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LINEARIZATION AND BOUNDARY TRAJECTORIES OF NONSMOOTH CONTROL SYSTEMS

H. FRANKOWSKA AND B. KAŠKOSZ

1. Introduction. This paper deals with boundary trajectories of non-smooth control systems and differential inclusions.

Consider a control system

$$(1.1)\qquad \begin{cases} x' = f(t, x, u(t)),\ u(t) \in U,\ t \in [0, 1] \\ x(0) = \xi \end{cases}$$

and denote by $R(t)$ its reachable set at time t. Let (z, u_*) be a trajectory-control pair. If for every t from the time interval $[0, 1]$, $z(t)$ lies on the boundary of $R(t)$ then z is called a boundary trajectory. It is known that for systems with Lipschitzian in x right-hand side, z is a boundary trajectory if and only if $z(1)$ belongs to the boundary of the set $R(1)$. If z is not a boundary trajectory, that is, $z(1) \in \operatorname{Int} R(1)$ then the system is said to be locally controllable around z at time 1.

A first-order necessary condition for boundary trajectories of smooth systems comes from the Pontriagin maximum principle, (see e.g. [**12**]). The principle says that if f is continuously differentiable with respect to x, (plus satisfies some more technical assumptions), and z is a boundary trajectory then there exists a nontrivial solution p to the system:

$$(1.2)\qquad -p'(t) = p(t)\frac{\partial f}{\partial x}(t, z(t), u_*(t)),\ p(0) \neq 0$$

$$(1.3)\qquad \langle p(t), z'(t)\rangle = \max_{u \in U} \langle p(t), f(t, z(t), u)\rangle \quad \text{a.e. in } [0, 1].$$

On the other hand, the solution $w \equiv 0$ of the linear system:

$$(1.4)\qquad \begin{cases} w' = \dfrac{\partial f}{\partial x}(t, z(t), u_*(t))w + v,\ v \in f(t, z(t), U) - z'(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [0, 1], \\ w(0) = 0 \end{cases}$$

is a boundary trajectory of (1.4) if and only if a nontrivial solution p of (1.2), (1.3) does exist. Hence local controllability around $w \equiv 0$ at time 1 of the linear system (1.4), (the linearization of (1.1) along z, u_*), is a sufficient condition for $z(1) \in \operatorname{Int} R(1)$.

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In 1976 F. H. Clarke studied boundary trajectories when f is a Lipschitzian function in x, (not necessarily continuously differentiable), ([**3**], [**2**]). He introduced the generalized Jacobian $\partial_x f$ of f with respect to x and proved that in the Lipschitzian case the same maximum principle (1.2), (1.3) is valid with the linear system (1.2) replaced by the "linear" differential inclusion:

$$(1.5) \qquad -p'(t) \in p(t)\partial_x f(t, z(t), u_*(t)).$$

His approach is based on a powerful tool of nonsmooth (and smooth) analysis; the Ekeland variational principle and several techniques allowing to reduce the problem to an unconstrained Bolza problem. (See also [**5**]). Another approach through smooth approximations and derivate containers was developed at the same time by Warga [**18**].

However, the existing proofs do not allow to directly relate controllability of the "linearized" nonsmooth system and local controllability of (1.1) around z at time 1. This creates a gap between the usual approach to smooth systems, (see for example [**12**]), and the techniques used to study nonsmooth systems. The aim of this paper is to fill the gap.

We express the violation of conditions (1.3), (1.5) in terms of controllability of a family of linear systems, (linearizations of (1.1) along z). Using a generalization of the classical open mapping principle by Warga [**19**], we prove that controllability of linearized systems implies local controllability of the original system. This in turn, implies that every boundary trajectory of (1.1) satisfies the maximum principle (1.3), (1.5).

The maximum principle is not a new result, (it is a special case of Theorem 5.1.2 of [**2**]), but the proof presented here brings a better understanding of relationships between local controllability of a nonsmooth system, controllability of the linearized system, and the maximum principle.

We consider also a dynamical system governed by a differential inclusion:

$$(1.6) \qquad x' \in F(t, x); \quad x(0) = \xi.$$

Again, we use the approach via a linearization. To linearize the inclusion (1.6) along a given trajectory, we use a selection theorem of Łojasiewicz [**13**]. The linearization introduced in this paper is essentially different than those studied in [**6**], [**7**]. We prove that if the linearization is controllable, then $z(1) \in \text{Int } R(1)$ and derive from that the maximum principle for boundary trajectories of Kaškosz-Łojasiewicz [**10**].

The focus of this paper is on boundary trajectories as opposed to local controllability, on the relations between controllability of the linearized and the original systems and the maximum principle. We do not treat the more general case of a trajectory z for which $g(z(1))$ belongs to the boundary of $g(R(1))$ for a given mapping g, and the initial condition is

$x(0) \in C$ for a given set C. Actually, the method presented here, via linearization and the open mapping principle, is not very suitable to treat this case as problems occur with obtaining transversality conditions for $p(0)$, $p(1)$. We refer the reader interested in a more general version of the maximum principle to [**2**, Theorem 5.1.2].

The outline of the paper is as follows. In the next section we study linear differential inclusions and the duality between the maximum principle and controllability. In Section 3 we apply the results to convex-valued differential inclusions. We introduce a linearization of an inclusion and use it to prove the maximum principle. Section 4 is devoted to the nonsmooth (nonconvex) control system. Finally, in Section 5, for the reader's convenience, we give a proof of the selection theorem for convex-valued differential inclusions.

2. Some properties of linear control systems. In this section we investigate controllability of a generalized linear control system.

We recall that a set-valued map U from the interval $[0, 1]$ to a Banach space E is integrably bounded if for some Lebesgue integrable function $k:[0, 1] \to \mathbf{R}_+$, $U(t) \subset k(t)B$ a.e. in $[0, 1]$, where B denotes the closed unit ball in E.

Let $M_{n,n}$ be the space of $n \times n$ matrices with the usual norm. The following theorem establishes a duality relation between maximum principle and controllability of linear systems.

THEOREM 2.1. *Let $V:[0, 1] \to M_{n,n}$, $U:[0, 1] \to \mathbf{R}^n$ be measurable, integrably bounded set-valued maps with closed images such that for almost all $t \in [0, 1]$ $V(t)$ is convex and let $\bar{u}(t) \in U(t)$ be a measurable selection. Then either*

(a) *for some $\alpha > 0$ and all measurable selections $G(t) \in V(t)$, the reachable set at time 1 of the linear control system*

$$(2.1) \qquad w' = G(t)w + u, \quad u \in U(t) - \bar{u}(t), \quad w(0) = 0$$

contains the ball $\alpha B := \{x \in \mathbf{R}^n : \|x\| \leqq \alpha\}$, or

(b) *there exist $p \in W^{1,1}(0, 1)$ different from zero and a measurable selection $A(t) \in V(t)$, $t \in [0, 1]$, such that for almost all $t \in [0, 1]$*

$$(2.2) \qquad -p'(t) = A(t)^*p(t); \quad \max_{u \in U(t)} \langle p(t), u \rangle = \langle p(t), \bar{u}(t) \rangle.$$

The next result says that the property (a) of Theorem 2.1 is stable under perturbations of the dynamics.

