteratively pick any two new vertices  $x, y \in X$  and then an arbitrary vertex  $z$  which is suitable as the middle point for a cherry on  $xy$ . Note that the number of choices for  $z$  is at least  $n - 2 - 2cn$ , the number of common neighbours of  $x, y \in X \subseteq W$ , minus  $|X| - 1$ , the number of vertices previously used as middle points. This is positive by (3.2) and  $c \ll 1$ , so we can always proceed. Note for future reference that every vertex

is incident to at most three edges removed in Step 1. Also, Step 1 adds  $|Z_1| = |M| + 2(\lfloor |X|/2 \rfloor) \leq |M| + |X|$  copies of  $K_2$  to  $\mathcal{D}$ .

Clearly, Step 2 can always be processed. Consider the moment when we apply Step 2 to some  $u \in U$ . In the current graph, the induced subgraph  $G[\Gamma(u) \cap W]$  has minimum degree at least  $|\Gamma(u) \cap W| - cn - 3$ , which is at least  $|\Gamma(u) \cap W|/2$  since  $|\Gamma(u)| \geq n/8 - 3$ . So by Dirac's theorem, this subgraph has a matching covering all but at most one vertex, that is, all edges between  $u$  and  $W$  except at most one are decomposed as triangles in Step 2. Let  $U'$  be the set of those  $u \in U$  for which an exceptional edge occurs. Thus we have  $|U'| \leq |U|$  copies of  $K_2$  connecting  $U$  to  $W$  that are added to  $\mathcal{D}$  in Step 2. There are trivially at most  $\binom{|U|}{2}$  edges with both endpoints in  $U$ . So Step 2 adds  $|Z_2| \leq \binom{|U|}{2} + |U|$  copies of  $K_2$  to  $\mathcal{D}$ . Note that all edges incident to  $U$  are decomposed after Step 2.

Since all vertices of  $W$  but at most one had even degrees before Step 2, we have that  $S'$  has at most  $|U'| + 1 \leq |U| + 1$  vertices. As in Step 1, a simple greedy algorithm finds all cherries as stated in Step 3(a). (Note that  $S'$ , as the set of all odd-degree vertices, has even size.)

The minimum degree of  $G[W]$  after Step 3(a) is at least  $0.99n$ , since each  $w \in W$  has at most  $2|U| + 6$  incident edges removed (at most  $2|U|$  from Step 2 and at most 3 in each of Steps 1 and 3(a)). Thus we can find the required 4- or 5-cycle in Step 3(b).

Clearly, we add  $|Z_3| \leq |S'| + 5 \leq |U| + 6$  copies of  $K_2$  to  $\mathcal{D}$  in Step 3.

Note that, at the end of Step 3, the graph  $G[W]$  has minimum degree at least, say,  $0.98n$  while all its degrees are even. By Theorem 1.4, all remaining edges can be decomposed using only triangles, so Step 4 indeed removes all remaining edges.

Step 4 adds no additional  $K_2$ 's, so the total number of  $K_2$ 's in  $\mathcal{D}$  is

$$|Z_1| + |Z_2| + |Z_3| \leq |M| + |X| + \binom{|U|}{2} + 2|U| + 6,$$

finishing the proof of the claim. □

Now we compute the cost of  $\mathcal{D}$ . Using the notation from above, we have

$$\begin{aligned} w &\leq \pi_3(G) - \binom{n}{2} \\ &\leq -|Y_U| - |Y_W| + |Z_1| + |Z_2| + |Z_3| \\ &\leq -|Y_U| - |Y_W| + |M| + |X| + \binom{|U|}{2} + 2|U| + 6. \end{aligned} \tag{3.5}$$

Substituting the bounds from (3.3) and (3.4) and rearranging the terms, we get

$$w \leq \left( 2 \binom{|U|}{2} + 2|U| - cn|U| + 6 \right) + (3 - |X|) \left( \frac{|X|}{2} + |M| \right) + (\mathbb{1}_{|X|=1} - 2)|M|. \tag{3.6}$$

