# A Note on the Transformability of Spherically Symmetric Metrics

By Paul Kustaanheimo

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*Summary.* It is shown that every spherically symmetric metric can be transformed into the isotropic form. As illustration an example is given.

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Max Wyman has in *Mathematical Reviews*, 10 (1949), 579, reviewed a paper by Bertil Qvist and the author[1] and in the review declared that:

"It should be noted that the authors state that every spherically symmetric metric can be written in the so-called isotropic form. This assertion is incorrect as it is based on a proof given by Tolman [*Relativity, Thermodynamics and Cosmology* (Oxford, 1934), p. 240] which is wrong. The right-hand side of formula (94 . 7) as given by Tolman is not a perfect differential when $\lambda$ is a function of $t$."

In view of the above remark I venture to put forward a few observations.

Einstein and Straus have stated[2] that "A general centrally-symmetric field can be brought into the (conformally Euclidean, not necessarily static) form

$$ds^2 = -e^{\mu}\,\delta_{ik}\,dx_i\,dx_k + e^{\nu}\,dt^2 \qquad i, k = 1, 2, 3, \tag{1}$$

where $\mu$ and $\nu$ are functions of $r$ and $t$".

This transformation of a general spherically symmetric metric into the isotropic form can be performed in the following way.

As is known, every spherically symmetric metric may be written in the standard form

$$ds^2 = e^{\nu}\,dt^2 - r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2) - e^{\lambda}\,dr^2, \tag{2}$$

where $\nu$ and $\lambda$ are certain functions of $r$ and $t$ alone.

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[1] Paul Kustaanheimo and Bertil Qvist, "A note on some general solutions of the Einstein field equations in a spherically symmetric world", *Soc. Sc. Fenn. Comm. Phys.-Math.* XIII, No. 16 (1948).

[2] Albert Einstein and Ernst G. Straus, "The Influence of the Expansion of Space on the Gravitation Fields Surrounding the Individual Stars", *Reviews of Modern Physics*, Vol. 17, Nos. 2 and 3 (1945), 121.

We make in (2) the differential substitution

$$d\bar{r} = A\,dr + B\,dt, \tag{3}$$

$$d\bar{t} = C\,dr + D\,dt, \tag{4}$$

where $A$, $B$, $C$ and $D$ are for the present arbitrary functions of $r$ and $t$, which have to satisfy only the integrability conditions

$$\frac{\partial A}{\partial t} = \frac{\partial B}{\partial r}, \tag{5}$$

$$\frac{\partial C}{\partial t} = \frac{\partial D}{\partial r}. \tag{6}$$

Since by (3) and (4)

$$dr = \frac{D\,d\bar{r} - B\,d\bar{t}}{AD - BC}, \qquad dt = \frac{-C\,d\bar{r} + A\,d\bar{t}}{AD - BC},$$

we have from (2)

$$ds^2 = \frac{e^{\nu} A^2 - e^{\lambda} B^2}{(AD-BC)^2}\, d\bar{t}^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \frac{e^{\lambda} D^2 - e^{\nu} C^2}{(AD-BC)^2}\, d\bar{r}^2 - 2\,\frac{e^{\nu} AC - e^{\lambda} BD}{(AD-BC)^2}\, d\bar{r}\, d\bar{t}. \tag{7}$$

The expression (7) is of the form (1) if

$$e^{\nu} AC - e^{\lambda} BD = 0,$$

$$\frac{1}{\bar{r}}\, AD - \frac{1}{\bar{r}}\, BC = \frac{1}{r}\sqrt{e^{\lambda} D^2 - e^{\nu} C^2},$$

or

$$\frac{A}{\bar{r}} = \frac{r^{-1} e^{\lambda} D}{\sqrt{e^{\lambda} D^2 - e^{\nu} C^2}}, \tag{8}$$

$$\frac{B}{\bar{r}} = \frac{r^{-1} e^{\nu} C}{\sqrt{e^{\lambda} D^2 - e^{\nu} C^2}}. \tag{9}$$

Since (3) is a total differential only when also

$$\frac{d\bar{r}}{\bar{r}} = \frac{A}{\bar{r}}\, dr + \frac{B}{\bar{r}}\, dt$$

is a total differential, we may use instead of (5) the equivalent condition

$$\frac{\partial}{\partial t}\frac{A}{\bar{r}} = \frac{\partial}{\partial r}\frac{B}{\bar{r}}.$$

Then we get, according to (8) and (9),

$$\frac{\partial}{\partial t}\frac{r^{-1} e^{\lambda} D}{\sqrt{e^{\lambda} D^2 - e^{\nu} C^2}} = \frac{\partial}{\partial r}\frac{r^{-1} e^{\nu} C}{\sqrt{e^{\lambda} D^2 - e^{\nu} C^2}}. \tag{10}$$

