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SOME RECENT RESULTS ON INVARIANT SUBSPACES

BY
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1. **Introduction.** This expository paper surveys work on invariant subspaces and related topics which has been done in the past few years. We recommend, naturally, that the reader consult [52] for work done prior to 1973 and [54] for a discussion of some of the consequences of Lomonosov's Lemma; (Lomonosov's paper has now also appeared in English ([36])).

For simplicity we consider operators (i.e., bounded linear transformations) on separable complex Hilbert space only, although many of the results could be stated for Banach spaces. If $\mathcal{S}$ is any subset of $\mathcal{B}(\mathcal{H})$ (the algebra of operators on $\mathcal{H}$) we use Lat $\mathcal{S}$ to denote the lattice of all subspaces which are invariant under all the operators in $\mathcal{S}$. The *invariant subspace problem* is the question: does every operator have a non-trivial (i.e., different from $\{0\}$ and $\mathcal{H}$) invariant subspace? Definitions of all terms used below can be found in [52].

2. **Three unusual theorems on existence of invariant subspaces.** Suppose that the Hilbert space $\mathcal{H}$ is decomposed as a direct sum (not necessarily orthogonal) in two different ways: $\mathcal{H}=\mathcal{K}+\mathcal{L}=\mathcal{M}+\mathcal{N}$ with $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ closed subspaces. Must there exist subspaces $\mathcal{K}_0$, $\mathcal{L}_0$, $\mathcal{M}_0$, and $\mathcal{N}_0$ of $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ respectively such that $\mathcal{K}_0+\mathcal{L}_0=\mathcal{M}_0+\mathcal{N}_0$ (and such that $\mathcal{K}_0+\mathcal{L}_0$ is not $\{0\}$ or $\mathcal{H}$)? This question seems to be about the geometry of Hilbert space, and certainly appears tractable at first glance. If we let P denote the projection of $\mathcal{H}$ onto $\mathcal{K}$ along $\mathcal{L}$ and let Q denote the projection of $\mathcal{H}$ onto $\mathcal{M}$ along $\mathcal{N}$ then we are asking whether the idempotents P and Q have a common non-trivial invariant subspace. Chandler Davis [19] constructed three self-adjoint idempotents which have no common non-trivial invariant subspaces. Eric Nordgren, Heydar Radjavi, and I were surprised to discover the following.

2.1. THEOREM ([43]). *Every operator has a non-trivial invariant subspace if and only if every pair of idempotents has a common non-trivial invariant subspace.*

The proof of Theorem 2.1 is very elementary. In one direction it rests on the observation that $\begin{pmatrix} A & A \\ 1-A & 1-A \end{pmatrix}$ is an idempotent for all operators A; a

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common invariant subspace for $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} A & A \\ 1-A & 1-A \end{pmatrix}$ leads to an invariant subspace for A. Conversely, if $\mathcal{S}$ is an invariant subspace of $(PQP) \mid P\mathcal{H}$, it is easily shown that $\mathcal{S}+\overline{(1-P)Q\mathcal{S}}$ is a common invariant subspace for P and Q.

The fact that the proof of Theorem 2.1 is so simple does not convince me that it won't be useful in attacking the invariant subspace problem; (remember Hilden's proof of Lomonosov's Theorem ([52], p. 158)?). The formulation in terms of direct-sum decompositions seems different enough to me that I'm willing to spend more time testing different decompositions (looking for a *negative* result).

An interesting variant of Lomonosov's Theorem was obtained by Daughtry.

2.2. THEOREM ([17]). *If $AK-KA$ has rank 1 for some compact operator K, then A has a non-trivial invariant subspace.*

Lomonosov's Theorem, of course, is the case where $AK-KA$ has rank 0 for some compact K other than 0. Daughtry's proof requires Lomonosov's Lemma ([52], p. 156) and is short and elegant. A question that Theorem 2.2 suggests is: if $AK-KA$ has rank 2 for some compact K must A have an invariant subspace? As Daughtry [17] points out, this is equivalent to the invariant subspace problem since $AK-KA$ has rank at most 2 if K is chosen to have rank 1.

