An Analytical Study of Plane Rolling Mechanisms.

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1. *Introductory.*

In the Appendix III. of his "Mechanics of Machinery," Le Conte attacks the problem of rolling curves by an elegant analytical method in which the cartesian forms of the involute and the trochoids are derived from elementary differential equations. Weisbach ("Mechanics of Engineering and of Machinery," Vol. III., chap. II.), while using mainly the geometrical method, discusses one application analytically (see §3 of this paper) in which polar forms are introduced. Many other writers, for example Barr ("Kinematics of Machinery," chaps. III., IV.), deal wholly with the geometry of the subject.

In the present paper rolling curves are discussed analytically from the standpoint of polar and pedal forms. While it is not proposed to consider the relative merits of the two methods, it may be noted that some more general results follow from analysis; for example, that the log spiral and ellipse are derivable from a single formula, and are only two members of a family of self rolling-curves, and that the epicycloid will gear with other curves besides an epi- or hypo-cycloid.

2. *Pure Rolling.*

POLAR FORMS.

Two curves have pure rolling on each other only when their point of contact lies in the line of centres. For, P's velocity must be the same round O and O′, and therefore must be perpendicular both to PO and PO′.

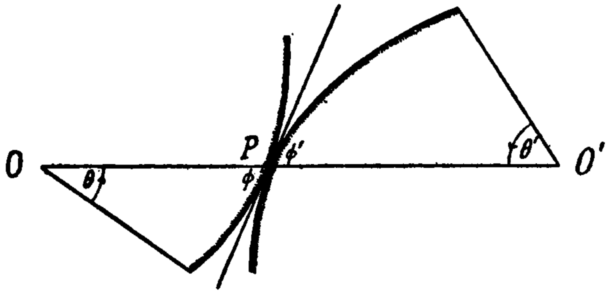

Fig. 1.

The analytical expression of this condition is

$$r+r'=d, \text{ and } \phi=\phi',$$

Instead of $\phi=\phi'$ we shall use $r\dfrac{d\theta}{dr}=r'\dfrac{d\theta'}{dr'}$.

To find a curve which will roll with a given curve $r=f(\theta)$, that is, to find a relation between r', θ' we have only to eliminate r, θ. We may write $\theta=f^{-1}(r)$ as the formula inverse to $r=f(\theta)$, so that

$$\frac{dr}{d\theta}=f'(\theta)=f'(f^{-1}(r)),$$

and
$$\frac{1}{r'}\frac{dr'}{d\theta'}=\frac{1}{r}\frac{dr}{d\theta}=\frac{1}{d-r'}f'(f^{-1}(d-r')),$$

whence
$$\theta'=\int\frac{d-r'}{r'f'(f^{-1}(d-r'))}dr'.$$

If the first curve is defined in the form $\theta=\phi(r)$

then
$$\theta'=\int\frac{(d-r')\phi'(d-r')}{r'}dr'.$$

The following examples illustrate these formulae :—

1. *The log. spiral.* $\qquad r=e^{a\theta}$

$$f'(\theta)=ae^{a\theta}\,;\ f'(f^{-1}(r))=ar.$$

$$\therefore\quad \theta'=\int\frac{d-r'}{r'a(d-r')}dr'=\int\frac{dr'}{ar'}.$$

$$\therefore\quad r'=e^{a\theta'}.$$

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2. *The ellipse turning about a focus.* $\frac{l}{r} = 1 + e\cos\theta$,

$$\frac{l}{r^2}\frac{dr}{d\theta} = e\sin\theta = \sqrt{\left\{e^2 - \frac{(l-r)^2}{r^2}\right\}},$$

$$f'(f^{-1}(r)) = \frac{r}{l}\sqrt{\{e^2r^2 - (l-r)^2\}},$$

$$\therefore \quad \theta' = \int \frac{l(d-r')dr'}{r'(d-r')\sqrt{\{e^2(d-r')^2 - (l-d+r')^2\}}}.$$

This integral is easily evaluated in the form

$$\frac{l}{\mathrm{L}}\cos^{-1}\frac{1}{le}\left(\frac{\mathrm{L}^2}{r'} - \mathrm{D}\right),$$

where $\mathrm{L}^2 = (d-l)^2 - e^2d^2$,

and $\mathrm{D} = d(1-e^2) - l$.