First, suppose that  $|U| > 0$ . Then the estimate  $|U| \leq 2\delta n/c$  yields that

$$2 \binom{|U|}{2} + 2|U| - cn|U| + 6 \leq -cn|U|/2 \leq -cn/2.$$

Since  $w \geq 2$ , we must have that  $|X| \leq 1$ . Observe that  $n$  is odd as otherwise  $w \geq n/2$  and, by  $|M| \leq n/2$ , the cases  $|X| \in \{0, 1\}$  also contradict (3.6). So every vertex of degree  $n - 1$  has even degree, meaning that every vertex of  $S$  is in some pair from  $Y_W$  or  $Y_U$ . Hence  $2|M| \leq 2|Y_W| + |Y_U|$ . Substituting this into the right-hand side of (3.5) and using our bound on  $|Y_U|$  from (3.4), we obtain

$$w \leq -\frac{|Y_U|}{2} + |X| + \binom{|U|}{2} + 2|U| + 6 \leq \frac{3}{2} \binom{|U|}{2} + 2|U| - \frac{cn|U|}{2} + 7,$$

which again is negative for  $|U| > 0$  and large  $n$ , contradicting  $w \geq 2$ .

Thus  $U$  is empty and, by the assumption of Case 1,  $S$  is also empty (and so are  $X$  and  $M$ ). This gives that the initial graph  $G$  has minimum degree at least  $(1 - c)n$ ,  $|Z_1| = |Z_2| = 0$ ,  $S' = \emptyset$ , and no  $K_2$ 's are added to  $\mathcal{D}$  in Step 3(a).

If  $n$  is even, then every vertex of  $G$  has at least one missing edge,

$$e(G) \leq \binom{n}{2} - \frac{n}{2},$$

and

$$\pi_3(G) \leq \binom{n}{2} - \frac{n}{2} + |Z_3| \leq \binom{n}{2} - \frac{n}{2} + 5,$$

which is strictly less than  $\pi_3(K_n)$ , a contradiction.

Let  $n$  be odd and let  $r := \binom{n}{2} - e(G)$  be the number of missing edges in  $G$ . Suppose that  $r > 0$ , as otherwise  $G = K_n$  and we are done. The upper bound on  $\pi_3(G)$  given by  $\mathcal{D}$  is  $\rho_r + \binom{n}{2} - r$ , where we define  $\rho_r$  as the unique element of  $\{0, 4, 5\}$  with  $\binom{n}{2} - \rho_r - r \equiv 0 \pmod{3}$ . Therefore  $r \leq 3$  as otherwise  $\pi_3(G) \leq \binom{n}{2} + 1$ , contradicting  $w \geq 2$ . On the other hand all the degrees of  $\bar{G}$  are even so  $r = 3$  and the only non-empty component of  $\bar{G}$  is a triangle. However, this contradicts  $w \geq 2$  because

$$\pi_3(G) = \begin{cases} \binom{n}{2} - 1 & n \equiv 1, 3 \pmod{6}, \\ \binom{n}{2} + 1 & n \equiv 5 \pmod{6}. \end{cases}$$

Case 2.  $U = \emptyset$  and  $S \neq \emptyset$ .

Some things simplify in this case (as we do not need to deal with  $U$ ). On the other hand we have to be a bit more careful with calculations, as the new extremal graphs ( $K_n$  minus a matching) fall into this case. In particular, removing a 4- or 5-cycle may be too wasteful here. So we construct a decomposition  $\mathcal{D}$  of  $G$  as follows. Recall that  $M$  is a maximum matching in  $G[S]$  and  $X$  is the set of vertices of  $S$  not matched by  $M$ .

Step 1. Make the graph triangle-divisible by removing the following as  $K_2$ 's. If  $X = \emptyset$ , then remove all but one edge  $xy \in M$  and a path of length  $\rho + 1 \in \{1, 2, 3\}$  whose endpoints are  $x$  and  $y$  (thus, for  $\rho = 0$ , we remove just the matching  $M$ ). If  $X$  is non-empty, then remove  $M$  and the edge sets of some  $|X|/2 - 1$  paths of length 2 and one path of length  $\rho + 2 \in \{2, 3, 4\}$  so that their degree-1 vertices partition  $X$  and their degree-2 vertices are pairwise distinct.

Step 2. Decompose the rest perfectly into triangles.

Note that  $S$ , the set of all odd-degree vertices of  $G$ , has even size (and also  $|X| = |S| - 2|M|$  is even). Since the minimal degree of  $G$  is at least  $(1 - c)n$ , a simple greedy algorithm achieves Step 1 (and Theorem 1.4 takes care of Step 2).

