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ON THE UNIQUENESS OF THE COEFFICIENT RING IN A GROUP RING

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1. Introduction and notation. Let R_1 and R_2 be commutative rings with identities, G a group and R_1G and R_2G the group ring of G over R_1 and R_2 respectively. The problem that motivates this work is to determine what relations exist between R_1 and R_2 if R_1G and R_2G are isomorphic. For example, is the coefficient ring R_1 an invariant of R_1G? This is not true in general as the following example shows. Let H be a group and

$$G = \bigoplus_{\alpha=1}^{\infty} H_\alpha \quad \text{with} \quad H_\alpha \simeq H.$$

If R_1 is a commutative ring with identity and $R_2 = R_1H$, then

$$R_1G \simeq R_1(H \oplus G) \simeq R_1H(G) \simeq R_2G,$$

but R_1 needn't be isomorphic to R_2.

Several authors have investigated the problem when $G = <x>$, the infinite cyclic group, partly because of its closeness to $R[x]$, the ring of polynomials over R. An exposition of many of the known results on the problem appear as Chapter IV in [**13**]. Even in this special case the results have been fragmentary. By imposing conditions on R_1 and on G several cases of the problem are treated extending many of the known results.

In the following we will always assume all coefficient rings are commutative with identity. If $\alpha \in RG$ with $\alpha = \sum_{g \in G} \alpha(g)g$, $\alpha(g) \in R$, we write

$$\text{supp } \alpha = \{g \in G | \quad \alpha(g) \neq 0\},$$

the augmentation map $RG \rightarrow R$ sending $\alpha \rightarrow \sum \alpha(g)$ will be denoted by δ_R and have kernel $\Delta_R(G)$ or $\delta(G)$. If H is a normal subgroup of G, and we extend the natural map $G \rightarrow G/H$ to a map $RG \rightarrow R(G/H)$, this new map has kernel $\Delta_R(G, H)$. $\Delta_R(G, H)$ is generated by $\{1\text{-}h| \quad h \in H\}$. For the group G, G' denotes its commutator subgroup and $\Theta(G)$ the set of orders of all finite subgroups. The ring R will have Jacobson radical $J(R)$, Nill radical $N(R)$, characteristic $\text{ch}(R)$, and units $U(R)$.

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As usual, $\mathbf{Z}$ denotes the integers, $\mathbf{Q}$ the rationals, $\mathbf{Z}_n$ either the ring of integers modulo n or a cyclic group with n elements and ζ_d a primitive D^{th} root of unity.

2. Reduction to abelian groups.

LEMMA 2.1. (Coleman) *Let I be an ideal of RG. Then the residue class algebra RG/I is commutative* $\Leftrightarrow \Delta(G, G') \subseteq I$.

THEOREM 2.2. *Let R_1 and R_2 be commutative rings with unity and G a group. Then*

$$R_1G \simeq R_2G \Rightarrow R_1(G/G') \simeq R_2(G/G').$$

Proof. Let $\sigma:R_1G \rightarrow R_2G$ be the given isomorphism. For $i = 1, 2$ let

$$\mathscr{I}_i = \{I \leq R_iG: \quad R_iG/I \text{ is commutative}\}.$$

By Lemma 2.1, $\Delta_{R_i}(G, G')$ is the unique minimal ideal in $\mathscr{I}_i$.
Then

$$\frac{R_1G}{\Delta_{R_1}(G, G')} \simeq \frac{R_2G}{\sigma(\Delta_{R_1}(G, G'))}$$

implying

$$\sigma(\Delta_{R_1}(G, G')) \supseteq \Delta_{R_2}(G, G').$$

By considering σ^{-1}, we see

$$\sigma(\Delta_{R_1}(G, G')) = \Delta_R(G/G').$$

Hence

$$R_1(G/G') \simeq \frac{R_1G}{\Delta_1(G, G')} \simeq \frac{R_2G}{\Delta_2(G, G')} \simeq R_2(G/G').$$

COROLLARY 2.3. *If G is a group with $G' = G$, then*

$$R_1G \simeq R_2G \Rightarrow R_1 \simeq R_2.$$

Throughout the following we will assume that all groups are abelian.

3. Abelian p-groups.

Definition. An element $a \in R$ is *regular* if a is not a zero divisor in R.

PROPOSITION 3.1. *Let R be a ring of characteristic p^e. Then $n \in \mathbf{Z}^+$, n is not regular in $R \Leftrightarrow n$ is a multiple of p.*

Proof. Suppose n is a multiple of p. Say $n = p^r t$:$(t, p) = 1$. If $r \geqq e$, then $n = p^{r-e}p^e t = 0$ in R. $1 \cdot n = 0$ and n is a zero divisor in R. So n is not regular. If $r < e$, $n = p^{e-r}p^r t = 0$. Conversely, suppose n is not a multiple of p. So $(n, p) = 1 = (n, p^e)$. Thus $\exists\, s, q \in \mathbf{Z}$:$ns + qp^e = 1 \Rightarrow ns = 1$ in R. So n is a unit in R.