Many people have attempted to construct operators without invariant subspaces. A class of potential examples was introduced by E. Bishop. For α any irrational number in $[0, 1]$ the *Bishop operator* B_α corresponding to α sends the function $f \in \mathcal{L}^2(0, 1)$ into the function whose value at $t \in [0, 1]$ is $tf(t+\alpha)$, where $t+\alpha$ is computed modulo 1. For more than 10 years it was not known whether any B_α had an invariant subspace. On the other hand, it was also not known whether any Bishop operator had a cyclic vector; it was not even known if the function identically 1 was cyclic for any Bishop operator! A. M. Davie [18] recently made a powerful attack on existence of invariant subspaces for Bishop operators. Davie's work does not completely settle the question, however. In fact, Davie's results heighten the mystery of Bishop's operators. Davie proves that B_α has an invariant subspace for almost every α! The precise theorem is as follows.

2.3. THEOREM ([18]). *If $\alpha \in [0, 1]$ and there is no sequence $\{p_n/q_n\}$ of rational numbers with $q_n \geq 2$ and $|\alpha - (p_n/q_n)| < q_n^{-n}$, then the Bishop operator B_α has a non-trivial hyperinvariant subspace.*

The set of α satisfying the hypothesis of Theorem 2.3 has measure 1 and contains all algebraic numbers. The proof of Theorem 2.3 involves a number-

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theoretic result of Dirichlet, the Denjoy–Carleman Theorem, varying of a technique of Wermer's, and a great deal of ingenuity!

3. **Strongly reductive operators.** Recall that the operator A is *reductive* if $\mathcal{M} \in \operatorname{Lat} A$ implies $\mathcal{M}^{\perp} \in \operatorname{Lat} A$. The Dyer–Porcelli Theorem states that every operator has an invariant subspace if and only if every reductive operator is normal, and there are many special cases in which it is known that reductivity implies normality; (see section 6 below).

Moore [37] defined an operator T to be *strongly reductive* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|(1-P)TP\| < \delta$ for a Hermitian projection P implies $\|PT - TP\| < \varepsilon$. Informally, each approximately invariant subspace is approximately reducing.

Strongly reductive operators are obviously reductive, so a natural question is whether or not strongly reductive operators must be normal. It is clear that Hermitian operators are strongly reductive; Harrison [29] characterizes strongly reductive normal operators.

3.1. Theorem ([29]). *If T is a normal operator, then the following are equivalent:*

(i) *T is strongly reductive,*
(ii) *$\sigma(T)$ neither divides the plane nor has interior,*
(iii) *T^* is the uniform limit of a sequence of polynomials in T.*

Theorem 3.1 is, as Harrison notes, related to a corollary of Sarason's Theorem ([56], [52, p. 180]), which states that a normal operator T is reductive if and only if T^* is the weak limit of a net of polynomials in T.

To prove that strongly reductive operators are normal it must be shown that strongly reductive operators have invariant subspaces. A partial result was obtained by Apostol and Fong [7].

3.2. Theorem ([7]). *If T is strongly reductive and if the uniformly closed inverse-closed algebra generated by $\{1, T\}$ contains an operator whose essential norm is less than its norm, (i.e., some S with $\inf\{\|S+K\| : K \text{ compact}\} < \|S\|$), then T has a non-trivial invariant subspace.*

The proof of Theorem 3.2 is very interesting. First it is observed that the Apostol–Foias–Voiculescu [4] characterization of quasi-triangular operators (also discussed in [21]) implies that T is quasi-triangular. Then the Aronszajn–Smith technique is used to produce sequences $\{P'_n\}$ and $\{P''_n\}$ of finite-rank projections each converging weakly such that $\|(1-P'_n)TP'_n\|$ and $\|(1-P''_n)TP''_n\|$ both approach 0. The strong reductivity obviously implies that T commutes with the weak limits of $\{P'_n\}$ and $\{P''_n\}$. Since these weak limits are Hermitian, we are done—*if* one of the weak limits is not a multiple of the identity. In the Aronszajn–Smith proof, and in subsequent improvements, compactness is used to get sequences $\{P'_n\}$ and $\{P''_n\}$ such that one of their weak limits is not 0 or 1. In the present Apostol–Fong proof it is shown that choosing a unit vector e and making

$$(P'_n e, e) \leq \alpha \leq (P''_n e, e)$$

for a suitable positive number α, where α is determined by the operator whose essential norm is less than its norm, implies that the resulting $\{P'_n\}$ and $\{P''_n\}$ cannot both converge to multiples of the identity.

Lomonosov may have killed Aronszajn–Smith; if so, Apostol and Fong have resurrected them, and Apostol–Foias–Voiculescu [5] have raised them to new heights.