THEOREM 2.2. *Under all assumptions of Theorem 2.1 assume that the conclusion* (b) *does not hold and let $l \in L^1(0, 1; \mathbf{R})$. Then there exist $\beta > 0$, $\rho > 0$ and measurable selections $u_j(t) \in U(t)$, $t \in [0, 1]$, $j = 1, \ldots, k$, such that for every $G \in L^1(0, 1; M_{n,n})$, $v_j \in L^1(0, 1; \mathbf{R}^n)$ satisfying*

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(2.3) $\|G(t)\| \leqq l(t); \int_0^1 \operatorname{dist}(G(t), V(t))dt \leqq \rho$

(2.4) $\|V_j(t)\| \leqq l(t); \int_0^1 \|u_j(t) - \bar{u}(t) - v_j(t)\|dt \leqq \rho$

we have

$$(2.5)\quad \begin{cases} \beta B \subset \Big\{ w_\lambda(1):\lambda \in \mathbf{R}^k_+, \ \sum_{j=1}^k \lambda_j = 1, \ w_\lambda(0) = 0, \\ \qquad w'_\lambda(t) = G(t)w_\lambda(t) + \sum_{j=1}^k \lambda_j v_j(t), \quad \text{a.e. } in\ [0, 1] \Big\}. \end{cases}$$

For every integrable function $G:[0, 1] \to M_{n,n}$ let X_G denote the fundamental solution of the linear system

$$X'(t) = G(t)X(t); \quad X(0) = \mathrm{Id}.$$

Recall that the reachable set at time 1 of the system (2.1) is equal to

$$(2.6)\quad R_G(1) = X_G(1) \int_0^1 X_G^{-1}(t)(U(t) - \bar{u}(t))dt.$$

The set $R_G(1)$ is convex and compact (see for example [**15**], [**12**]). To prove the above theorems we need the following simple:

LEMMA 2.3. *Under all assumptions of Theorem* 2.1 *we have*

(i) *The set*

$$Q = \{G \in L^1(0, 1; M_{n,n}):G(t) \in V(t) \quad \text{a.e.}\}$$

is weakly sequentially compact.

(ii) *The set*

$$P = \{ (X_G, X_G^{-1}):G \in Q\}$$

is compact in $C(0, 1; M_{n,n})$.

(iii) *For every sequence* $G_i \in Q$ *weakly converging to some* G,

$$\lim_{i\to\infty} X_{G_i} = X_G, \ \lim_{i\to\infty} X_{G_i}^{-1} = X_G^{-1},$$

where limits are taken in $C(0, 1; M_{n,n})$.

We provide a proof of the lemma for the reader's convenience.

Proof. The set Q is convex and closed and, by the Mazur lemma, it is weakly closed. From the Dunford-Pettis criterion it follows that Q is also weakly precompact. Hence (i). For all $G \in Q$, $t \in [0, 1]$,

$$(X_G^{-1})'(t) = -X_G(t)^{-1}G(t).$$

The Gronwall inequality implies that P is a bounded set of equicontinuous functions. From the Ascoli theorem it follows that P is precompact in $C(0, 1; M_{n,n})$. Consider a sequence $(X_{G_i}, X_{G_i}^{-1})$ converging uniformly to some (X, Y). By (i) we may assume that G_i converges weakly to some $G \in Q$. Observe that for all i and $t \in [0, 1]$

$$(2.7)\quad \begin{cases} X_{G_i}(t) = \mathrm{Id} + \int_0^t G_i(s)X_{G_i}(s)ds \\ X_{G_i}(t)^{-1} = \mathrm{Id} - \int_0^t X_{G_i}(s)^{-1}G_i(s)ds. \end{cases}$$

Passing to the limit we obtain that for all $t \in [0, 1]$

$$\begin{cases} X(t) = \mathrm{Id} + \int_0^t G(s)X(s)ds \\ Y(t) = \mathrm{Id} - \int_0^t Y(s)G(s)ds. \end{cases}$$

It implies that $X = X_G$, $Y = X_G^{-1}$ and proves (iii). Since this is true for an arbitrary converging subsequence of P we obtain (ii).

Proof of Theorem 2.1. Replacing $U(t)$ by $U(t) - \bar{u}(t)$ and using (2.6) we may restrict ourselves to the case $\bar{u} = 0$. If (a) does not hold then there exist a sequence of measurable selections $A_i(t) \in V(t)$ and $r_i \in FrR_{A_i}(1)$ satisfying

$$(2.8)\quad \lim_{i\to\infty} r_i = 0.$$

By the separation theorem for some $\bar{p}_i \in S^{n-1}$ and all $i \geqq 1$

$$(2.9)\quad \sup\{\langle \bar{p}_i, e\rangle : e \in R_{A_i}(1)\} = \langle \bar{p}_i, r_i\rangle.$$

Using (2.6) we obtain that for every measurable selection $u(t) \in U(t)$

$$(2.10)\quad \begin{cases} \langle \bar{p}_i, X_{A_i}(1)\int_0^1 X_{A_i}(t)^{-1}u(t)dt\rangle \\ = \int_0^1 \langle X_{A_i}(t)^{*-1}X_{A_i}(1)^*\bar{p}_i, u(t)\rangle dt \leqq \langle \bar{p}_i, r_i\rangle. \end{cases}$$

Taking a subsequence if needed we may assume by the compactness of S^{n-1} and Lemma 2.3 (ii) that for some $\bar{p} \neq 0$ and $A \in Q$,

$$\lim_{i\to\infty} \bar{p}_i = \bar{p},\ \lim_{i\to\infty} (X_{A_i}, X_{A_i}^{-1}) = (X_A, X_A^{-1}).$$

Passing to the limit in (2.10) and using (2.8) we derive that for all measurable selections $u(t) \in U(t)$.

$$(2.11)\quad \int_0^1 \langle X_A(t)^{*-1}X_A(1)^*\bar{p}, u(t)\rangle dt \leqq 0.$$

This implies that the function

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$$p(t) = X_A(t)^{*-1}X_A(1)^*\overline{p}, \quad t \in [0, 1]$$

satisfies the relations (2.2).

Proof of Theorem 2.2. As in the proof of the previous theorem we consider only the case $\overline{u} = 0$. Let $\alpha > 0$ be as in the claim (a) of Theorem 2.1 and $b_1, \ldots, b_s \in \alpha B$, $\gamma > 0$, $\beta > 0$ be such that for all $b'_j \in b_j + \gamma B$

(2.12) $\beta B \subset \text{co}\{b'_j : j = 1, \ldots, s\}$.