The decomposition  $\mathcal{D}$  has exactly  $|M| + |X| + \rho$  copies of  $K_2$ . Also,  $e(G) = \binom{n}{2} - |Y_W|$ . Thus

$$w \leq \pi_3(G) - \binom{n}{2} \leq -|Y_W| + |M| + |X| + \rho. \tag{3.7}$$

Using (3.3) and that  $|X| \neq 1$  (since  $|X|$  is even), we obtain

$$w \leq (3 - |X|) \left( \frac{|X|}{2} + |M| \right) - 2|M| + \rho. \tag{3.8}$$

Moreover,  $|X| \leq 2$  as otherwise  $2 \leq w \leq \rho - 2 - 3|M|$ , contradicting  $\rho \leq 2$ . Thus  $X$  has either 0 or 2 elements.

Suppose that  $X = \emptyset$ . First, let  $n$  be even. Then every vertex not in  $S$  is incident to at least one non-edge of  $G$ ,  $|Y_W| \geq (n - 2|M|)/2$ , and by (3.7),

$$n/2 \leq w \leq 2|M| + \rho - n/2.$$

If  $2|M| \leq n - 2$ , then all inequalities here become equalities and thus  $|M| = (n - 2)/2$ ,  $|Y_W| = 1$ ,  $\rho = 2$ ,  $w = n/2$ , and  $n \equiv 0, 2 \pmod{6}$ . However, then the graph after Step 1 has exactly

$$\binom{n}{2} - 1 - \frac{n - 2}{2} - 2$$

edges, which is not divisible by 3, a contradiction. Thus  $2|M| = n$ , the copies of  $K_2$  in the decomposition contain a perfect matching of  $G$ , and  $\pi_3(G) \leq \pi_3(K_n)$  with equality only if  $G = K_n$ , as desired. So suppose that  $n$  is odd. Since every vertex of  $S$  has to be incident to a missing edge of  $G$ , we have  $|Y_W| \geq |S|/2 = |M|$  and the bound in (3.7) becomes  $w \leq \rho$ . It follows that we have equality throughout,  $|Y_W| = |M|$ ,  $w = \rho = 2$ ,  $n \equiv 1, 3 \pmod{6}$ , and  $\binom{n}{2} - |M| - \rho \equiv 0 \pmod{3}$ ; the last gives that  $|M| \equiv 2 \pmod{3}$ . Thus  $G$  is as required.

Finally, it remains to consider the case when  $|X| = 2$ . This time, (3.8) yields that

$$2 \leq w \leq \rho - |M| + 1 \leq 3.$$

Therefore  $|M| \leq 1$ , and  $n \equiv 1, 3 \pmod{6}$  as otherwise  $w \geq 4$ . If  $|M| = 1$ , then we have equality everywhere, giving  $w = \rho = 2$ ,  $|S| = 4$  and  $|Y_W| = 3$ . However, then the graph after Step 1 has

$$\binom{n}{2} - |Y_W| - |M| - |X| - \rho = \binom{n}{2} - 8$$

edges, which is not divisible by 3, a contradiction. Thus  $M$  is empty,  $\rho \in \{1, 2\}$  and  $S = X$ . By (3.7),  $|Y_W| \leq 2$  and hence  $|Y_W| = 1$ . In other words,  $G = K_n^-$ . However, then the graph after Step 1 has

$$\binom{n}{2} - 1 - (2 + \rho)$$

edges, which is not divisible by 3. (Alternatively, Theorem 1.4 gives that  $\pi_3(K_n^-) - \binom{n}{2} < 2 = w$ .) This contradiction finishes Case 2 and the proof of the lemma.  $\square$

**Lemma 3.5.** *There exist constants  $\delta > 0$  and  $n_1 \in \mathbb{N}$  such that the following holds. Let  $G$  be a graph on  $n \geq n_1$  vertices maximizing  $\pi_3(G)$  among all graphs that are  $\delta n^2$ -close to  $K_n$ . Then  $G \in \mathcal{E}'_n$ .*

**Proof.** Let  $n_0$  and  $\delta$  be the constants from Lemma 3.4. We claim that, for example,  $n_1 := 2n_0$  is enough for the conclusion of Lemma 3.5 to hold. Indeed, take any extremal graph  $G$  of order  $n \geq n_1$ . If  $G$  satisfies the assumption on minimum degree of Lemma 3.4, then we are done. Hence assume that the minimum degree of  $G$  is less than  $n/8$ . Let  $G_n := G$ , and iteratively define a sequence of graphs  $G_{n-1}, G_{n-2}, \dots$  as follows. Given a graph  $G_i$  of order  $i$ , if it has a vertex  $x$  of degree less than  $i/8$ , let  $G_{i-1} := G_i - x$  be obtained from  $G_i$  by removing the vertex  $x$ ; otherwise stop. Note that the process does not reach  $i < n/2$  for otherwise  $G$  has roughly at least  $(n/2) \times (n/4)$  non-edges, which is a contradiction to  $G$  being  $\delta n^2$ -close to  $K_n$ .