LEMMA 3.2. (Cornell [**8**]). *Let G be an abelian group and R a commutative ring with $J(R) = 0$. Suppose that all elements of $\Theta(G)$ are regular in R. Then $J(RG) = 0$.*

LEMMA 3.3. (Passman [**21**]). *Let G be an abelian group and R a commutative ring with $N(R) = 0$. If all elements of $\Theta(G)$ are regular in R, then $N(RG) = 0$.*

PROPOSITION 3.4. *Let R be a commutative ring of characteristic p^e and G an abelian group. If S_p denotes the p-Sylow subgroup of G, then $R/J(R)(G/S_p)$ is semisimple.*

Proof. $R/J(R)$ is commutative and semisimple. If $n \in \Theta(G/S_p)$, n is not a multiple of p, and so by 3.1 it is regular in $R/J(R)$. The result now follows from 3.2.

LEMMA 3.4. *If G is a p-group and R a ring with $J(R) = 0$ and $p = 0$ in R, then $J(RG) = \Delta(G)$.*

Proof. Since G is an abelian p-group and R is of characteristic p, $\Delta(G)$ is nil as it is generated by nilopotent elements. Thus $\Delta(G) \subseteq J(RG)$.

Let $\alpha \in J(RG)$, and $r \in R$. There exists $\beta \in RG$ such that $\delta_R(\beta) = r$. $1-\alpha\beta \in U(RG)$. So

$$\delta_R(1 - \alpha\beta) = 1 - \delta_R(\alpha)r \in U(R).$$

But r arbitrary implies

$$\delta(\alpha) \in J(R) \quad \text{and} \quad J(RG) \subseteq \ker \delta_R = \Delta(G).$$

LEMMA 3.5. *Let R be a ring, I an ideal of R, G a group and H a normal subgroup of G. Let the natural epimorphisms η and π be given by*

$$RG \xrightarrow{\eta} R(G/H) \xrightarrow{\pi} R/I(G/H).$$

Then $\text{Ker}\, \pi\eta = \Delta_R(G, H) + IG$.

Proof. Clearly $\ker \eta + IG \subseteq \ker \pi\eta$. Let $\alpha \in \ker \pi\eta$ with $\alpha = \sum\alpha(g)\, g$. Write

$$G = \bigcup_{i \in I} H_{g_i} \cdot \pi\eta(\alpha) = \sum \pi\eta(\alpha(g)g) = 0$$

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i.e.,

$$\overline{\sum \alpha(g)}\ \bar{g} = 0 = \sum_{i=1}^{k} \sum_{g' \in Hg_i} \overline{\alpha(g')}\ \bar{g}_i = 0$$

for some finite set $g_1, g_2 \dots g_k$ of the g_i's. Thus

$$\sum_{g \in Hg_i} \alpha_g \in I, \quad i = l, \dots, k.$$

Write

$$s_i = \sum_{g \in Hg_i} \alpha_g \quad \text{and} \quad \beta = \sum s_i g_i \in IG.$$

Then

$$\alpha = (\alpha - \beta) + \beta$$

$$\begin{aligned} \eta(\alpha - \beta) &= \eta(\alpha) - \eta(\beta) \\ &= \sum \alpha(g)\bar{g} - \sum_{i=1}^{k} s_i \bar{g}_i \\ &= \sum_{i=1}^{k} \left(\sum_{g \in Hg_i} \alpha_g \right) \bar{g}_i - \sum s_i \bar{g}_i = 0. \end{aligned}$$

Thus $\alpha \in \ker \eta + IG$ and the result follows.

THEOREM 3.6. *Let G be an abelian group, S_p its p-Sylow subgroup and R a ring of characteristic p.*

(a) $N(RG) = N(R)G + \Delta_R(G, S_p)$;

(b) $J(RG) \subset J(R)G + \Delta_R(G, S_p)$ *with equality if G is torsion or if $J(R) = N(R)$ (e.g. if R is artinian).*

Proof. $N(R)G + \Delta_R(G, S_p)$ is generated by nilpotent elements, hence contained in $N(RG)$. Putting $\bar{R} = R/N(R)$ and $\bar{G} = G/S_p$, we have by (3.5)

$$RG/(N(R)G + \Delta_R(G, S_p)) \simeq \overline{RG}.$$

But by (3.3)

$$0 = N(\overline{RG}) = N(RG/(N(R)G + \Delta_R(G, S_p)))$$

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and (a) follows. Similarly, as $J((R/J(R))\bar{G} = 0$ by (3.2), it follows again using 3.5 that

$$J(R)G \subset (J(R)G + \Delta_R(G, S_p)).$$

If $J(R) = N(R)$, equality follows from (a). When G is torsion equality follows since $J(R)G \subset J(RG)$.