For all $A \in Q$ and $j = 1, \ldots, s$ let $u^j_A(t) \in U(t)$ be measurable selections such that for all j

(2.13) $b_j = X_A(1) \int_0^1 X_A(t)^{-1} u^j_A(t) dt$

and set

$$\epsilon_A = \gamma | 2\left(1 + \max_{G \in Q} \|X_G\|^2_C + \max_{1 \leqq j \leqq s} \|u^j_A\|_{L^1}\right)$$

$$N_A = \{X_G(1)X_G^{-1} : G \text{ satisfies (2.3)},$$
$$\|X_G(1)X_G^{-1} - X_A(1)X_A^{-1}\|_C < \epsilon_A\}.$$

Then for all measurable v_j, $G \in N_A$ satisfying (2.3), (2.4) with $\rho = \epsilon_A$, $u_j = u^j_A$ and $V = A$ we have

$$\left\| X_G(1) \int_0^1 X_G(t)^{-1} v_j(t) dt - X_A(1) \int_0^1 X_A(t)^{-1} u^j_A(t) dt \right\|$$
$$\leqq \|X_G(1)X_G^{-1} - X_A(1)X_A^{-1}\|_C \|u^j_A\|_{L^1}$$
$$+ \|X_G(1)\| \, \|X_G^{-1}\|_C \|u^j_A - v_j\|_{L^1} \leqq \gamma.$$

By Lemma 2.3 (ii) the set

$$S = \{X_A(1)X_A : A \in Q, \|A(t)\| \leqq l(t) \quad \text{a.e.}\}$$

is compact in $C(0, 1; M_{n,n})$. Thus the open covering $\{N_A : A \in Q\}$ contains a finite subcovering $\{N_{A_i} : i = 1, \ldots, m\}$. For all i, j set

$$u_{i,j} = u^j_{A_i}$$

and let

$$\rho = \min_{1 \leqq i \leqq m} \epsilon_{A_i}.$$

We claim that the family $\{u_{i,j}\}$ satisfies the conclusion of the theorem. Indeed, let G and $v_{i,j}$ be such that (2.3), (2.4) hold true with u_j, v_j replaced by $u_{i,j} v_{i,j}$ respectively. Then, by the choice of ρ for all $1 \leqq j \leqq s$ we can find i such that the solution w of the linear equation

$$w' = G(t)w + v_{i,j}; \quad w(0) = 0$$

satisfies $\|w(1) - b_j\| \leqq \gamma$. Since the right-hand side of (2.5) is convex the inclusion (2.12) ends the proof.

3. Boundary trajectories of convex valued differential inclusions. Let F be a set-valued map from $[0, 1] \times \mathbf{R}^n$ to $\mathbf{R}^n$. We associate with it the differential inclusion

(3.1) $\quad x' \in F(t, x)$.

A function $x \in W^{1,1}(0, 1)$, (the Sobolev space), is called a trajectory of (3.1) if for almost all $t \in [0, 1]$,

$$x'(t) \in F(t, x(t)).$$

For a point $\xi \in \mathbf{R}^n$, we denote by $R(t, \xi)$ the reachable set of (3.1) from ξ at time t, i.e.,

(3.2) $\quad R(t, \xi) := \{x(t)\text{: } x \text{ is a trajectory of (3.1)}, x(0) = \xi\}$.

Boundary trajectories are those trajectories of (3.1) which satisfy

(3.3) $\quad$ for all $t \in [0, 1]$, $x(t)$ lies on the boundary of $R(t, x(0))$.

In this section we provide a necessary condition for boundary trajectories. Fix a trajectory z of (3.1) and assume that:

(H_1)
- (i) F has nonempty, compact images.
- (ii) For all $x \in \mathbf{R}^n$, $F(\cdot, x)$ is measurable.
- (iii) For some $\epsilon > 0$, $k \in L^1(0, 1)$ and almost all $t \in [0, 1]$, $F(t, \cdot)$ is $k(t)$-Lipschitzian on $z(t) + \epsilon B$ with respect to the Hausdorff metric and $F(t, x) \subset k(t)B$.

The following consequence of a Filippov theorem (see [**1**, p. 120]) is well known:

PROPOSITION 3.1. *If* (H_1) *holds true then* z *is a boundary trajectory if and only if* $z(1)$ *lies on the boundary of* $R(1, z(0))$.

In this section we impose an additional assumption on F:

(H_2) F has convex images.

The following theorem is due to Łojasiewicz ([**13**], [**10**]):

THEOREM 3.2. *If the hypotheses* (H_1), (H_2) *hold true, then for every measurable selection* $u(t) \in F(t, z(t))$, $t \in [0, 1]$, *there exists a function* $f(t, x)$ *from* $[0, 1] \times \mathbf{R}^n$ *to* $\mathbf{R}^n$ *such that*

(i) $f(t, x) \in F(t, x)$ *for all* $t \in [0, 1]$, $x \in \mathbf{R}^n$

(ii) $f(t, z(t)) = u(t)$ *for almost all* $t \in [0, 1]$

(iii) $f(\cdot, x)$ *is measurable for each fixed* $x \in \mathbf{R}^n$

(iv) $f(t, \cdot)$ *is* $4nk(t)$*-Lipschitzian on* $z(t) + \epsilon B$ *for almost all* $t \in [0, 1]$.

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In the last section we provide a proof of the theorem for the reader's convenience.

Let $\partial f(t, z(t))$, $t \in [0, 1]$, denote the (Clarke) generalized Jacobian of the function $f(t, \cdot)$ at $z(t)$ (see **[2]**, **[3]**). We recall that $\partial f(t, \cdot)$ is upper semicontinuous and reduces to the derivative $(\partial/\partial x)f(t, z(t))$, when $f(t, \cdot)$ happens to be continuously differentiable at $z(t)$.

THEOREM 3.3. *Let $z \in W^{1,1}(0, 1)$ be a boundary trajectory of* (3.1) *and assume that* (H_1) *and* (H_2) *hold true. Further let f be a selection satisfying* (i), (iii), (iv) *of Theorem* 3.2 *and such that*

$$f(t, z(t)) = z'(t) \quad \text{a.e.}$$

Then there exists $p \in W^{1,1}(0, 1)$, $p \neq 0$, such that for almost all $t \in$ [0, 1]

$$\text{(3.4)} \quad \begin{cases} -p'(t) \in p(t)\partial f(t, z(t)) \\ \langle p(t), z'(t) \rangle = \max\limits_{e \in F(t,z(t))} \langle p(t), e \rangle. \end{cases}$$

To prove the theorem we need the following generalization of the classical open mapping principle: Let $k \geqq 1$ be an integer and set

$$\text{(3.5)} \quad \mathcal{T} := \left\{ (\theta_1, \ldots, \theta_k) \in \mathbf{R}^k, \quad \theta_j \geqq 0, \sum_{j=1}^{k} \theta_j \leqq 1 \right\}.$$