Let  $G_s$  with  $|G_s| = s \geq n/2 \geq n_0$  be the graph for which the above process terminates. By Lemma 3.4, we have that  $\pi_3(G_s) \leq s^2/2 + 1$ . By decomposing all edges in  $E(G) \setminus E(G_s)$  as  $K_2$ 's, we obtain

$$\pi_3(G_n) \leq \pi_3(G_s) + 2(n - s) \cdot \frac{n}{8} \leq \frac{s^2}{2} + 1 + (n - s) \cdot \frac{n}{4}.$$

This is a convex function in  $s$  so it is maximized on the boundary of  $n/2 \leq s \leq n - 1$ . If  $s = n/2$ , we get

$$\pi_3(G_n) \leq n^2/4 + 2 < \binom{n}{2} \leq \pi_3(K_n).$$

If  $s = n - 1$ , we get

$$\pi_3(G_n) \leq \pi_3(G_s) + 2(n - s) \cdot \frac{n}{8} \leq \frac{(n - 1)^2}{2} + 1 + \frac{n}{4} \leq \binom{n}{2} - \frac{n}{4} + 2 < \pi_3(K_n).$$

In both cases, we get a contradiction to  $G_n$  being extremal. □

**Proof of Theorem 1.5.** Choose sufficiently small constants in this order  $1 \gg \delta \gg 1/n_0 > 0$ . In particular,  $n_0$  is sufficiently large to satisfy Corollary 3.2 for this  $\delta$  as well as Lemmas 3.3 and 3.5. Let  $G$  be an arbitrary graph of order  $n \geq n_0$  with  $\pi_3(G) \geq \ell(n)$ . By Corollary 3.2,  $G$  is  $\delta n^2$ -close to either  $T_2(n)$  or  $K_n$ .

If  $G$  is close to  $T_2(n)$  then it must be  $T_2(n)$  by Lemma 3.3. If  $G$  is close to  $K_n$  then it must be in  $\mathcal{E}'_n$  by Lemma 3.5. By comparing the costs of optimal decompositions, we conclude that  $G \in \mathcal{E}_n$ . □

#### 4. Extension to an arbitrary cost $\alpha$

The goal of this section is to prove Theorem 1.6. Everywhere in this section, let  $n$  be sufficiently large.

First, note that the case  $\alpha \geq 6$  is trivial. Indeed, the cost of a triangle is not better than a cost of three edges. Thus, for every graph  $G$ , an optimal decomposition is to decompose all edges of  $G$  as  $K_2$ 's. The unique graph maximizing the number of edges is  $K_n$ , so it is also the unique maximizer of  $\pi_3^\alpha$  for every  $\alpha \geq 6$ .

Next let us make some easy general observations which apply when  $\alpha < 6$ . First,

$$\pi_3^\alpha(G) = \alpha v(G) + 2(e(G) - 3v(G)) = 2e(G) - (6 - \alpha)v(G),$$

where  $v(G)$  denotes the maximum number of edge-disjoint triangles contained in  $G$ . Also, if  $\alpha_1 \leq \alpha_2 < 6$ ,  $v(G_1) \geq v(G_2)$  and  $\pi_3^{\alpha_1}(G_1) > \pi_3^{\alpha_1}(G_2)$  for some graphs  $G_1$  and  $G_2$ , then

$$\pi_3^{\alpha_2}(G_1) - \pi_3^{\alpha_2}(G_2) = \pi_3^{\alpha_1}(G_1) - \pi_3^{\alpha_1}(G_2) + (\alpha_2 - \alpha_1)(v(G_1) - v(G_2)) > 0. \tag{4.1}$$

In particular, if  $K_n$  is the maximizer of  $\pi_3^{\alpha_1}$ , it is also a maximizer for  $\pi_3^{\alpha_2}$ .