COROLLARY 3.7. *Let R_i be a ring of characteristic a power of p, and let G_i be an abelian group with p-Sylow subgroup S_i for $i = 1, 2$. Put $\bar{R}_i = R_i/N(R_i)$ and $\bar{G}_i = G_i/S_i$, $i = 1, 2$. Then $R_1G_1 \simeq R_2G_2 \Rightarrow \bar{R}_1\bar{G}_1 \simeq \bar{R}_2\bar{G}_2$.*

4. Finite Abelian G.

LEMMA 4.1. *Let E and F be fields of characteristics p or 0 such that $F \simeq E(\zeta_n)$ and $E \simeq F(\zeta_t)$. Then $E \simeq F$.*

Proof. $E \simeq F(\zeta_t) \simeq E(\zeta_n, \zeta_t)$. Hence $\zeta_n, \zeta_t \in E$ *and* $F \simeq E(\zeta_n) = E$.

Definition. If E and F are fields put $E \leqq F$ if $F \simeq E(\zeta_n)$ for some n. By 4.1 this defines a partial ordering on the isomorphism classes of fields.

THEOREM 4.2. *Let F_1 and F_2 be fields and G_1 and G_2 torsion abelian groups. Then*

$$F_1G_1 \simeq F_2G_2 \Rightarrow F_1 \simeq F_2.$$

Proof. The residue class fields of F_iG_i are all cyclotomic extensions of F_i, so F_i is characterized, up to isomorphism, as the unique minimal element, in the partial ordering of fields defined above, among these.

We can generalize this result as follows:

LEMMA 4.3. *Let I be an ideal in the noetherian ring R, $\bar{R} = R/I$ and G a finitely generated abelian group. Suppose $RG \simeq \bar{R}G$ then $I = 0$ and $\bar{R} = R$.*

Proof. Let $\phi:RG \rightarrow \bar{R}G$ be the given isomorphism. Extend the natural map $R \rightarrow R/I$ to $\rho:RG \rightarrow \bar{R}G$. From [**8**], p. 658, RG is Noetherian and thus the surjective map

$$f = \phi^{-1} \circ \rho:RG \rightarrow RG$$

is an injection. Hence $I = 0$.

THEOREM 4.4. *Suppose $FG \simeq RG$ where F is a field, R a ring and G a finite abelian group. Then $F \simeq R$.*

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Proof. Case (i): characteristic $F \nmid o(G)$. Then FG and RG are regular. So R is regular and $\forall n\ \Theta(G)$. n is a unit in R. ([**8**]). Thus $RG = \Delta(G) \oplus \Delta^*$ where $\Delta^* = \text{Ann.}(\Delta(G))$ and $R = \Delta^*$ ([**8**]). As RG is semisimple and G is finite abelian,

$$RG = \bigoplus_{i=1}^{k} E_i,$$

E_i fields cyclotomic over F. R is a direct summand of RG,

$$R = \bigoplus_{i=1}^{t} E_i, \quad t < k.$$

Now $E_nG \simeq FG \otimes_F E_n$ as F-algebras since

$$FG \otimes_F E_n = (F \oplus_= \dots \oplus F) \oplus_F E_n \cong \lambda(E_n) \text{ with } \lambda = o(G)$$

as F-modules. Hence $FG \otimes_F E_n \simeq E_nG$. Thus

$$RG = \bigoplus_{i=1}^{t} (E_iG) = \bigoplus_{i=1}^{t} (FG \otimes_F E_i)$$

$$= \bigoplus_{i=1}^{t} \left(\bigoplus_{j=l}^{k} E_j \otimes_F E_i \right).$$

Thus RG has tk components. But $RG \simeq FG$ which has exactly k simple components. Thus $t = 1$, i.e., $R = E_i$ is a field. So by 4.2, $F \simeq R$.

Case (ii): char $F = p$ and $p \mid o(G)$. Let Sp be the p-Sylow subgroup of G. From (3.7) we have

$$F(G/Sp) \simeq \frac{R}{N(R)}(G/Sp) \quad \text{and} \quad p \nmid o(G/Sp).$$

From case (i) $F \simeq R/N(R)$. Thus $RG \simeq FG \simeq R/N(R)G$. As FG is Noetherian then R is, too, and so Lemma 4.3 implies $N(R) = 0$, i.e., $F \simeq R$.

If A is a commutative ring with 1 and $\mathfrak{a}$ is a finite set of minimal ideals of A we define an equivalence relation on $\mathfrak{a}$ by $I_1, I_2 \in \mathfrak{a}$ are equivalent if $I_1 \simeq I_2$ as rings. Write $\mathfrak{a}/\sim\ = D_A$. When A is semi simple artinian, then $\mathfrak{a}$ consists of fields and we make D_A into a partially ordered set by $\bar{F}_1 \leq \bar{F}_2$ if and only if $F_2 \simeq F_1(\zeta_k)$ for some positive integer k. $\bar{F}_i$ denotes the equivalence class of $F_i \in \mathfrak{a}$.