3.3. THEOREM ([5]). *If T is strongly reductive, then T is normal.*

The main problem in proving Theorem 3.3 is in proving that T has an invariant subspace; the result can then be completed by using the Dyer–Porcelli technique or ideas from ([14], sections 6.10 and 6.11), as [5] points out. The proof that T has invariant subspaces begins with Theorem 3.2. If Theorem 3.2 does not produce an invariant subspace for T it must be the case that the essential norm of $p(T)$ is equal to the norm of $p(T)$ for all polynomials p; i.e., the natural projection of the algebra of polynomials in T into the Calkin algebra is isometric. Now it suffices to show that $TT^* - T^*T$ is compact. For when this is known the image of T in the Calkin algebra is normal and the uniformly closed algebra it generates is isometrically isomorphic to the continuous functions on its spectrum. This leads to an isometry of a $\mathscr{C}(\mathscr{X})$ into $\mathscr{B}(\mathscr{H})$ with T corresponding to a generator of $\mathscr{C}(\mathscr{X})$, so a well-known result of Dunford implies that T is a spectral operator of scalar type.

The problem of showing that $TT^* - T^*T$ is compact is very difficult. The authors of [5] cleverly reduce this to a deep result of [58] on decomposing "approximate equivalents" of representations of separable $\mathscr{C}^*$-algebras.

Theorems 3.1 and 3.3 together immediately yield a characterization of strongly reductive operators.

3.4. COROLLARY. *The operator T is strongly reductive if and only if T^* is a uniform limit of polynomials in T, (and this occurs if and only if T is a normal operator whose spectrum has no interior and does not separate the plane).*

Corollary 3.4 is striking in that T's behavior with respect to approximately invariant subspaces implies normality. I certainly expect this result and the ideas behind it to have further ramifications. Apostol, Foias, and Voiculescu have already extended their result to prove that strongly reductive separable uniformly-closed commutative algebras are $\mathscr{C}^*$-algebras [6]. This theorem also relies heavily on [58].

4. **Compact perturbations of normal operators.** It has been known for some time that certain perturbations of normal operators have invariant subspaces. The next theorem incorporated work of Macaev, Schwartz, Kitano, and others.

4.1. THEOREM ([52], Corollary 6.13). *If $T = A + B$ where A is normal, $\sigma(A)$ is contained in a twice-differentiable Jordan arc and $B \in \mathscr{C}_p$ for some $p \geq 1$, then T has a non-trivial hyperinvariant subspace.*

(Note that the case where $\sigma(T)$ is a singleton is covered by Lomonosov's Theorem).

There have been several improvements of Theorem 4.1 that should be mentioned. The theorem can be generalized by broadening the class of B's allowed or by relaxing the requirement that A be normal. A generalization of the first type has been found by Kitano [34], and generalizations of the second type by Apostol [2] and Jafarian [31]. An extension in both directions is given by Radjabalipour and Radjavi [50].

4.2. THEOREM ([50]). *If $T = A + B$ where A has spectrum contained in a twice-differentiable Jordan arc J such that*

$$\|(z-A)^{-1}\| \leq \frac{K}{\text{dist}(z, J)} \quad (\textit{for some } K)$$

for $z \notin J$, and where $B \in C_\omega$ (the Macaev ideal), then T has a non-trivial hyperinvariant subspace.

Results such as Theorems 4.1 and 4.2 above have corollaries for operators such that $T - T^* \in \mathscr{C}_p$ (i.e., perturbations of Hermitian operators) and for operators such that $1 - T^*T \in \mathscr{C}_p$ (i.e., perturbations of unitary operators); (cf. Corollaries 6.15 and 6.16 of [52]). A natural question (cf. [52], p. 194) is: if $\sigma(T)$ is suitably thin and T is close to normal in the sense that $T^*T - TT^* \in \mathscr{C}_p$, must T have an invariant subspace? The remarkable study by Brown, Douglas, and Fillmore ([11], [12]) of essentially normal operators gives some insight into this question. For it is an immediate consequence of ([11], 11.2 Corollary) that $T^*T - TT^*$ compact and T *not* the sum of a normal operator and a compact operator implies T has a hyperinvariant subspace. Thus a partial result in answer to the above question would follow from a sufficient condition that

(i) $\sigma(T)$ thin,
(ii) $T^*T - TT^* \in \mathscr{C}_p$ and
(iii) $T = N + K$, N normal and K compact

imply that $T = N_0 + K_0$ with N_0 normal and $K_0 \in \mathscr{C}_p$. (More generally, replace p by ω). Some such condition must exist.