The curve having rolling contact with the ellipse is thus given by

$$\frac{\mathrm{L}^2}{r'} = \mathrm{D} + le\cos\frac{\mathrm{L}}{l}\theta',$$

but if $\mathrm{L} = l$,

$$l^2 = d^2(1-e^2) - 2dl + l^2,$$

$$\therefore \quad d(1-e^2) = 2l = \mathrm{D} + l,$$

$$\therefore \quad \mathrm{D} = l.$$

Hence the second curve is an (equal) ellipse in one case.

3. *The circle.* $r = a\sin\theta$.

Here $$\theta' = \sin^{-1}\frac{d-r'}{a} - \frac{d}{\sqrt{(d^2-a^2)}}\sin^{-1}\frac{d^2-a^2-dr'}{ar'}.$$

4. *The straight line.* $r = c\sec\theta$

gives $$\frac{d^2-c^2}{r'} = d + c\cosh\left(\frac{\sqrt{(d^2-c^2)}}{c}.\theta'\right),$$

which takes the simpler form $\frac{c}{r'} = \sqrt{2} + \cosh\theta'$ when $d = c\sqrt{2}$.

[For a drawing see Barr, "Kinematics," p. 89.]

5. *The involute of a circle.* $\theta = \frac{\sqrt{(r^2-a^2)}}{a} - \cos^{-1}\frac{a}{r}$ rolls with

$$a\theta' = \sqrt{\{(d-r')^2 - a^2\}} - d\cosh^{-1}\frac{d-r'}{a} - \sqrt{(d^2-a^2)}\cosh^{-1}\left(\frac{d^2-a^2}{ar'} - \frac{d}{a}\right)$$

No relation between d and a can reduce this to the involute form. Hence there cannot be pure rolling in the gearing of involute teeth.

Pedal Forms.

The conditions are expressed by $r + r' = d$; $\frac{p}{r} = \frac{p'}{r'}$. Hence if $p = f(r)$ is the equation of one curve, we have only to eliminate p, r for that of the other.

$$\frac{p}{r} = \frac{f(r)}{r} = \frac{f(d-r')}{d-r'} = \frac{p'}{r'}.$$

$$\therefore \quad p' = \frac{r'}{d-r'} f(d-r').$$

This simple formula gives an immediate solution for several cases.

If $p = ar$, then $p' = \frac{r'}{d-r'} a(d-r') = ar'$, the case of log spirals.

Again, if $$\frac{b}{p} = \sqrt{\left(\frac{2a}{r} - 1\right)},$$

then $$\frac{b}{p'} = \frac{d-r'}{r'} \sqrt{\left(\frac{2a}{d-r'} - 1\right)}$$

$$= \frac{1}{r'} \sqrt{(d-r')(2a-d+r')}$$

$$= \sqrt{\left(\frac{2a}{r'} - 1\right)}$$

when $d = 2a$, which again establishes the rolling property of equal ellipses. The straight line and the involute can be dealt with in the same simple way. The pedal form therefore is of great advantage when the second curve can be recognised in pedal coordinates. It is of course possible to trace the curve by calculating from $p = r \times \sin\phi$ and plotting points in close succession, as in graphical integration. When feasible, however, it is preferable to convert into polars. For $p = f(r)$,

$$\theta = \int \frac{f(r)dr}{r\sqrt{(r^2 - \{f(r)\}^2)}} = \int \frac{\phi(r)}{r} dr,$$

$$\theta' = \int \frac{f(d-r')dr'}{r'\sqrt{((d-r')^2 - \{f(d-r')\}^2)}} = \int \frac{\phi(d-r')}{r'} dr'.$$

These form also define a pair of rolling curves.

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3. *Self-Rolling Curves.*

The fact that the log spiral and the ellipse in pure rolling and the involute and epicycloid in slip-rolling are self-rolling curves used by engineers for gearing suggests searching for other curves possessing the same property. The formulae of § 2 give this result at once since

$$\int \frac{\phi(r)}{r} dr \equiv \int \frac{\phi(d-r)}{r} dr$$

if $$\phi(r) \equiv \phi(d-r).$$

In particular, the forms $\int \frac{F(r)+F(d-r)}{r} dr$ and $\int \frac{F(r) \,.\, F(d-r)}{r} dr$ define an infinite number of self-rolling curves. A simple example is provided by $F(r) \equiv r$ in the second type,

$$\theta = \int \frac{r(d-r)}{r} dr = rd - \frac{r^2}{2}.$$

This curve would give a symmetrical reciprocal motion of rotation to two parallel shafts, and is easily traced by its relation to a parabola. The log spiral is given by $\phi(r) = c$, and the ellipse by

$$\phi(r) = c \div \sqrt{\left\{\frac{d^2}{2} - r^2 - (d-r)^2\right\}}.$$

It should be noted that other factors enter into the problem of applying pure rolling curves to gearing. For example, it is possible to design log spiral teeth on two wheels to gear with each other. It will be seen, however, that the frictional advantage of making contact on the line of centres is more than compensated by the difficulty of clearing non-gearing pairs of teeth and at the same time ensuring a strong form of tooth.