THEOREM 4.5. *Let R be a finite direct sum of fields, G a finite group and S a ring. Suppose $RG \simeq SG$. Then $R \simeq S$.*

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Proof. Case (i): RG is semisimple. RG semi simple implies SG is also. R regular implies RG and thus SG is regular. So if $n \in \Theta(G)$, n is a unit in S and S is a direct summand of $SG \simeq RG$ ([**8**]). We now see that S a finite direct sum of fields.

Let $R = F_1 \oplus \ldots \oplus F_n$, $S = E_1 \oplus \ldots \oplus E_m$ with F_i and E_j fields. Proceed by induction on n. If $n = 1$, then $F_1G \simeq SG$ and $F_1 \simeq S$ from Theorem 4.3.

Suppose $n > 1$ and the theorem is true for $n - 1$. As RG is semi simple so is F_iG and E_jG for all i and j.

In RG we consider the set $\mathscr{M}_1$ of minimal ideals and the associated partially ordered set D_{RG}. Similarly we consider D_{SG}. Let $\bar{F}$ be a minimal element in D_{RG}. Then $\overline{\sigma(F)}$ is a minimal element in D_{SG}. For let $\sigma(F) = E_j(\zeta_d)$ and suppose there exists $\overline{E_t(\zeta_l)} < \overline{E_j(\zeta_d)}$, i.e.,

$$E_j(\zeta_d) \simeq E_t(\zeta_l)(\zeta_r) \quad \text{with } \zeta_r \notin E_t(\zeta_l).$$

Hence

$$F \simeq \sigma^{-1}(E_t(\zeta_k, \zeta_r)) \simeq F_k(\zeta)(\zeta'_r) \quad \text{for some } k.$$

But $F \neq F_k(\zeta)$ since otherwise $\zeta'_r \in F$ and $\zeta_r \in E_t(\zeta_l)$. Hence

$$F \simeq \sigma^{-1}(E_t(\zeta_k, \zeta_r)) \simeq F_k(\zeta)(\zeta'_r) \quad \text{for some } k.$$

But $F \neq F_k(\zeta)$ since otherwise $\zeta'_r \in F$ and $\zeta_r \in E_t(\zeta_l)$. Hence $\overline{F_k(\zeta)} < \bar{F}$ contradicting the minimality of $\bar{F}$.

Since $\bar{F}$ is minimal in D_{RG}, F is isomorphic to a field in R, i.e., $F \simeq F_i$ for some i. (F in RG implies $F = F_k(\zeta_d)$ for some k. But $\bar{F}$ minimal implies $\zeta_d \in F_k$ and F is isomorphic to an ideal in R.) As $\sigma(F) = K$ has $\bar{K}$ minimal in S, $\bar{K} \simeq E_j$ for some j. Write $R = F_k \oplus R_1$, $S = E_j \oplus S_1$. Then

$$RG \simeq F_kG \oplus R_1G \simeq SG \simeq E_jG \oplus S_1G.$$

But F_kG, by a rearrangement of the original isomorphism, if necessary (RG and SG have the same number of single components, similarly for F_kG and E_jG), we can assume $R_1G \simeq S_1G$. But R_1 contains n-1 minimal ideals and so by induction $R_1 \simeq S_1$. Hence $R \simeq S$.

Case 2: RG not semi simple. For p a prime, let

$$R'(p) = \{x \in R | \quad px = 0\} \quad \text{and} \quad S'(p) = \{x \in S | \quad px = 0\}.$$

$S'(p)$ is an ideal in S and

$$\{x \in RG | \quad px = 0\} = R'(p)G,$$

$$\{x \in SG | \quad px = 0\} = S'(p)G$$

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then $R'(p)G \simeq S'(p)G$.

If P is the p-Sylow subgroup of G, by (3.7) we have that

$$R'(p)(G/P) \simeq \frac{S'(p)}{N(s'(p))}(G/P).$$

Apply case (1) to conclude

$$R'(p) \simeq \frac{S'(p)}{n(S'(p))}.$$

But as in the proof of the previous theorem, we have that $S'(p)$ is Noetherian and so by Lemma 4.3, $R'(p) \simeq S'(p)$.

Let $p_1, p_2, \dots, p_k$ be the distinct primes dividing $o(G)$, and let E_{p_i} denote the identity in $R'(p_i)$ or $S'(p_i)$. Write

$$e = E_{p_1} + \dots + E_{p_k}.$$

Then e is an idempotent, and

$$RG \simeq ((1 - e)R \oplus eR)G \simeq ((1 - e)S \oplus eS)G.$$

Hence

$$((1 - e)R)G \simeq \frac{RG}{(eR)G} \simeq \frac{SG}{(eS)G} \simeq ((1 - e)S)G.$$

By case (1), again $(1 - e)R \simeq (1 - e)S$ and thus $R \simeq S$.