A related question has been asked by Radjabalipour and Radjavi [49]: if T satisfies the hypotheses of Theorem 4.1 above, and if $\mathcal{M}$ is hyperinvariant for T, must $T \mid \mathcal{M}$ satisfy the hypotheses of Theorem 4.1?

Sufficient conditions that a compact perturbation of a normal operator be decomposable (in the sense of [14]) are given in [49] and [48].

5. **Transitive operator algebras.** A *transitive algebra* is a subalgebra of $\mathcal{B}(\mathcal{H})$ which is weakly closed and has no invariant subspaces other than $\{0\}$ and $\mathcal{H}$, and the *transitive algebra problem* is the question of whether there exist any transitive algebras other than $\mathcal{B}(\mathcal{H})$. A number of partial solutions to the transitive algebra problem can be found in ([52], Chapter 8). Lomonosov's work [36] has a number of corollaries about transitive algebras, some of which are discussed in [54] and several of which we now consider.

5.1. THEOREM ([38], [30]). *If T is a contraction in the Sz.–Nagy–Foias class $\mathcal{C}_0$ (cf. [57]), and if $1-T^*T$ and $1-TT^*$ are compact, then the weakly closed algebra generated by T contains a non-zero compact operator.*

The proof of Theorem 5.1 relies on Muhly's characterization [39] of the compact operators in the commutant of a $\mathcal{C}_0$ operator. Theorem 5.1 was inspired by [13], which deals with the special case where $1-T^*T$ and $1-TT^*$ have finite rank.

A transitive algebra result is an immediate consequence of Theorem 5.1 and a corollary of Lomonosov's lemma; ([52], Theorem 8.23).

5.2. COROLLARY ([38], [30]). *If $T\in\mathcal{C}_0$ and $1-T^*T$ and $1-TT^*$ are compact, then the only transitive algebra containing T is $\mathcal{B}(\mathcal{H})$.*

Another recently discovered relative of Lomonosov's Theorem is a generalization of Foias' theorem ([24], [52, Theorem 8.9]) on algebras with no invariant operator ranges.

5.3. THEOREM ([45]). *If $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ all of whose proper invariant operator ranges are ranges of compact operators, then either $\mathcal{A}$ has a finite-dimensional invariant subspace other than $\{0\}$ or $\mathcal{A}$ is weakly dense in $\mathcal{B}(\mathcal{H})$.*

Theorem 5.3 is proven using graph transformations. If $\mathcal{A}$ has no finite-dimensional invariant subspaces then $\mathcal{A}$ is transitive, and it is shown that the graph transformations for $\mathcal{A}$ must all be bounded.

A fairly natural generalization of Lomonosov's Theorem does not require the compact operator to commute with any operator.

5.4. THEOREM ([41]). *If $\mathcal{A}$ is a uniformly closed subalgebra of $\mathcal{B}(\mathcal{H})$ and K is an injective quasinilpotent compact operator such that $\mathcal{A}K\subset K\mathcal{A}$, then $\mathcal{A}$ has a non-trivial invariant subspace.*

Hilden's proof of Lomonosov's theorem ([52], p. 165) is easily modified to yield Theorem 5.4. Handling of the case where K has point spectrum requires a little more work; this leads to the following generalization.

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5.5. THEOREM ([41]). *If $\mathcal{A}$ is a uniformly closed subalgebra of $\mathcal{B}(\mathcal{H})$ such that $\mathcal{A}K = K\mathcal{A}$ for some non-zero compact operator K, then $\mathcal{A}$ has a non-trivial invariant subspace.*

Another variant of Hilden's proof was found by Pearcy and Shields [46].

5.6. THEOREM ([46]). *If A is a quasinilpotent operator different from* 0, *and if there exists a bounded sequence $\{J_n\}$ of operators different from* 0 *with J_0 compact and $AJ_n = J_{n+1}A$ for all n, then A has a non-trivial hyper-invariant subspace.*

6. **Reductive operator algebras.** Recall that a subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is *reductive* if $\mathcal{M} \in \operatorname{Lat} \mathcal{S}$ implies $\mathcal{M}^\perp \in \operatorname{Lat} \mathcal{S}$, and the *reductive algebra problem* is the question: is every weakly closed reductive algebra self-adjoint? See ([52], Chapter 9) for basic results on reductive algebras.