The practical problem, however, is to find the forms of both curves for a prescribed variable velocity ratio.

Since $r\frac{d\theta}{dr} = r'\frac{d\theta'}{dr'}$ and $dr = -dr'$ we have

$$\text{velocity ratio} = \frac{r}{r'} = -\frac{d\theta'}{d\theta}.$$

Hence the datum may be taken as $\theta' = F(\theta)$ and then $\frac{r}{r'} = -F'(\theta)$, from which, using $r + r' = d$, we get

$$r - \frac{r}{F'(\theta)} = d,$$

$$r' - r'F'\{F^{-1}(\theta')\} = d,$$

as the equations defining the required curves. As these results in a modified form are illustrated in Weisbach ("Mechanics of Engineering," Vol. III., pp. 190–195) further consideration need not be given them here.

4. *Slip Rolling.*

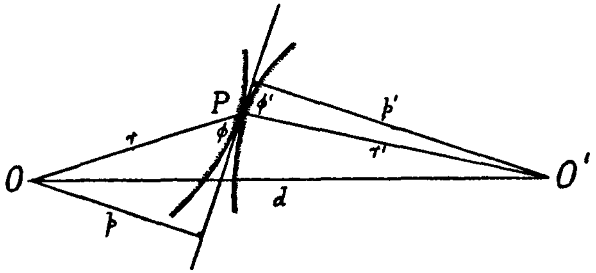

Fig. 2.

Polar coordinates give the following relations

$$d^2 = r^2 + r'^2 + 2rr'\cos(\phi - \phi'),$$

$$\tan\phi = \frac{rd\theta}{dr}\ ;\ \tan\phi' = r'\frac{d\theta'}{dr'}\ ;\ r = f(\theta).$$

To find r' in terms of θ' we must eliminate ϕ, ϕ', r, θ and this requires another equation expressing some further condition (in this case arbitrary). But even this simplest of these conditions does not appear to render the elimination tractable. We therefore proceed to consider pedal coordinates, which free us at first from a differential equation.

Here $$p = r\sin\phi\ ;\ \sqrt{(r^2 - p^2)} = r\cos\phi.$$

Hence the fundamental equation becomes

$$d^2 = r^2 + r'^2 + 2pp' + 2\sqrt{(r^2 - p^2)\,(r'^2 - p'^2)},$$

from which, in conjunction with $p = f(r)$ and the special equation of condition, p and r are to be eliminated. The following are examples of this condition :—

1. If the direction of the common normal be fixed

$$p + p' = c.$$

2. If the common normal pass through a fixed point on OO′ (the condition for a fixed velocity ratio)

$$\sqrt{(r^2 - p^2)} = c\sqrt{(r'^2 - p'^2)}.$$

3. If P describe an ellipse

$$r + r' = c.$$

4. If P lie on a line distant k from OO′

$$p\sqrt{(r'^2 - p'^2)} - p'\sqrt{(r^2 - p^2)} = kd.$$

5. If P lie on a line perpendicular to OO′, distant c from O

$$r^2 - r'^2 = 2dc - d^2.$$

To prove that the involute gears with any other involute.

Take condition I. and the pedal equation of the involute

$$r^2 - p^2 = a^2.$$

Then
$$d^2 = (p^2 + a^2) + r'^2 + 2a\sqrt{(r'^2 - p'^2)} + 2pp'$$
$$= (c - p')^2 + a^2 + r'^2 + 2a\sqrt{(r'^2 - p'^2)} + 2p'(c - p').$$
$$\therefore \quad (r'^2 - p'^2) + 2a\sqrt{(r'^2 - p'^2)} + c^2 + a^2 - d^2 = 0.$$
$$\therefore \quad \sqrt{(r'^2 - p'^2)} = -a \pm \sqrt{(d^2 - c^2)}.$$
$$\therefore \quad r'^2 - p'^2 = (a \mp b)^2; \quad b^2 \equiv d^2 - c^2.$$

It is easily seen from a figure that $b - a$, $b + a$ are the radii of the "root" circles for external and internal contact. As these radii depend upon the arbitrary constant c the number of rolling involutes is infinite.