##### 4.1 The case $\alpha < 3$

Next we discuss the case  $\alpha < 3$ . Let  $n$  be large and let  $G$  be a  $\pi_3^\alpha(n)$ -extremal graphs. Since

$$\pi_3^3(G) \geq \pi_3^\alpha(G) \geq \pi_3^\alpha(T_2(n)) = \pi_3^3(T_2(n)) = (1/2 + o(1))n^2,$$

Corollary 3.2 gives that  $G$  is  $o(n^2)$ -close to  $K_n$  or  $T_2(n)$ . Since  $\alpha < 3$ , we have that  $\pi_3^\alpha(T_2(n)) \geq (1 + \Omega(1))\pi_3^\alpha(K_n)$  and thus  $G$  is close to  $T_2(n)$ . Now, Lemma 3.3 implies that  $\pi_3^\alpha(G) \leq \pi_3^3(G) \leq \pi_3^3(T_2(n)) = \pi_3^\alpha(T_2(n))$ , with equality if and only if  $G = T_2(n)$ , as desired.

##### 4.2 The case $3 < \alpha < 4$

This subsection proves Theorem 1.6 for  $3 < \alpha < 4$ .

First let us show that every  $\pi_3^\alpha$ -maximizer  $G$  is in  $K_n$  or  $K_n^=$ . Suppose for a contradiction that  $G$  violates this. In particular, we have  $\pi_3^\alpha(G) \geq \pi_3^\alpha(K_n)$ . By (4.1), we have that  $\pi_3^3(G) \geq \pi_3^3(K_n)$ . For

$n \rightarrow \infty$ , it holds by Table 1 that  $\pi_3^\alpha(K_n) \geq (1 + \Omega(1)) \pi_3^\alpha(T_2(n))$ . Hence  $G$  needs to be close to  $K_n$  and Lemma 3.5 applies to  $G$ . In particular, this means that  $n \equiv 1, 3 \pmod{6}$ . Lemma 3.5 gives that all  $\pi_3^3$ -extremal graphs are obtained from  $K_n$  by removing a matching of size congruent to 2 modulo 3. It follows from (4.1) that, among these graphs,  $\pi_3^\alpha$  is strictly maximized by  $K_n^-$  since this graph has the largest  $v$ .

Theorem 1.4 gives that  $3v(K_n^-) = \binom{n}{2} - 6$ . Since  $\pi_3^\alpha(G) \geq \pi_3^\alpha(K_n^-)$  and  $\pi_3^3(G) < \pi_3^3(K_n^-)$ , this implies by (4.1) that  $v(G) > v(K_n^-)$ . Since also  $v(G) < v(K_n)$  (otherwise  $\pi_3^\alpha(G) < \pi_3^\alpha(K_n)$ ), we conclude that  $3v(G) = \binom{n}{2} - 3$ , that is, exactly three pairs of vertices of  $G$  are not included in some triangle from an optimal decomposition of  $G$ . This implies that  $G$  is a complete graph without one edge, or a path on three vertices, or a triangle. Among these three candidates (that have the same  $v$ ),  $K_n^-$  has the largest size and thus maximizes  $\pi_3^\alpha$ . So  $K_n^-$  is the only possible candidate for  $G$ . However,  $\pi_3^\alpha(K_n^-) > \pi_3^\alpha(K_n)$  if  $\alpha < 4$ . This contradiction finishes the proof for  $3 < \alpha < 4$ .

Thus every  $\pi_3^\alpha$ -maximizer is in  $\{K_n, K_n^-\}$ . It remains to compare these two graphs. Calculations based on Theorem 1.4 show that

$$\frac{\pi_3^\alpha(K_n^-) - \pi_3^\alpha(K_n) + 4}{6 - \alpha} = v(K_n) - v(K_n^-) = \begin{cases} 0 & n \equiv 0, 2, 4, 5 \pmod{6}, \\ 2 & n \equiv 1, 3 \pmod{6}. \end{cases}$$

Thus  $\pi_3^\alpha(K_n) > \pi_3^\alpha(K_n^-)$  if  $n \equiv 0, 2, 4, 5 \pmod{6}$  and  $\pi_3^\alpha(K_n^-) > \pi_3^\alpha(K_n)$  otherwise, as required.

**4.3 The case  $4 \leq \alpha < 6$**

In this case we provide a direct proof, without using flag algebras or fractional decompositions. Let  $n$  be large and let  $G$  be any graph of order  $n$  such that  $\pi_3^\alpha(G) = \pi_3^\alpha(n)$ . Let  $\mathcal{D}$  be a decomposition of  $G$  with minimum weight consisting of  $t$  triangles and  $\ell$  edges.