5. Torsion free groups.

THEOREM 5.1. *Let R be a regular ring, G a group with torsion subgroup T and suppose that for $n \in \Theta(T)$, n is a unit in R. Then RT is the unique maximal regular ring of RG with 1_{RG}.*

Proof. Case 1: G torsion free. Let L be a regular subring of RG with $1_{RG} \in L$ and let $\alpha \in L$. As L is regular there exist $\beta, \gamma \in L$ with

$$\alpha^2\beta = \alpha \quad \text{and} \quad (1 - \alpha)^2\gamma = 1 - \alpha.$$

Let P be a prime ideal of R. Then in R/PG, $\bar{\alpha}(\bar{\alpha}\bar{\beta} - \bar{1}) = 0$ and $(\overline{1 - \alpha})((\overline{1 - \alpha})\bar{\gamma} - \bar{1}) = 0$.

But R/PG is an integral domain and so either

a) $\bar{\alpha} = 0$ or $\overline{1 - \alpha} = 0$

or

b) $\bar{\alpha}\bar{\beta} = \bar{1}$ and $(\overline{1 - \alpha})\bar{\gamma} = \bar{1}$.

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If a) holds then $\alpha \in PG$ or $1 - \alpha \in PG$ while if b) holds we must have $\bar{\alpha} = \bar{c}h$ and $\overline{1 - \alpha} = ug$ for $\bar{c}\, \bar{u} \in U(R/P)$ and $h, g \in G$. Then $\overline{1 - ch} = \bar{u}g$ implying $h = g = c$ and $\bar{\alpha} = \bar{c} = 0$, i.e., $\alpha - c \in PG$. In any case there exists $c \in R$ with $\alpha - c \in PG$. Write $\alpha = \sum \alpha(g)g$. Then

$$\alpha - c = \alpha(1) - c + \sum_{g \neq 1} \alpha(g)g \in PG.$$

But P is an arbitrary prime ideal so that $\alpha(g)$ for $g \neq 1$ is nilpotent. Thus $\alpha(g) = 0$ if $g \neq 1$ and $\alpha = \alpha(1)$. Hence $L \subseteq R$.

Case 2: General G. Again let L be a regular subring of RG with $1_{RG} \in L$ and let $\alpha \in L$. Find $\beta, \gamma \in L$ with $\alpha^2\beta = \alpha$ and $(1 - \alpha)^2\gamma = 1 - \alpha$. Let H be the subgroup of G generated by $\mathrm{Supp}(\alpha) \cup \mathrm{Supp}(\beta) \cup \mathrm{Supp}(\gamma)$. Since H is finitely generated, the torsion subgroup H^* of H is a direct summand of H, say $H = H^* \oplus W$ with W torsion free. We have

$$\alpha, \beta, \gamma \in RH \simeq RH^*(W).$$

Since RH^* is regular by case 1, $\alpha, \beta, \gamma \in RH^* \subset RT$. Hence $L \subset RT$.

COROLLARY 5.2. *Let R_1 and R_2 be regular. If $\sigma:R_1G \to R_2G$ is an isomorphism then $\sigma(R_1T) = R_2T$. If in particular, G is torsion free, then $\sigma(R_1) = R_2$.*

COROLLARY 5.3. *Let R_1 and R_2 be artinian and G torsion free. Then $R_1G = R_2G$ implies*

$$\frac{R_1}{J(R_1)} \simeq \frac{R_2}{J(R_2)}.$$

Proof. Let

$$\eta_i:R_iG \to \frac{R_i}{J(R_i)}\, G$$

be the natural maps for $i = 1, 2$. As R_i is artinian, $J(R_i)$ is nilpotent and $J(R_i)G \subseteq J(R_iG)$. But $R_i/J(R_i)\, G$ is semi-simple so

$$J(R_iG) \subseteq \ker \eta_i = J(R_i)G.$$

Thus

$$\frac{R_i}{J(R_i)}\, G \simeq \frac{R_iG}{J(R_iG)}$$

and the result follows by 5.2.

We do not know if R_1, R_2 artinian G torsion free and $R_1G \simeq R_2G$ implies $R_1 \simeq R_2$.

Definition. A ring is called *reduced* if its nil radical is 0.

LEMMA 5.4. *Suppose R is a ring without non-trivial idempotents and G is a torsion free group. Then*

$$U(RG) = U(R) \times (1 + N(R) \cdot \Delta_R(G)) \times G.$$

If, in particular R is reduced, then $U(RG) = U(R)G$.