THEOREM 3.4 ([**19**, Theorem 2.3]). *Let φ be a function from $\mathbf{R}^k$ to $\mathbf{R}^n$ such that on a neighborhood of zero the derivative φ' exists and is Lipschitzian. Assume that for some $\delta \in]0, 1]$, $\beta > 0$ and all $\theta \in \delta\mathcal{T}$*

$$\text{(3.6)} \quad \beta B \subset \varphi'(\theta)\mathcal{T}.$$

Then for every continuous function $\Psi:\delta\mathcal{T} \to \mathbf{R}^n$ satisfying

$$\sup_{x \in \delta\mathcal{T}} \|\varphi(x) - \Psi(x)\| \leqq \delta\beta/32$$

we have

$$\text{(3.7)} \quad \Psi(0) + \frac{\delta\beta}{16}B \subset \Psi(\delta\mathcal{T}).$$

Consider any measurable in the first variable functions

$$f_j:[0, 1] \times \mathbf{R}^n \to \mathbf{R}^n, \ j = 1, \ldots, k$$

and set $f_0 = f$. We associate with every $\theta = (\theta_1, \ldots, \theta_k) \in \mathbf{R}^k$ the ODE

$$\text{(3.8)} \quad \begin{cases} x' = \left(1 - \sum\limits_{j=1}^{k} \theta_j\right) f(t, x) + \sum\limits_{j=1}^{k} \theta_j f_j(t, x) \\ x(0) = z(0). \end{cases}$$

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The Gronwall inequality (see for example [**1**, pp. 119-121]) implies

LEMMA 3.5. *Assume that for some* $\eta > 0$, $l \in L^1(0, 1)$, *and all* $j = 0, \dots, k$

$$\|f_j(t, z(t))\| \leqq l(t);$$

$f_j(t, \cdot)$ *is* $l(t)$*-Lipschitzian on* $z(t) + \eta B$ *almost everywhere in* $[0, 1]$.
Let $\gamma > 0$ *be such that*

$$4k\gamma \exp\left(\int_0^1 l(t)dt\right) \int_0^1 l(t)dt = \eta.$$

Then the function $\pi:\gamma B \rightarrow W^{1,1}(0,1)$ *associating with every* θ *the (unique) solution* $x_\theta \in W^{1,1}(0,1)$ *of* (3.8) *is Lipschitzian. Moreover for all* $\theta \in \gamma B$ *and* $t \in [0, 1]$

$$\|x_\theta(t) - z(t)\| \leqq \eta/2.$$

The next lemma follows from the classical theorem about differentiation of solutions of ODE with respect to a parameter.

LEMMA 3.6. *Under all assumptions of Lemma* 3.5 *assume that for some* $\mu \in L^1(0, 1)$ *and all* $j = 0, \dots, k$ *the derivative* $(\partial/\partial x)f_j(t, \cdot)$ *exists and is* $\mu(t)$*-Lipschitzian on* $z(t) + \eta B$. *Let* γ, π *and* x_θ *be as in Lemma* 3.5. *Then for all* $\theta \in \gamma B$ *the derivative* $\pi'(\theta)$ *exists and* π' *is Lipschitzian on* γB. *Moreover for all* $q \in \mathbf{R}^k$, $\pi'(\theta)q$ *is equal to the solution of the linear system*

$$(3.9)\quad \begin{cases} w' = \dfrac{\partial f}{\partial x}(t, x_\theta(t))w \\ \quad + \displaystyle\sum_{j=1}^{k} \theta_j\left(\frac{\partial}{\partial x}f_j(t, x_\theta(t)) - \frac{\partial}{\partial x}f_j(t, x_\theta(t))\right)w \\ \quad + \displaystyle\sum_{j=1}^{k} q_j(f_j(t, x_\theta(t)) - f(t, x_\theta(t))) \quad \text{a.e. in } [0, 1] \\ w(0) = 0. \end{cases}$$

Proof of Theorem 3.3. It is not restrictive to assume that $z(0) = 0$ and that for almost all $t \in [0, 1]$

$$(3.10)\quad \sup\{\|e\|:e \in F(t, z(t))\} + 2nk(t)\epsilon \leqq k(t).$$

For all $t \in [0, 1]$ set

$$V(t) = \partial f(t, z(t)); \quad U(t) = F(t, z(t))$$
$$\bar{u}(t) = z'(t); \quad l(t) = 4nk(t).$$

If there is no $p \in W^{1,1}(0, 1)$, $p \neq 0$, satisfying (3.4) then the statement (b) of Theorem 2.1 does not hold and we may apply Theorem 2.2. Let

$\beta, \rho \in]0, 1]$ and measurable selections $u_j(t) \in U(t)$, $j = 1, \ldots, k$ be as in the claim of Theorem 2.2.

Step 1. We replace here the differential inclusion (3.1) by a family of ODE (3.8) with nonnegative θ_j. By Theorem 3.2 for all $j = 1, \ldots, k$ there exists a measurable in the first-variable function

$$f_j:[0, 1] \times \mathbf{R}^n \to \mathbf{R}^n$$

satisfying (i), (iii), (iv) of Theorem 3.2 and such that

$$(3.11) \quad f_j(t, z(t)) = u_j(t) \quad \text{a.e. in } [0, 1].$$

Set $f_0 = f$.

From (3.10)-(3.11) we obtain that for a.e. $t \in [0, 1]$ and all $j = 0, \ldots, k$, $x \in z(t) + (\epsilon/2)B$,

$$\|f_j(t, x)\| \leqq \|f_j(t, z(t))\| + 4nk(t)\frac{\epsilon}{2} \leqq k(t).$$

Thus we may apply Lemma 3.5 with $\eta = \epsilon/2$. Let $\gamma \in]0, 1]$ and π be as in its claim. Since $F(t, x)$ are convex, for all $\theta \in \gamma\mathcal{T}$, $\pi(\theta)$ is a trajectory of (3.1). For all $\theta \in \gamma\mathcal{T}$ set

$$(3.12) \quad \Psi(\theta) = \pi(\theta)(1) \in R(1, 0).$$

Step 2. By (3.12) and Theorem 3.4 it remains to show the existence of $\delta > 0$ and a function $\varphi:\gamma B \to \mathbf{R}^n$ satisfying all the assumptions of Theorem 3.4.