If  $G$  is a complete graph, then we are done. Hence we assume there exists some pair of vertices  $x, y \in G$  such that  $xy \notin E(G)$ . Let  $G'$  be obtained from  $G$  by adding the edge  $xy$ . Let  $\mathcal{D}'$  be an optimal decomposition of  $G'$  containing  $t'$  triangles and  $\ell'$  edges. Recall that finding an optimal decomposition is equivalent to maximizing a triangle packing, that is,  $t' = v(G')$ . Hence  $t' \geq t$ .

If  $xy$  is used as an edge in  $\mathcal{D}'$ , then removing  $xy$  from  $\mathcal{D}'$  gives a decomposition of  $G$  with cost  $\pi_3^\alpha(G') - 2$ , contradicting the maximality of  $G$ . Therefore  $xy$  must appear in a triangle  $xyz \in \mathcal{D}'$ . We now construct a decomposition  $\mathcal{D}^*$  of  $G$  by removing  $xyz$  from  $\mathcal{D}'$  and adding the edges  $xz$  and  $yz$ . Since the total cost of  $\mathcal{D}^*$  is  $\alpha(t' - 1) + 2(\ell' + 2)$ , we have

$$\pi_3^\alpha(G) \leq \text{cost}(\mathcal{D}^*) = \alpha(t' - 1) + 2(\ell' + 2) = \alpha t' + 2\ell' - \alpha + 4 \leq \alpha t' + 2\ell' = \pi_3^\alpha(G'),$$

which contradicts the maximality of  $\pi_3^\alpha(G)$  if at least one of the inequalities is strict. Hence  $\alpha = 4$ ,  $xy$  must be in a triangle in  $\mathcal{D}'$ , and  $\pi_3^\alpha(G') = \pi_3^\alpha(n)$ .

This means that it is possible to keep adding edges to  $G$ , which results in a sequence of graphs  $G, G', \dots, K_n$  where an optimal decomposition of each of these graphs has cost  $\pi_3^\alpha(n)$ , i.e. they are all  $\pi_3^\alpha$ -extremal graphs. Note that we can add missing edges to  $G$  in any order, always obtaining a sequence of extremal graphs.

This allows us to reverse the process and examine a sequence of edge removals from  $K_n$ .

Suppose that  $G$  is obtained from  $K_n$  by removing the edge  $xy$ , i.e.  $G'$  is  $K_n$ . Note that if  $\ell' > 0$ , i.e. the optimal decomposition of  $K_n$  contains an edge, then there exists an option for  $\mathcal{D}'$  that contains the edge  $xy$ , which was already ruled out. This means that  $K_n$  is triangle-divisible, which is the case if and only if  $n \equiv 1, 3 \pmod{6}$ .

Now assume that  $G$  is missing more than one edge. Hence  $K_n^-$  must be also extremal. By the above,  $n \equiv 1, 3 \pmod{6}$ ,  $K_n$  is triangle-divisible, and  $\pi_3^4(n) = 4v(K_n)$ , where  $v(K_n) = \frac{1}{3} \binom{n}{2}$ .

Suppose that  $G$  is obtained from  $K_n$  by removing two edges  $uv$  and  $xy$ . First suppose that  $u = x$ . Let  $\mathcal{D}^*$  be a decomposition of  $G$  into triangles and one edge  $vy$ . This gives

$$\pi_3^4(G) \leq \text{cost}(\mathcal{D}^*) = 4(v(K_n) - 1) + 2 < 4v(K_n) = \pi_3^4(n),$$

contradicting the maximality of  $\pi_3^4(G)$ . Hence  $xy$  and  $uv$  form a matching. Note that  $x, y, u$  and  $v$  have odd degrees in  $G$ , so  $\ell \geq 2$ , for else we are unable to fix the parity of the vertices  $x, y, u$  and  $v$ . Now  $\binom{n}{2} - \ell - 2$  needs to be divisible by 3, so  $\ell \geq 4$ . There indeed exists a decomposition with  $\ell = 4$  by taking edges  $xu, xv, yu$  and  $yv$  and the rest as triangles. This gives

$$\pi_3^4(G) = 4(v(K_n) - 2) + 2 \cdot 4 = \pi_3^4(n).$$

Therefore  $G$  is extremal.

Suppose that  $G$  is obtained from  $K_n$  by removing three edges  $uv, xy$  and  $zw$ . Since  $G'$  must be  $K_n$  without a matching,  $uv, xy$  and  $zw$  also form a matching. Let  $\mathcal{D}^*$  be a decomposition of  $G$  into triangles and edges  $ux, yz$  and  $vw$ . This gives

$$\pi_3^4(G) \leq \text{cost}(\mathcal{D}^*) = 4(v(K_n) - 2) + 6 < 4v(K_n) = \pi_3^4(n),$$

contradicting the maximality of  $\pi_3^4(G)$ . This implies that  $G$  cannot be obtained from  $K_n$  by deleting three or more edges, thus finishing the proof of this case and of Theorem 1.6.