Dyer and Procelli [22] and Dyer, Pedersen, and Procelli [23] proved that every operator has an invariant subspace if and only if every reductive operator is normal. This work has been extended by Azoff, Fong, and Gilfeather as follows.

6.1. THEOREM ([9]). *If $\mathcal{A}$ is a reductive algebra then $\mathcal{A}$ has a direct integral decomposition $\mathcal{A} \sim \int_\Lambda \oplus \mathcal{A}(\lambda)\mu\,(d\lambda)$ such that $\mathcal{A}(\lambda)$ is a transitive algebra for almost every λ.*

6.2. COROLLARY ([22], [23], [9]). *Every operator has an invariant subspace if and only if every reductive operator is normal.*

6.3. COROLLARY ([9]). *The following are equivalent:*

(i) *every abelian subalgebra of $\mathcal{B}(\mathcal{H})$ has a non-trivial invariant subspace,*
(ii) *every weakly closed abelian reductive algebra is selfadjoint.*

6.4. COROLLARY ([9]). *The following are equivalent:*

(i) *$\mathcal{B}(\mathcal{H})$ is the only weakly closed transitive algebra containing its commutant,*
(ii) *every weakly closed reductive algebra containing its commutant is selfadjoint.*

6.5. COROLLARY ([9]). *The following are equivalent:*

(i) *every operator other than a multiple of the identity has a non-trivial hyperinvariant subspace,*
(ii) *a reductive algebra which is the commutant of a single operator is selfadjoint.*

There have been some further results on commutants of reductive algebras.

6.6. THEOREM. *If T commutes with the reductive algebra $\mathcal{A}$, then T^* commutes with $\mathcal{A}$ provided:*

(i) ([53]) *T is compact, or*
(ii) ([53]) *T is n-normal, or*
(iii) ([27]) *T is polynomially compact, or*
(iv) ([40]) *T is essentially unitary and is in $\mathcal{C}_0$, or*
(v) ([28]) *T is a zero of a locally non-zero abelian analytic function, or*
(vi) ([27]) *T is quasi-similar to a normal operator.*

Note that each of the cases of Theorem 6.6 includes the statement that T reductive and T having the additional property listed implies T is normal.

Fong ([27]) has strengthened Corollary 6.5 as follows.

6.7. THEOREM ([27]). *If every operator has a hyperinvariant subspace, then $\mathcal{A}$ reductive implies the commutant of $\mathcal{A}$ is selfadjoint.*

A result in the negative direction was found by Loebl and Muhly.

6.8. THEOREM ([35]). *There exist ultraweakly closed reductive algebras which are not selfadjoint.*

The examples given in [35] all have selfadjoint weak closures. Nonetheless Theorem 6.8 might give hope that a counterexample to the reductive algebra problem can be found. It should be noted, as Arveson has pointed out, that it is not known whether there are any ultraweakly closed transitive algebras other than $\mathcal{B}(\mathcal{H})$.

Other sufficient conditions that reductive algebras be self-adjoint are given in ([38]) and ([51]), and some sufficient conditions that a reductive operator be normal are given in ([32]), ([47]), ([25]), and ([42]).

Fong ([26], [27]) considered the question: if every *hyper*invariant subspace of T is reducing, must T be normal? For normal operators, of course, every hyperinvariant subspace is reducing, so an affirmative answer would produce an amazing characterization of normal operators. Fong ([26], [27]) shows that the question has an affirmative answer in most of the cases where it is known that reductivity implies normality.

7. **Remarks.** There are many other recent results on invariant subspaces and related questions. I must mention Arveson's deep study [8] of reflexivity of algebras containing m.a.s.a.'s (also see [44]), Voiculescu's remarkable proof [58] that the reducible operators are dense, and the Abrahamse and Douglas [1] generalization of Beurling's theorem to bundle shifts. I have listed several

other recent papers which concern related topics in the references below, and there are other results that might well have been included.

The subject is alive and well.

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Added in proof: Per Enflo has recently constructed a non-reflexive Banach space on which there is an operator without non-trivial invariant subspaces. The problem remains open for Hilbert space, and Enflo states that his techniques will not shed any light on the Hilbert space problem.

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