For all $h > 0$, $t \in [0, 1]$ set

$$d_h(t) = \sup\left\{\text{dist}\left(\frac{\partial}{\partial x}f(t, x), V(t)\right):x \in z(t) + hB \text{ and the derivative } \frac{\partial}{\partial x}f(t, x) \text{ does exist}\right\}.$$

Since $\partial f(t, \cdot)$ is upper semicontinuous at $z(t)$,

$$\lim_{h \to 0^+} \int_0^1 d_h(t)dt = 0.$$

Let $0 < h < \min\{\epsilon/4, 1\}$ be such that

$$(3.13) \quad \int_0^1 d_h(t)dt \leqq \rho/4; \quad h\int_0^1 l(t)dt \leqq \rho/4.$$

Pick $0 < \delta < \min\{h/2k, 1\}$ such that

$$(3.14) \quad 16k\delta \exp\left(\int_0^1 l(t)dt\right)\int_0^1 l(t)dt < \epsilon$$

and for all $\theta \in \delta B$, $t \in [0, 1]$

$$(3.15) \quad \|x_\theta(t) - z(t)\| \leqq h/2.$$

Set

$$v = \rho\delta\beta h/\left[32(1 + 2k)\exp\left(\int_0^1 l(t)dt\right)\int_0^1 (1 + l(t)dt\right]$$

and consider a mollifier $\chi:\mathbf{R}^n \to [0, 1]$, i.e., $\chi \in C^\infty$ of support in the unit ball B, satisfying

$$\int_{\mathbf{R}^n} \chi(y)dy = 1.$$

For all $j = 0, \ldots, k$, $t \in [0, 1]$, $x \in z(t) + (\epsilon/4)B$ set

$$g_j(t, x) = \int f_j(t, x - vy)\chi(y)dy.$$

Then,

(3.16) $\quad \|g_j(t, x) - f_j(t, x)\| \leqq l(t)v.$

By the mean-value theorem, for every $x \in z(t) + (h/2)B$,

$$\frac{\partial}{\partial x}g_0(t, x) \in \text{co}\left\{\frac{\partial}{\partial x}f(t, x):x \in z(t) + hB\right\}.$$

The convexity of $V(t)$ implies that for all $x \in z(t) + (h/2)B$

(3.17) $\quad \text{dist}\left(\frac{\partial}{\partial x}g_0(t, x), V(t)\right) \leqq d_h(t).$

Observe that $\{g_j\}_{j=0}^k$ satisfy all the assumptions of Lemma 3.6 with $\eta = \epsilon/4$. By (3.14) and Lemma 3.6 we may set $\gamma = \delta$. By the choice of v, (3.16) and the Gronwall inequality for all $\theta \in \delta B$ the solution

$$\hat{x}_\theta \in W^{1,1}(0, 1)$$

of ODE

$$x' = g_0(t, x) + \sum_{j=1}^k \theta_j(g_j(t, x) - g_0(t, x)), \quad x(0) = 0$$

satisfies

$$\|\hat{x}_\theta - x_\theta\|_C \leqq (1 + 2k)v\exp\left(\int_0^1 l(t)dt\right)\int_0^1 l(t)dt$$
$$\leqq \min\{\delta\beta/32, h/2\}.$$

We define the function φ by $\varphi(\theta) = \hat{x}_\theta(1)$. Then for all $\theta \in \delta B$,

$$\|\Psi(\theta) - \varphi(\theta)\| \leqq \delta\beta/32.$$

By Lemma 3.6 φ' is Lipschitzian on a neighborhood of zero. To apply Theorem 3.4 it remains to check the inclusion (3.6). Indeed, by (3.15) for all $\theta \in \delta B$,

$$\|\hat{x}_\theta - z\|_C \leqq h.$$

To compute the derivative $\varphi'(\theta)$ set

$$G^{\theta}(t) := \frac{\partial}{\partial x} g_0(t, \hat{x}_{\theta}(t)) + \sum_{j=1}^{k} \theta_j \left(\frac{\partial}{\partial x} g_j(t, \hat{x}_{\theta}(t)) - \frac{\partial}{\partial x} g_0(t, \hat{x}_{\theta}(t)) \right)$$

and

$$v_j^{\theta}(t) = g_j(t, \hat{x}_{\theta}(t)) - g_0(t, \hat{x}_{\theta}(t)).$$

By Lemma 3.6 for all $q \in \mathscr{T}$ the derivative $\varphi'(\theta)q$ is equal to $w(1)$, where w is the solution of linear ODE

$$w' = G^{\theta}(t)w + \sum_{j=1}^{k} q_j v_j^{\theta}(t); \quad w(0) = 0.$$

Furthermore, by (3.17), (3.13) and the choice of δ,

$$\int_0^1 \operatorname{dist}(G^{\theta}(t), V(t))dt \leqq \int_0^1 d_h(t)dt + \sum_{j=1}^{k} \theta_j \left(\left\| \frac{\partial}{\partial x} g_j(\cdot, \hat{x}_{\theta}(\cdot)) \right\|_{L^1} + \left\| \frac{\partial}{\partial x} g_0(\cdot, \hat{x}_{\theta}(\cdot)) \right\|_{L^1} \right) \leqq \rho/4 + 2k\delta \|l\|_{L^1} \leqq \rho$$

and for all $j = 1, \dots, k$,

$$\begin{aligned} &\int_0^1 \|v_j^{\theta}(t) - f_j(t, z(t)) + f(t, z(t))\| dt \\ &\leqq \int_0^1 \|g_j(t, \hat{x}_{\theta}(t)) - g_j(t, z(t))\| dt \\ &+ \int_0^1 \|g_j(t, z(t)) - f_j(t, z(t))\| dt \\ &+ \int_0^1 \|g_0(t, \hat{x}_{\theta}(t)) - g_0(t, z(t))\| dt \\ &+ \int_0^1 \|g_0(t, z(t)) - f(t, z(t))\| dt \\ &\leqq (h + v + h + v)\|l\|_{L^1} \leqq \rho. \end{aligned}$$

Inclusion (2.5) of Theorem 2.2 ends the proof.

4. Nonconvex control systems. The right-hand side of the differential inclusion considered in the previous section was assumed to be convex. The main reason for this assumption is to ensure the existence of singlevalued selections of the set-valued map $F(t, x)$ which are Lipschitzian in x. This problem disappears if one considers a parametrized inclusion, that is, an ordinary control system. In this case, convexity of the set of velocities is not necessary and the same approach can be used to derive a maximum principle for a nonconvex, nonsmooth control system.

Consider the following control system:

$$(4.1) \quad \begin{cases} x'(t) = f(t, x(t), u(t)) \\ x(0) = \xi \end{cases}$$

where $x \in \mathbf{R}^n$, $t \in [0, 1]$, $f:[0, 1] \times \mathbf{R}^n \times \mathbf{R}^k \to \mathbf{R}^n$. Control functions $u(\cdot)$ are assumed to be Lebesgue measurable on $[0, 1]$ and satisfy $u(t) \in \Omega(t)$ a.e. in $[0, 1]$, where $\Omega(t)$ is a given multifunction from $[0, 1]$ to $\mathbf{R}^k$. Let $z(\cdot)$ be a trajectory of the system (4.1) defined on $[0, 1]$ and X be a compact subset of $\mathbf{R}^n$ containing $z(t)$ for all t in its interior.

Denote by $L \times B^k$ the σ-algebra of subsets of $[0, 1] \times \mathbf{R}^k$ generated by products of Lebesgue measurable subsets of $[0, 1]$ and Borel subsets of $\mathbf{R}^k$. Assume that the system satisfies the following conditions.

(H_1) The function $f(\cdot, x, \cdot)$ is $L \times B^k$ measurable for each fixed $x \in X$.