Proof. $U(RG) = U(R) \times V$ where

$$V = \{v \in \sum e_g g \in U(RG) | \sum e_g = 1\ e_g \in R\}.$$

If R is an integral domain, $V = G$. Hence, if P is a prime ideal of R

$$e_g e_h \equiv \delta_{g,h} e_g \bmod P$$

where $\delta_{g,h}$ is the Krondeker delta function. Taking the intersection of all prime ideals gives this congurence modulo $N(R)$. But orthogonal idempotents lift modulo the nil ideal $N(R)$. As R has only trivial idempotents, we must have

$$v = gw \text{ with } g \in G \quad \text{and} \quad w \equiv 1 \bmod (N(R)(RG\Delta_R(G))).$$

Because

$$N(R)(RG\Delta_R(G)) = N(R)\Delta_R(G),$$

we conclude $w \in 1 + N(R)\Delta_R(G)$ which implies the lemma.

PROPOSITION 5.5. *Let R be a reduced ring with no non-trivial idempotents and G a torsion free abelian group. Then any local subring of RG, containing 1_{RG}, is contained in R.*

Proof. Let L be a local subring of RG containing 1_{RG}. If $a \in L$, $1 - a \in L$, and either a is a unit or $1 - a$ is a unit. We can assume a is a unit. By Lemma 5.4, $a = ug$ with $u \in R$, $g \in G$. Also $a + a^{-1}$ or $1 - (a + a^{-1})$ is a unit. If $1 - (a + a^{-1}) = v\,g_2$ with $v \in U(R)$, $g_2 \in G$ then

$$1 - ug - u^{-1}g^{-1} = vg_2$$

which implies $g = g^{-1} = g_2 = e$ and $a = u$, $u \in U(R)$. Similarly if $(a + a^{-1})$ is a unit. Thus $L \subseteq R$.

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COROLLARY 5.6. *If I is a local reduced ring, R a ring and G a torsion free abelian group then $IG \overset{\sigma}{\simeq} RG$ implies $\sigma(I) \subseteq R$.*

Proof. $N(RG) = N(R)G$, so that R is reduced. IG has no non trivial idempotents (see e.g. [**25**], p. 40) and so R does not. The result now follows from Proposition 5.5.

COROLLARY 5.7. *Let R_1 and R_2 be local rings G_1, G_2 torsion free abelian groups and $R_1G \simeq R_2G$. Then $R_1/N_1(R_1) \simeq R_2/N_2(R_2)$.*

Proof. Let $\sigma:R_1G_1 \to R_2G_2$ be the given isomorphism. Write $N_i = N(R_i)$. Then

$$\frac{R_1G}{N_1G} = \frac{R_1G}{N(R_1G)} \simeq \frac{R_2G}{N(R_2G)} = \frac{R_2G}{N(R_2)G} \quad \text{and}$$

$$\bar{\sigma}:\frac{R_1}{N_1}G_1 \simeq \frac{R_2}{N_2}G_2.$$

But R_i/N_i is local reduced. By Corollary 5.6

$$\bar{\sigma}(R_i/N_i) \subseteq R_2/N_2.$$

Similarly

$$\bar{\sigma}^{-1}(R_2/N_2) \subseteq R_1/N.$$

Hence $R_1/N_1 \simeq R_2/N_2$.

THEOREM 5.8. *Let R be a reduced ring with no non trivial idempotents, S a ring and G a torsion free abelian group. Suppose $RG \overset{\sigma}{\simeq} SG$ and $\sigma(R) \subseteq S$. Then there exist subgroups H, K of G such that*

(i) $G \simeq H$
(ii) $G = HK$ (*internal direct sum*)
(iii) $S = \sigma(RK)$.

Proof. As

$$0 = N(R)G = N(RG) = \sigma^{-1}(N(SG)) = \sigma^{-1}(N(S)G),$$

$N(S) = 0$. If $e \in SG$ and $e^2 = e$, $\sigma^{-1}(e) \in RG$ and $\sigma^{-1}(e) \in R$ ([**24**]). Thus $\sigma^{-1}(e) = 0$ or 1 and $e = 0$ or 1, and S is a reduced ring with no non trivial idempotents. If $g \in G$, $\sigma^{-1}(g) = U_gh_g$ with $U_g \in U(R)$, $h_g \in G$ from Lemma 5.4. i.e.,

$$g = \sigma(U_g)\sigma(h_g), \quad \sigma(U_g) \in \sigma(R) \subset S.$$

Let $\alpha_g = \sigma(U_g^{-1})$ then α_g is such that $\sigma^{-1}(\alpha_g g) = h_g \in G$. Thus if $g \in G$

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there exists an $\alpha_g \in \sigma(R)$ such that $\sigma^{-1}(\alpha_g g) = h_g \in G$. Let

$$H = \{h \in G : \sigma^{-1}(\alpha) = h \text{ for some } \alpha \in \sigma(R)\, g \in G\}.$$

H is a subgroup of G and $\sigma(H) \subseteq \sigma(R)G$ implying $\sigma(RH) \subseteq \sigma(R)G$.