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The equation (10) really contains only one unknown function, the ratio $C/D$. Putting $R=C/D$ and carrying out the differentiations in (10), we get

$$e^{\lambda+\nu} R \frac{\partial R}{\partial t} - e^{\lambda+\nu} \frac{\partial R}{\partial r} + \left(\tfrac{1}{2}e^{\nu} \frac{\partial}{\partial r} e^{\nu} - \frac{1}{r} e^{2\nu}\right) R^3$$
$$+ \left(\tfrac{1}{2}e^{\lambda} \frac{\partial}{\partial t} e^{\nu} - e^{\nu} \frac{\partial}{\partial t} e^{\lambda}\right) R^2 + \left(\tfrac{1}{2}e^{\nu} \frac{\partial}{\partial r} e^{\lambda} - e^{\lambda} \frac{\partial}{\partial r} e^{\nu} + \frac{1}{r} e^{\lambda+\nu}\right) R$$
$$+\tfrac{1}{2}e^{\lambda} \frac{\partial}{\partial t} e^{\lambda} = 0, \tag{11}$$

which is of the general form

$$P(r, t, R) \frac{\partial R}{\partial t} + Q(r, t) \frac{\partial R}{\partial r} - S(r, t, R) = 0$$

and has an infinite number of solutions in all cases considered in the theory of relativity.

After taking any one of the solutions of (11) as the ratio $R=C/D$, it is always possible to find a function $D$ which satisfies (6), an equation that may be written

$$\frac{1}{R} \frac{\partial}{\partial r} \ln D - \frac{\partial}{\partial t} \ln D = \frac{\partial}{\partial t} \ln R, \tag{12}$$

where $R=C/D$ is a known function of $r$ and $t$. Finally, from (8) and (9) we get $A/\bar{r}$ and $B/\bar{r}$, which, together with $C$ and $D$, when substituted into (3) and (4), give the transformation

$$\frac{d\bar{r}}{\bar{r}} = \frac{A}{\bar{r}} dr + \frac{B}{\bar{r}} dt, \quad d\bar{t} = C\,dr + D\,dt.$$

By means of this transformation, (2) changes into the isotropic form

$$ds^2 = \frac{e^{\lambda+\nu}}{e^{\lambda} D^2 - e^{\nu} C^2} d\bar{t}^2 - \frac{r^2}{\bar{r}^2} [d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta\, d\phi^2)].$$

If in particular the metric (2) is static or quasistatic, *i.e.* if $\lambda$ is a function of $r$ alone, we may immediately take $C/D=0$ as a solution of (10), and then $D=1$ as a solution of (6).

The transformation (3) ... (4) now takes the form

$$\frac{d\bar{r}}{\bar{r}} = e^{\frac{1}{2}\lambda} \frac{dr}{r}, \quad d\bar{t} = dt,$$

which is given in Tolman, *op. cit.*, p. 240, formula (94.7).

As an application we transform the metric

$$ds^2 = \frac{r^2}{t^2} dt^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \frac{r^2}{t} dr^2 \tag{13}$$

into isotropic form.

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The metric (13) is representable in the standard form (2), by choosing

$$\lambda = 2\ln r - \ln t, \tag{14}$$

$$\nu = 2\ln r - 2\ln t. \tag{15}$$

The equation (11) thus takes the form

$$R\frac{\partial R}{\partial t} - \frac{\partial R}{\partial r} - \tfrac{1}{2} = 0,$$

the general solution of which is

$$2R + r = F(R^2 - t), \tag{16}$$

where $F(\ )$ is an arbitrary function of one variable. We may in particular choose $F(\ )$ to be identically zero. Then we have

$$R = \frac{C}{D} = -\frac{r}{2}, \tag{17}$$

and so (6) takes the form

$$-\frac{r}{2}\frac{\partial}{\partial t}D = \frac{\partial}{\partial r}D, \tag{18}$$

one solution being

$$D = 4t - r^2. \tag{19}$$

The transformation (3) ... (4) thus has the form

$$\frac{d\bar{r}}{\bar{r}} = \frac{2dr - rt^{-1}dt}{\sqrt{4t - r^2}},$$

$$d\bar{t} = (4t - r^2)(-\tfrac{1}{2}r\,dr + dt),$$

or

$$\bar{r} = e^{2\arcsin\frac{1}{2}r/\sqrt{t}},$$

$$\bar{t} = \tfrac{1}{8}(4t - r^2)^2,$$

transforming the metric (13) into the isotropic form

$$ds^2 = \frac{4r^2}{t(4t - r^2)^3}\,d\bar{t}^2 - \frac{r^2}{\bar{r}^2}\,[d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta\,d\phi^2)],$$

or

$$ds^2 = \frac{\sin^2\frac{1}{2}\ln\bar{r}}{\bar{t}\sqrt{2\bar{t}}}\,d\bar{t}^2 - \frac{2}{\bar{r}^2}\sqrt{2\bar{t}}\,\tan^2\tfrac{1}{2}\ln\bar{r}\,[d\bar{r}^2 + \bar{r}^2(d\theta^2 + \sin^2\theta\,d\phi^2)].$$

ASTRONOMICAL OBSERVATORY,
HELSINKI, FINLAND.