(H_2) There exists a function $k(\cdot) \in L^1(0, 1)$, such that for almost all $t \in [0, 1]$ and all $u \in \Omega(t)$, $x, y \in X$:

$$|f(t, x, u) - f(t, y, u)| \leqq k(t)|x - y|,$$
$$|f(t, x, u)| \leqq k(t).$$

(H_3) The sets $\Omega(t)$ are bounded for almost all $t \in [0, 1]$ and the graph of the multifunction $\Omega(\cdot)$ is $L \times B^k$ measurable.

Denote again by $R(1, \xi)$ the set of points which can be reached at time 1 by trajectories of (4.1). We shall prove the following theorem:

THEOREM 4.1. *Assume that $z(1)$ belongs to the boundary of the set $R(1, \xi)$ and let $u_*(\cdot)$ be a control function generating $z(\cdot)$. Then the pair $(z(\cdot), u_*(\cdot))$ satisfies the maximum principle, that is, there exists an absolutely continuous function $p(\cdot) \in W^{1,1}(0, 1)$ such that for almost all $t \in [0, 1]$ the following conditions hold*:

$$-p'(t) \in p(t)\partial f(t, z(t), u_*(t))$$
$$\langle p(t), z'(t) \rangle = \sup_{u \in \Omega(t)} \langle p(t), f(t, z(t), u) \rangle.$$

Theorem 4.1 is a special case of Theorem 5.2.1 of [**2**], but the proof which we give below is quite different.

To prove the theorem we need one more lemma. Let

$$f_j(\cdot, \cdot):[0, 1] \times X \to \mathbf{R}^n, \ j = 0, \ldots, k$$

be any functions which are measurable in the first variable and such that for almost all $t \in [0, 1]$, all $x, y \in X$ and $j = 0, \ldots, k$:

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(4.2) $|f_j(t, x) - f_j(t, y)| \leqq k(t)|x - y|, \ |f_j(t, x)| \leqq k(t).$

Assume that $z(\cdot)$ is the solution of the equation:

$$z' = f_0(t, z(t)), \ z(0) = \xi.$$

Consider again the system:

$$x' = \sum_{j=0}^{k} \lambda_j f_j(t, x), \ x(0) = \xi, \tag{4.3}$$

$\lambda = (\lambda_0, \dots, \lambda_k) \in \Delta$, where

$$\Delta = \left\{ (\lambda_0, \dots, \lambda_k) \,|\lambda_j \geqq 0, \ j = 0, \dots, k, \ \sum_{j=0}^{k} \lambda_j = 1 \right\}$$

$$= \left\{ \left(1 - \sum_{j=1}^{k} \lambda_j, \lambda_1, \dots, \lambda_k\right) \,|\, (\lambda_1, \dots, \lambda_k) \in \mathscr{T} \right\}.$$

Let $\epsilon > 0$ be such that $z(t) + \epsilon B \subset X$ for $t \in [0, 1]$. It follows from Lemma 3.5 that for a neighborhood N of the point $(1, 0, \dots, 0)$ in Δ,

$$N = \{ (\lambda_0, \lambda_1, \dots, \lambda_k) \in \Delta | \ (\lambda_1, \dots, \lambda_k) \in \gamma\mathscr{T} \},$$

and for every λ in N the trajectory $x_\lambda(\cdot)$ of (4.3) exists on the whole interval $[0, 1]$ and satisfies:

$$x_\lambda(t) \in z(t) + \frac{\epsilon}{2} B \quad \text{for every } t.$$

We shall need the following

LEMMA 4.2. *Let $\sigma > 0$ be fixed. Then for each $\lambda \in N$ there exist measurable scalar functions*

$$u_j^\lambda(\cdot):[0, 1] \to \{0, 1\}, \quad j = 0, \dots, k,$$

which satisfy:

$$\sum_{j=0}^{k} u_j^\lambda(t) = 1 \quad \text{a.e. } in \ [0, 1]$$

and such that for the trajectory $\tilde{x}_\lambda(\cdot)$ of the system:

$$\tilde{x}'_\lambda(t) = \sum_{j=0}^{k} u_j^\lambda(t) f_j(t, \tilde{x}_\lambda(t)), \ \tilde{x}_\lambda(0) = \xi \tag{4.4}$$

the following conditions hold:

(i) $|x_\lambda(1) - \tilde{x}_\lambda(1)| < \sigma$ *for every* $\lambda \in N$,

(ii) *the mapping* $\lambda \to \tilde{x}_\lambda(1)$ *from N into R^n is continuous.*

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We provide a proof of the lemma at the end of this section. First we show how the same approach as in the previous section allows to prove the maximum principle for the system (4.1).

Proof of Theorem 4.1. Without any loss of generality we may assume that $\xi = 0$. Put

$$V(t) = \partial f(t, z(t), u_*(t)), \quad U(t) = \text{cl } f(t, z(t), \Omega(t))$$

where "cl" stands for the closure. If the maximum principle does not hold then we may apply Theorem 2.2 with

$$\bar{u}(t) = f(t, x(t), u_*(t)), \quad l(t) = k(t).$$

Let $\beta, \rho \in]0, 1]$ and measurable selections $u_j(t) \in U(t)$, $j = 1, \dots, k$ be as in the claim of Theorem 2.2. From [**4**, Lemma 3] there exist control functions $c_j(\cdot)$, $j = 1, \dots, k$ such that

$$\|f(t, z(t), c_j(t)) - u_j(t)\| \leqq \rho/2 \quad \text{a.e. in } [0, 1].$$

Thus for every measurable G as in (2.3) and $v_j \in L^1(0, 1; \mathbf{R}^n)$, $\|v_j(t)\| \leqq l(t)$, satisfying

$$\int_0^1 \|v_j(t) - f(t, z(t), c_j(t))\| dt \leqq \rho/2$$

the following condition holds:

$$(4.5) \quad \begin{cases} \beta B \subset \Big\{ w_\lambda(1) \,|\, \lambda \in \mathbf{R}^k_+, \displaystyle\sum_{j=1}^k \lambda_j = 1, \\ w'_\lambda(t) = G(t) w_\lambda(t) + \displaystyle\sum_{j=1}^k \lambda_j v_j(t) \quad \text{a.e. in } [0, 1];\ w_\lambda(0) = 0 \Big\}. \end{cases}$$

We join now to the proof of Theorem 3.2 setting

$$f_0(t, x) = f(t, x, u_*(t)),\ f_j(t, x) = f(t, x, c_j(t)),\ j = 1, \dots, k,$$
$$F(t, x) = \text{co}\{f_j(t, x) : j = 0, \dots, k\}$$

and replacing ρ by $\rho/2$ and β by $\beta/2$. By the proof we know that there exist $\delta > 0$ and a function $\varphi : \delta B \to \mathbf{R}^n$ satisfying all the assumptions of Theorem 3.4 and such that for all $\theta \in \delta B$ and the solution x_θ of (3.8) we have

$$(4.6) \quad \|x_\theta(1) - \varphi(\theta)\| \leqq \delta\beta/64.$$

On the other hand Lemma 4.2 applied with $\sigma = \delta\beta/64$ implies the existence of a continuous mapping $\Psi : \mathscr{T} \cap \delta\beta$ to the reachable set at time 1 of the control system (4.1) such that

$$\|\Psi(\theta) - x_\theta(1)\| \leqq \delta\beta/64.$$

Thus for all $\theta \in \mathscr{T} \cap \delta B$

$$\|\Psi(\theta) - \varphi(\theta)\| \leqq \delta\beta/32$$

and the proof follows from Theorem 3.4.