Let

$$K = \{g \in G : \sigma(g) \in S1_G = S\}.$$

K is a subgroup of G. Clearly $H \cap K = \{1\}$. Let $g \in G$ and $\sigma(g) = ug_1$ (Lemma 5.4), $g = \sigma^{-1}(u)\sigma^{-1}(g_1)$. Write $\sigma^{-1}(u) = vg_2$, $\sigma^{-1}(g_1) = wg_3$ with $v, w \in U(R)$, $g_2, g_3 \in G$. So $g = vg_2wg_3$ and $vw = 1$. As $g_2 = gg_3^{-1}$,

$$\begin{aligned}\sigma(g_2) &= ug_1\sigma(g_3^{-1}) \\ &= ug_1\sigma(w\sigma^{-1}(g_1^{-1})) \\ &= ug_1\sigma(w)g_1^{-1} \\ &= u\sigma(w) \in U(S),\end{aligned}$$

and $g_2 \in K$,

$$g_3 = w^{-1}\,\sigma^{-1}(g_1) = v\sigma^{-1}(g_1) = \sigma^{-1}(\sigma(v)g_1)$$

and $g_3 \in H$.

This shows G is the direct sum of H and K establishing (ii). $\sigma(RH) \subseteq \sigma(R)G$ while $\sigma^{-1}(\sigma(R)G) \subseteq RH$ implying $\sigma(RH) = \sigma(R)G$. Then

$$\sigma|_{RH} : RH \to \sigma(R)G$$

implies $H \simeq G$ via $\bar{\sigma}(h) = g$ if $\sigma(h) = \alpha g$. This shows (i). $\sigma(RK) \subseteq S$.

$$SG = \sigma(RG) = \sigma(R(KH)) = \sigma((RK)H) \subseteq \sigma(RK)G \subseteq SG.$$

This shows $SG = \sigma(RK)G$. If $s \in S$, $S = \sum \alpha_i g_i$ with $\alpha_i \in \sigma(RK)$, $g_i \in G$. But each $\alpha_i \in S$. So $s = \alpha_1$ with $g_1 = e$ and $s = \sigma(RK)$.

COROLLARY 5.9. *If F is a field, S a ring and G a torsion free abelian group then $FG \simeq SG \Leftrightarrow$ there exist subgroups H, K of G with $G \simeq H \oplus K$, $H \simeq G$ and $S \simeq \sigma(FK)$.*

Proof. If the right hand side holds,

$$FG \simeq F(K \oplus H) \simeq FK(H) \simeq SH \simeq SG.$$

Conversely, from 5.6, $\sigma(F) \subset S$. Theorem 5.8 now implies the result.

Similarly using 5.6 and 5.8, it follows that

COROLLARY 5.10. *If R is a local reduced ring, S a ring and G a torsion* *free abelian group then* $FG \simeq SG \Leftrightarrow$ *there exist subgroups* H *and* K *of* G *with* $H \oplus K \simeq G$ $H \simeq G$ *and* $S = \sigma(RK)$.

COROLLARY 5.11. *If* S *is a ring,* G *a torsion free abelian group then* $ZG \simeq SG \Leftrightarrow$ *there exist subgroups* H *and* K *of* G *with* $H \oplus K \simeq G$, $H \simeq G$ *and* $S = \sigma(ZK)$.

THEOREM 5.12. *Let*

$$R = \bigoplus_{i=1}^{n} F_i$$

be a direct sum of fields, S *a ring and* G *a torsion free abelian group. Then* $RG \simeq SG$ *if and only if there exist subrings* $S_1, S_2, \ldots, S_n$ *of* S, *subgroups* $H_1, K_1, H_2, K_2, \ldots, H_nK_n$ *of* G *with*

(i) $S = S_1 \oplus S_2 \oplus \ldots \oplus S_n$

(ii) $G \simeq H_i, \quad i = 1, 2, \ldots, n$

(iii) $G \simeq H_i \oplus K_i, \quad i = 1, \ldots, n$

(iv) $S_i \simeq F_iK_i$.

Proof. ($\Leftarrow$) This follows as in Corollary 5.9. ($\Rightarrow$). Let

$$RG = \bigoplus_{i=1}^{n} F_iG \xrightarrow{\sigma} SG$$

be the given isomorphism. Since G is torsion free every idempotent of RG belongs to R. Let $e_1, e_2, \ldots, e_n$ be the orthogonal primitive idempotents of R numbered so that $e_iR = F_i$. Then $\{\sigma(e_i) = f_i, i = 1, \ldots, n\}$ is the unique set of orthogonal primitive idempotents in S. Let $S_i = f_iS$. Then

$$\sigma(F_iG) \simeq \sigma(e_iRG) \simeq f_iSG = S_iG, \quad i = 1, \ldots, n.$$