To prove that the epicycloid gears with a hypocycloid or an epicycloid.

The condition required is that the common normal meets the line of centres in a certain fixed point. If the epicycloid is defined by

$$r^2 = a^2 + \frac{4b(a+b)}{(a+2b)^2}p^2 \tag{1}$$

then the value of c is $a \div (d - a)$, so that

$$(d - a)\sqrt{(r^2 - p^2)} = a\sqrt{(r'^2 - p'^2)} \tag{2}$$

Eliminating r from these equations

$$a^2(d-a)^2 + \frac{4b(a+b)(d-a)^2}{(a+2b)^2}p^2 = (d-a)^2p^2 + a^2(r'^2 - p'^2)$$
$$p^2(d-a)^2 = (a+2b)^2\{(d-a)^2 - r'^2 + p'^2\}.$$

Put
$$p(d - a) = \pm(a + 2b)x \tag{3}$$

Next eliminate r between (1) and the fundamental equation,

$$d^2 = r^2 + r'^2 + 2\sqrt{(r^2 - p^2)(r'^2 - p'^2)} + 2pp'$$

$$d^2 = a^2 + \frac{4b(a+b)}{(a+2b)^2}p^2 + r'^2 + \frac{2a}{d-a}(r'^2 - p'^2) + 2pp'.$$

Inserting the value of p given by (3), we have

$$(r'^2 - d^2 + a^2)(d-a)^2 + 4b(a+b)x^2 + 2a(d-a)(r'^2 - p'^2)$$
$$\pm 2p'(d-a)(a+2b)x = 0.$$

But $$r'^2 = (d-a)^2 + p'^2 - x^2.$$

$$\therefore \quad x^2[4b(a+b) - d^2 + a^2] \pm 2xp'(d-a)(a+2b) + p'^2(d-a)^2 = 0.$$

Solve as a quadratic in x

$$x(a+2b) \pm p'(d-a) = \pm xd \qquad (4)$$

$$x(a+2b \pm d) = \pm p'(d-a)$$

$$r'^2 = (d-a)^2 + p'^2\left\{1 - \frac{(d-a)^2}{(a+2b \pm d)^2}\right\}$$

$$r'^2 = (d-a)^2 - \frac{4b(d-a-b)}{(d-a-2b)^2}p'^2 \quad \text{(lower sign)}$$

$$r'^2 = (d-a)^2 + \frac{4(b+d)(a+b)}{(d+a+2b)^2}p'^2 \quad \text{(upper sign).}$$

The first equation shows that the gearing curve is a hypocycloid if $d > a$ (external contact) and an epicycloid if $d < a$ (internal contact), the radius of the base circle being $d - a$ and that of the rolling circle b.

In the second equation $b = 0$ gives an epicycloid which does not gear with the original curve $r = a$; and the upper sign must be rejected in the right hand member of 4, where we can only say that at least one sign must give a correct result.

The General Problem.—To prove analytically that when any curve rolls on two pure rolling curves it traces out curves which gear with each other.

Let p_1, r_1 and p_2, r_2 (or $d - r_1$) refer to the given pure rolling (or "pitch") curves; P, ρ to the curve in contact at A rolling outside the first and inside the second pitch curve; and p, r and p', r' to the curves generated by B.

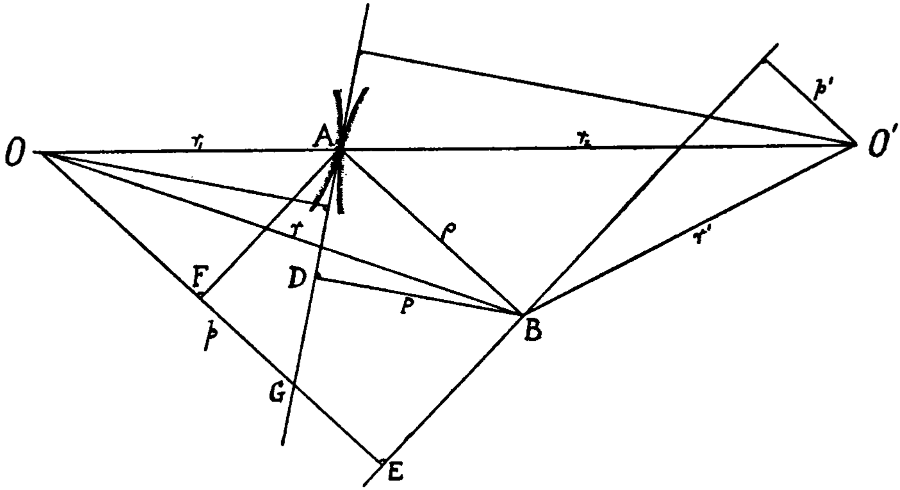

Fig. 3.