Proof of Lemma 4.2. Notice that as long as a trajectory $\tilde{x}_\lambda(t)$ of (4.4) remains in X the condition (4.2) implies that:

$$\begin{aligned} &|x_\lambda(t) - \tilde{x}_\lambda(t)| \\ &\leqq \int_0^t k(\tau)\,|x_\lambda(\tau) - \tilde{x}_\lambda(\tau)|d\tau \\ &+ \left|\int_0^t \left[\sum_{j=0}^k u_j^\lambda(\tau)f_j(\tau, x_\lambda(\tau)) - \sum_{j=0}^k \lambda_j f_j(\tau, x_\lambda(\tau))\right]d\tau\right|. \end{aligned}$$

From the Gronwall inequality it follows that in order to prove the claim (i) of Lemma 4.2 it is enough to find each $\lambda \in N$ functions $u_j^\lambda(t)$ in such a way that:

$$\left|\int_0^t \left[\sum_{j=0}^k u_j^\lambda(\tau)f_j(\tau, x_\lambda(\tau)) - \sum_{j=0}^k \lambda_j f_j(\tau, x_\lambda(\tau))\right]d\tau\right| < \sigma_1 \tag{4.7}$$

for $t \in [0, 1]$, where

$$\sigma_1 = \frac{\sigma}{1 + Ke^K}, \quad K = \int_0^1 k(\tau)d\tau.$$

We can assume without any loss of generality that $\gamma < \epsilon/2$ and then (4.7) implies that $\tilde{x}_\lambda(\cdot)$ exists on the whole interval [0, 1], remains in X and satisfies (i).

Divide the interval [0, 1] into subintervals: $I_0 = [0, t_1]$, $I_1 = [t_1, t_2], \dots,$ $I_r = [t_r, 1]$, $0 < t_1 < t_2 < \dots, t_r < 1$, small enough so that:

$$|x_\lambda(t') - x_\lambda(t'')| < \sigma_1/4K \quad \text{for every } \lambda \in N \tag{4.8}$$

whenever t', t'' are in the same subinterval. It is enough now to define for each $\lambda \in N$ functions $u_j^\lambda(t)$ in such a way that

$$\left|\int_{t''}^{t'} \left[\sum_{j=0}^k u_j^\lambda(\tau)f_j(t, x) - \sum_{j=0}^k \lambda_j f_j(\tau, x)\right]d\tau\right| < \sigma_2, \tag{4.9}$$

$$\sigma_2 = \frac{\sigma_1}{2(r + 1)},$$

for all $x \in X$, whenever t', t'' are in the same subinterval. Indeed, take $\lambda \in N$, $t \in [0, 1]$, let $t \in I_s$. Put

$$x_i = x_\lambda(t_i),\ i = 1, \dots, r,\ x_0 = x_\lambda(0).$$

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From (4.2) we have:

$$\left|\int_0^t \left[\sum_{j=0}^k u_j^\lambda(\tau) f_j(\tau, x_\lambda(\tau)) - \sum_{j=0}^k \lambda_j f_j(\tau, x_\lambda(\tau))\right] d\tau\right|$$

$$\leqq \sum_{i=0}^{s-1} \left|\int_{I_i} \left[\sum_{j=0}^k u_j^\lambda(\tau) f_j(\tau, x_i) - \sum_{j=0}^k \lambda_j f_j(\tau, x_i)\right] d\tau\right|$$

$$+ \left|\int_{t_s}^t \left[\sum_{j=0}^k u_j^\lambda(\tau) f_j(\tau, x_s) - \sum_{j=0}^k \lambda_j f_j(\tau, x_s)\right] d\tau\right|$$

$$+ \sum_{i=0}^s \int_{I_i} 2k(\tau)\, |x_\lambda(\tau) - x_i| d\tau.$$

Hence (4.9) and (4.8) imply (4.7) and therefore (i).

We shall construct functions $u_j^\lambda(t)$ which satisfy (4.9) on each of the subintervals I_i, $i = 0, \dots, r$, following the procedure introduced by Gamkrelidze in **[9]**. Let $g_j^i(t, x)$ be continuous functions on $I_i \times X$ such that:

$$(4.10) \quad \int_{I_i} |g_j^i(t, x) - f_j(t, x)| dt < \frac{\sigma_2}{2(k + 2)} \quad \text{for } x \in X,\ j = 0, \dots, k.$$

The existence of such functions is shown in **[9]**. Take a subdivision of each I_i into mutually disjoint subintervals J_m^i, $m = 1, \dots, m_i$ in such a way that

$$(4.11) \quad |g_j^i(t', x) - g_j^i(t'', x)| < \frac{\sigma_2}{6} \quad \text{for } x \in X,\ j = 0, \dots, k$$

whenever, t', t'' are in the same subinterval J_m^i. Let also the length $|J_m^i|$ of each J_m^i satisfy:

$$(4.12) \quad |J_m^i| \leqq \frac{\sigma_2}{24M},$$

where M is a constant for which:

$$(4.13) \quad |g_j^i(t, x)| \leqq M \quad \text{for } t \in I_i,\ x \in X,\ j = 0, \dots, k.$$

Take $\lambda \in N$. We shall define functions $u_j^\lambda(t)$, $j = 0, \dots, k$, on I_k, $i = 0, \dots, r$, as follows. Divide each J_m^i, $m = 1, \dots, m_i$, into mutually disjoint subintervals $J_{m,j}^i$, $j = 0, \dots, k$, $J_{m,j}^i = [\tau_{m,j}^i, \tau_{m,j+1}^i)$, $j = 0, \dots, k$, $\tau_{m,0}^i \leqq \tau_{m,1}^i \leqq \dots \leqq \tau_{m,k+1}^i$, in such a way that:

$$|J_{m,j}^i| = \lambda_j |J_m^i|, \quad j = 0, \dots, k,\ m = 1, \dots, m_i,\ i = 0, \dots, r.$$

Define $u_j^\lambda(t)$ on each I_i, $i = 0, \dots, r$, by:

$$u_j^\lambda(t) = \begin{cases} 1 \text{ for } t \in J_{m,j}^i, \ m = 1, \ldots, m_i \\ 0 \text{ otherwise.} \end{cases}$$