### 5. Related results

A related question of Erdős (see e.g. [9]) asks for the largest  $t = t(n, m)$  such that every graph with  $n$  vertices and  $t_2(n) + m$  edges has at least  $t$  edge-disjoint triangles. Of course,  $t \leq m$ . Győri [11] (see [13] for a correction) showed, for large  $n$ , that  $t \geq m - O(m^2/n^2)$  if  $m = o(n^2)$ , and  $t = m$  if  $n$  is odd and  $m \leq 2n - 10$  or  $n$  is even and  $m \leq 3n/2 - 5$ . Moreover, the last two bounds on  $m$  are sharp.

More recently, Győri and Keszegh [14] proved that every  $K_4$ -free graph with  $t_2(n) + m$  edges has  $m$  edge-disjoint triangles.

Theorem 1.5 shows that the maximum of  $\pi_3(G)$  is attained for  $G = T_2(n)$  or  $G = K_n$ . However, if we restrict the set of graphs under consideration to graphs of a particular edge density, the decomposition is perhaps cheaper. Note that if the optimal decomposition of a graph  $G$  contains  $t$  triangles and  $\ell$  edges, then  $\pi_3(G) = 2e(G) - 3t$ . That is, we have that  $\pi_3(G) = 2e(G) - 3v(G)$ , where as before  $v(G)$  denotes the maximum number of edge-disjoint triangles in  $G$ . Then Theorem 1.3 implies an inequality between the edge density of  $G$  and its *triangle packing density*, which we denote by  $v_d(G) := 3v(G)/\binom{n}{2}$ .

**Corollary 5.1** (of Theorem 1.3). *Let  $G$  be a graph with  $d\binom{n}{2}$  edges. Then*

$$v_d(G) \geq d - 1 + o(1).$$

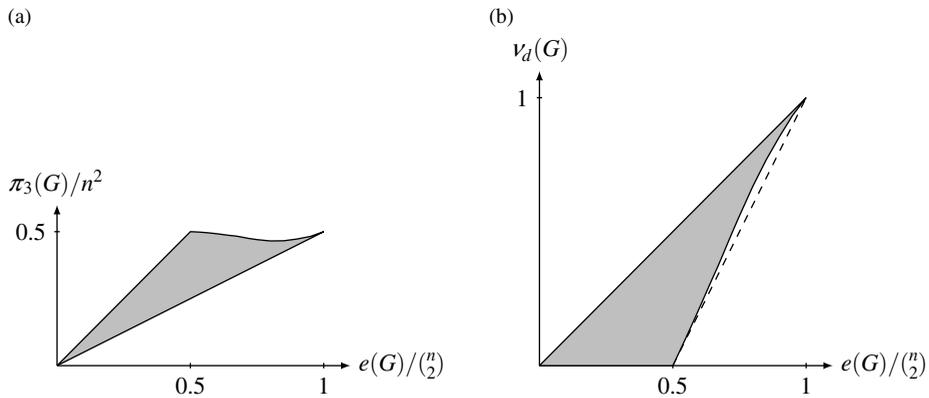
We also have that  $v_d(G) \leq d$ , which is tight for all graphs which are the union of edge-disjoint triangles.

A question reminiscent of the seminal result of Razborov on the minimal triangle density in graphs [25] (see also [22] and [20]) would be to determine the exact lower bound on  $v_d(G)$  in terms of  $d$  (answering asymptotically the question of Erdős stated above).

Some flag algebra computations yield numerical asymptotic lower bounds on  $v_d(G)$  with different edge densities between 0.5 and 1. The result, depicted in Figure 3, suggests that the true asymptotic shape of the region  $\{(d, v_d(G)) : 0 \leq d \leq 1, G \text{ graph}\}$  may indeed have a richer structure.

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**Figure 3.** Asymptotic bounds on possible values of  $\pi_3(G)$  and  $v_d(G)$ . The dashed line is simply  $y = 2x - 1$  for a better display of the shape.