Relations between p, r, P, ρ, p_1, r_1 :—

$$p = \text{OF} + \text{FE}.$$
$$= r_1 \sin(\phi_1 + \Phi - 90°) + \rho$$
$$= \rho - r_1 \cos(\phi_1 + \Phi). \qquad (1)$$
$$\sqrt{(r^2 - p^2)} = \text{EB} = \text{FA}$$
$$= r_1 \cos(\phi_1 + \Phi - 90°)$$
$$= r_1 \sin(\phi_1 + \Phi). \qquad (2)$$

Eliminating $\phi_1 + \Phi$ from (1), (2),

$$r_1^2 = r^2 + \rho^2 - 2\rho p. \qquad (3)$$

Again
$$p = \text{OG} + \text{GE}$$
$$= \text{OG} + \text{AB} - \text{EB}\cot\Phi$$
$$= p_1 \operatorname{cosec}\Phi + \rho - \cot\Phi \sqrt{(r^2 - p^2)}$$
$$= p_1 \frac{\rho}{\text{P}} + \rho - \frac{1}{\text{P}} \sqrt{(r^2 - p^2)(\rho^2 - \text{P}^2)}. \qquad (4)$$

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Relations connecting p', r', P, ρ, p_2, r_2 are got by changing the signs of $\qquad$ P, ρ, $\surd(\rho^2 - \mathrm{P}^2)$.

$$\therefore \quad r_2^2 = (d - r_1)^2 = r_1^2 + \rho^2 + 2\rho p'$$

$$p' = p_2 \frac{\rho}{\mathrm{P}} - \rho - \frac{1}{\mathrm{P}} \surd(r'^2 - p'^2)(\rho^2 - \mathrm{P}^2).$$

The common normal passes through A. Hence we must prove that the relations between the pr and p', r' curves are such that the fundamental equation

$$d^2 = r^2 + r'^2 + 2pp' + 2\surd(r^2 - p^2)(r'^2 - p'^2) \qquad (5)$$

is identically satisfied, when we have the condition

$$\sqrt{(r^2 - p^2)} \,.\, (d - r_1) = \sqrt{(r'^2 - p'^2)} \,.\, r_1 \qquad (6)$$

From the pitch curves we have

$$\frac{p_1}{r_1} = \frac{p_2}{d - r_1} \qquad (7)$$

Applying (6) and (7) to $4a$

$$p' = \frac{d - r_1}{r_1} p_1 \frac{\rho}{\mathrm{P}} - \rho - \frac{1}{\mathrm{P}} \cdot \frac{d - r_1}{r_1} \surd(r^2 - p^2)(\rho^2 - \mathrm{P}^2)$$

$$= \frac{d - r_1}{r_1}\left\{p_1 \frac{\rho}{\mathrm{P}} + \left(p - \rho - p_1 \frac{\rho}{\mathrm{P}}\right)\right\} - \rho.$$

$$\therefore \quad p' r_1 = (d - r_1)(p - \rho) - \rho r_1$$

$$= pd - pr_1 - \rho d \qquad (8)$$

Now express (5) in terms of r, ρ, p

$$r^2 + r'^2 = (r_1^2 - \rho^2 + 2\rho p) + (d - r_1)^2 - \rho^2 - 2\rho p'$$

$$= r_1^2 + (d - r_1)^2 - 2\rho^2 + 2\rho(p - p').$$

$$\therefore \quad r_1(r^2 + r'^2) = r_1(2r_1^2 - 2dr_1 + d^2 - 2\rho^2)$$

$$+ 2\rho(pr_1 - pd + pr_1 + \rho d) \text{ by (8)} \qquad (9)$$

Also $\qquad r_1(2pp' + 2\surd(r^2 - p^2)(r'^2 - p'^2)$

$$= 2p(pd - pr_1 - \rho d) + 2(d - r_1)(r^2 - p^2)$$

$$= 2dr_1^2 - 2p\rho d - 2d\rho^2 + 4d\rho p - 2r_1^3 + 2r_1\rho^2 - 4r_1\rho p \qquad (10)$$

Adding (9) and (10) the terms on the right cancel and leave (5) identically satisfied. Hence the proposition is proved.

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