The functions $u_j^\lambda(t)$ satisfy (4.9). Indeed, take t', t'' from the same I_i and fix $x \in X$. From (4.10) it follows:

$$(4.14) \quad \begin{cases} \left| \int_{t'}^{t''} \left[\sum_{j=0}^{k} u_j^\lambda(\tau) f(\tau, x) - \sum_{j=0}^{k} \lambda_j f_j(\tau, x) \right] d\tau \right| \\ \leqq \frac{\sigma_2}{2} + \left| \int_{t'}^{t''} \left[\sum_{j=0}^{k} u_j^\lambda(\tau) g_j^i(\tau, x) - \sum_{j=0}^{k} \lambda_j g_j^i(\tau, x) \right] d\tau \right|. \end{cases}$$

It is enough to show then that the last term in the expression above is bounded by $\sigma_2/2$. Let $t' \in J_{m'-1}^i$, $t'' \in J_{m''+1}^i$. Then from (4.12), (4.13) we obtain:

$$(4.15) \quad \begin{cases} \left| \int_{t'}^{t''} \left[\sum_{j=0}^{k} u_j^\lambda(\tau) g_j^i(\tau, x) - \sum_{j=0}^{k} \lambda_j g_j^i(\tau, x) \right] d\tau \right| \\ \leqq \sum_{m=m'}^{m''} \left| \int_{J_m^i} \left[\sum_{j=0}^{k} u_j^\lambda(\tau) g_j^i(\tau, x) - \sum_{j=0}^{k} \lambda_j g_j(\tau, x) \right] d\tau \right| + \frac{\sigma_2}{6}. \end{cases}$$

Similarly as in [**9**, Lemma 4.1] we derive from (4.11) and the definition of $u_j^\lambda(t)$ the following inequality on each J_m^i:

$$\left| \int_{J_m^i} \left[\sum_{j=0}^{k} u_j^\lambda(\tau) g_j^i(\tau, x) - \sum_{j=0}^{k} \lambda_j g_j^i(\tau, x) \right] d\tau \right| + \frac{\sigma_2}{3} |J_m^i|.$$

The latter together with (4.15), (4.14) imply (4.9) and the condition (i) is proven.

In order to prove (ii) notice first that the division of the interval [0, 1] into subintervals I_i and then J_m^i are independent of λ. Only the division of each J_m^i into $J_{m,j}^i$ depends on λ. Take $\lambda, \bar{\lambda} \in N$ such that:

$$|\lambda - \bar{\lambda}| < \delta.$$

It follows from the construction that for δ sufficiently small:

$$\mu\{t \in J_m^i | u_j^\lambda(t) \neq u_j^{\bar{\lambda}}(t)\} \leqq 2(k + 1)\delta$$

for

$$j = 0, \ldots, k, \ m = 1, \ldots, m_i, \ i = 0, \ldots, r,$$

where μ denotes the Lebesgue measure. Therefore, there exists a set $A \subset [0, 1]$ such that:

$$(4.16) \quad u_j^\lambda(t) = u_j^{\bar{\lambda}}(t) \quad \text{for } j = 0, \ldots, k, \ t \in [0, 1] \setminus A,$$
$$\mu(A) \leqq 2(k + 1)^2 \cdot \delta \cdot m_i \cdot (r + 1).$$

The condition (4.16) implies via Gronwall's inequality and (4.2) that:

$$|\tilde{x}_\lambda(1) - \tilde{x}_{\bar{\lambda}}(1)| \leqq 2(1 + Ke^K) \int_A k(t)dt.$$

Hence $\tilde{x}_\lambda(1) \to \tilde{x}_{\bar{\lambda}}(1)$ as $\lambda \to \bar{\lambda}$ and the proof of Lemma 4.2 is completed.

5. Proof of the selection theorem. Denote by $\mathcal{K}_n$ the collection of all non-empty, convex, compact subsets of R^n, by $\mathcal{B}_n$ the collection of all closed balls in R^n. For each $X \in \mathcal{K}_n$ define its Hamiltonian:

$$H(p, X) = \max_{x \in X} \langle p, x \rangle.$$

Let $s_n(X)$ denote the Steiner point of X, (see e.g. **[16]**, **[14]**), that is:

$$s_n(X) = n \int_{S^{n-1}} pH(p, X)\sigma(dp)$$

where S^{n-1} is the unit sphere equipped with the unit Lebesgue measure σ. The Steiner point has the following properties:

$$(5.1) \qquad \begin{cases} s_n(X) \in X \\ \|s_n(X) - s_n(Y)\| \leqq nh(X, Y) \quad \text{for all } X, Y \in \mathcal{K}_n, \end{cases}$$

where $h(\cdot, \cdot)$ denotes the Hausdorff distance.

Let $P(\cdot, \cdot, \cdot)$ be a map from the set

$$\tilde{\mathcal{A}} = \{ (X, a, A): X \in \mathcal{K}_n, a \in A, A \in \mathcal{K}_n \}$$

into $\mathcal{K}_n$ defined as follows:

$$(5.2) \qquad P(X, a, A) = X \cap (a + 2h(X, A)B).$$

Observe that:

$$(5.3) \qquad P(A, a, A) = \{a\}.$$

It is not difficult to prove that the intersection map $(X, Y) \to X \cap Y$ restricted to the set

$$\{ (X, Y): X \in \mathcal{K}_n, Y \in \mathcal{B}_n, X \cap Y \neq \emptyset \}$$

is continuous, also the map

$$(X, a, A) \to (a + 2h(X, A)B)$$

is obviously continuous, therefore the map $P(\cdot, \cdot, \cdot)$ is continuous.

It is proved in **[11]** that:

$$(5.4) \qquad h(P(X, a, A), P(Y, a, A)) \leqq Lh(X, Y)$$

for all $X, Y, A \in \mathcal{K}_n$, $a \in A$, where $L = (28/3)^{1/2}$. In particular, $3 < L < 4$.

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Define a map $p:\tilde{\mathscr{A}} \to R^n$ as follows:

(5.5) $\quad p(X, a, A) = s_n(P(X, a, A))$.

Then the map $p(\cdot, \cdot, \cdot)$ is continuous and by (5.3), (5.4) it satisfies:

(5.6) $\quad p(A, a, A) = a$

and

(5.7) $\quad \|p(X, a, A) - p(Y, a, A)\| \leqq 4nh(X, Y)$ for $X, Y \in \mathscr{K}_n$.

To prove Theorem 3.2 define $f(\cdot, \cdot):[0, 1] \times \mathbf{R}^n \to \mathbf{R}^n$ by:

(5.8) $\quad f(t, x) = p(F(t, x), u(t), F(t, z(t)))$ a.e. in [0, 1].

It follows from (5.1), (5.5) that $f(t, x)$ satisfies (i). From (5.7) we obtain that:

$$\|f(t, x) - f(t, y)\| \leqq 4nh(F(t, x), F(t, y))$$

which implies (iv). The condition (ii) follows from (5.6). The measurability of $f(t, x)$ in t follows easily from the Scorza-Dragoni and Luzin theorems. Hence $f(t, x)$ defined by (5.8) satisfies all conditions required in Theorem 3.2 and the proof is completed.

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Université de Paris-Dauphine,
Paris, France;
University of Rhode Island,
Kingston, Rhode Island