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## References

- [1] Alon, N., Fischer, E., Krivelevich, M. and Szegedy, M. (2000) Efficient testing of large graphs. *Combinatorica* **20** 451–476.
- [2] Barber, B., Kühn, D., Lo, A. and Osthus, D. (2016) Edge-decompositions of graphs with high minimum degree. *Adv. Math.* **288** 337–385.
- [3] Bollobás, B. (1998) *Modern Graph Theory*. Springer.
- [4] Chung, F. R. K. (1981) On the decomposition of graphs. *SIAM J. Algebraic Discrete Methods* **2** 1–12.
- [5] Delcourt, M. and Postle, L. (2019) Progress towards Nash-Williams’ conjecture on triangle decompositions. [arXiv:1909.00514](https://arxiv.org/abs/1909.00514)
- [6] Dross, F. (2016) Fractional triangle decompositions in graphs with large minimum degree. *SIAM J. Discrete Math.* **30** 36–42.
- [7] Dukes, P. and Horsley, D. (2020) On the minimum degree required for a triangle decomposition. *SIAM J. Discrete Math.* **34** 597–610.
- [8] Erdős, P. (1967) Some recent results on extremal problems in graph theory: results. In *Theory of Graphs (Internat. Sympos., Rome, 1966)*, pp. 117–123 (English), pp. 124–130 (French). Gordon & Breach.
- [9] Erdős, P. (1971) Some unsolved problems in graph theory and combinatorial analysis. In *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, pp. 97–109. Academic Press.
- [10] Erdős, P., Goodman, A. W. and Pósa, L. (1966) The representation of a graph by set intersections. *Canad. J. Math.* **18** 106–112.
- [11] Győri, E. (1988) On the number of edge-disjoint triangles in graphs of given size. In *Combinatorics (Eger, 1987)*, Vol. 52 of *Colloquia Mathematica Societatis János Bolyai*, pp. 267–276. North-Holland.
- [12] Győri, E. (1991) On the number of edge disjoint cliques in graphs of given size. *Combinatorica* **11** 231–243.
- [13] Győri, E. (1992) Edge disjoint cliques in graphs. In *Sets, Graphs and Numbers (Budapest, 1991)*, Vol. 60 of *Colloquia Mathematica Societatis János Bolyai*, pp. 357–363. North-Holland.
- [14] Győri, E. and Keszegh, B. (2017) On the number of edge-disjoint triangles in  $K_4$ -free graphs. *Combinatorica* **37** 1113–1124.
- [15] Győri, E. and Kostochka, A. V. (1979) On a problem of G. O. H. Katona and T. Tarján. *Acta Math. Acad. Sci. Hungar.* **34** 321–327.
- [16] Győri, E. and Tuza, Z. (1987) Decompositions of graphs into complete subgraphs of given order. *Studia Sci. Math. Hungar.* **22** 315–320.
- [17] Haxell, P. E. and Rödl, V. (2001) Integer and fractional packings in dense graphs. *Combinatorica* **21** 13–38.
- [18] Kahn, J. (1981) Proof of a conjecture of Katona and Tarján. *Period. Math. Hungar.* **12** 81–82.
- [19] Král’, D., Lidický, B., Martins, T. L. and Pehova, Y. (2019) Decomposing graphs into edges and triangles. *Combin. Probab. Comput.* **28** 465–472.

- [20] Liu, H., Pikhurko, O. and Staden, K. (2020) The exact minimum number of triangles in graphs of given order and size. *Forum Math. Pi* **8** e8.
- [21] Nash-Williams, C. (1970) An unsolved problem concerning decomposition of graphs into triangles. In *Combinatorial Theory and Its Applications III*, pp. 1179–1183. North-Holland.
- [22] Pikhurko, O. and Razborov, A. (2017) Asymptotic structure of graphs with the minimum number of triangles. *Combin. Probab. Comput.* **26** 138–160.
- [23] Pyber, L. (1992) Covering the edges of a graph by . . . . In *Sets, Graphs and Numbers (Budapest, 1991)*, Vol. 60 of *Colloquia Mathematica Societatis János Bolyai*, pp. 583–610. North-Holland.
- [24] Razborov, A. (2007) Flag algebras. *J. Symb. Logic* **72** 1239–1282.
- [25] Razborov, A. (2008) On the minimal density of triangles in graphs. *Combin. Probab. Comput.* **17** 603–618.
- [26] Simonovits, M. (1968) A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pp. 279–319. Academic Press.
- [27] Tuza, Z. (2001) Unsolved combinatorial problems, Part I. BRICS Lecture Series LS-01-1.
- [28] Yuster, R. (2005) Integer and fractional packing of families of graphs. *Random Struct. Algorithms* **26** 110–118.

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