From Corollary 5.9, there exist subgroups H_i, K_i of G with $H_i \simeq G$, $G \simeq H_i \oplus K_i$ and $S_i = \sigma(F_iK_i)$. Then

$$\begin{aligned} SG &= \sigma(F_1 G \oplus \ldots \oplus F_nG) = \sigma(F_1 G) \oplus \ldots \oplus \sigma(F_nG) \\ &= S_1G \oplus \ldots \oplus S_nG \\ &\simeq (S_1 \oplus \ldots \oplus S_n)G \subseteq SG. \end{aligned}$$

So $SG = (S_1 \oplus \ldots \oplus S_n)G$ and as $S_1 \oplus \ldots \oplus S_n \subseteq S$ we have

$$S_1 \oplus \ldots \oplus S_n = S.$$

6. Mixed groups. In this section, we give some applications and extensions of the previous theorems to mixed groups.

PROPOSITION 6.1. *Let R and S be finite direct sums of fields, G a group with RG, and SG semi-simple. If $RG \simeq SG$ then $RT \simeq ST$ where T denotes the torsion subgroup of G.*

Proof. Let $R = \oplus F_i$ with F_i a field. Then F_iT is regular (F_iG is regular if and only if G is locally finite and has no element of order p if char $F = p$. See e.g. [**23**]). So if $n \mid |T|$, n is a unit in F_i for all i. From Theorem 5.1 RT is the maximal subring of RG with 1_{RG}. Similarly for ST and $RT \simeq ST$.

PROPOSITION 6.2. *Let R_1 and R_2 be perfect rings of characteristics p, S_p the p-Sylow subgroup of group G and $R_1\ G \simeq R_2\ G$. Then*

$$\frac{R_1}{J(R_1)}(G/S_p) \simeq \frac{R_2}{J(R_2)}(G/S_p).$$

Proof. (For the definition of perfect ring see [**26**], p. 127.) Since R_i is perfect, $J(R_i)$ is T nilpotent and hence nil. From Corollary 3.6,

$$J(R_iG) = \Delta(G, S_p) + J(R_i)G \quad \text{and}$$

$$\frac{R_iG}{J(R_iG)} \simeq \frac{R_i}{J(R_i)}(G/S_p).$$

Since $R_1G \simeq R_2G$ implies $R_1G/J(R_1G) \simeq R_2G/J(R_2G)$ we have the result.

COROLLARY 6.3. *Let F_1 and F_2 be fields of characteristic p, S_p the p-Sylow subgroup, and T the torsion subgroup, of the group G. Then*

$$F_1G \simeq F_2G \Rightarrow F_1(T/S_p) \simeq F_2(T/S_p).$$

Proof. By Proposition 6.2, $F_1(G/S_p) \simeq F_2(G/S_p)$ with $F_1(G/S_p)$ semi-simple. As T/S_p is the torsion subgroup of G/S_p, Corollary 7.2 gives our conclusion.

THEOREM 6.4. *Let F_1, F_2 be fields and G_1, G_2 groups with $F_1G_1 \simeq F_2G_2$. Then $F_1 \simeq F_2$.*

Proof. If F_1 and F_2 are fields of characteristic p with p a prime or zero, then, using (6.3) we have

$$F_1(T_1/S_{p_1}) \simeq F_2(T_2/S_{p_2}).$$

From Theorem 4.2, the result now follows.

Theorem 6.4 is not valid if F_1 is a field and F_2 is the finite sum of fields as the following example shows.

Example. Let

$$G = \bigoplus_{i=1}^{\infty} \mathbf{Z}_3.$$

Then $G \simeq \mathbf{Z}_3 \oplus G$ and

$$\begin{aligned} \mathbf{Q}G &\simeq \mathbf{Q}(\mathbf{Z}_3)G \\ &\simeq (\mathbf{Q} + \mathbf{Q}(\zeta_3))G \\ &\simeq \mathbf{Q}G \oplus \mathbf{Q}(\zeta_3)G \\ &\simeq \mathbf{Q}G \oplus \mathbf{Q}(\zeta_3)G \oplus \mathbf{Q}(\zeta_3)G \\ &\simeq (\mathbf{Q} \oplus \mathbf{Q}(\zeta_3)) \oplus \mathbf{Q}(\zeta_3)G. \end{aligned}$$

If $R = \mathbf{Q} \oplus \mathbf{Q}(\zeta_3) \oplus \mathbf{Q}(\zeta_3)$, then $\mathbf{Q}$ and R are each finite direct sums of fields and R is not isomorphic to $\mathbf{Q}H$ for any subgroup H of G. In fact, R is not isomorphic to a group ring, over $\mathbf{Q}$, for any group, as $R \neq \mathbf{Q}\mathbf{Z}_5$ and $\dim R/\mathbf{Q} = 5$.

THEOREM 6.5. *Let G be an abelian group with finite torsion group T. Let R be a finite sum of fields and S a ring. Suppose $RG \simeq SG$.*

(a) *If S is artinian, then $R \simeq S/N(S)$.*

(b) *If G is finitely generated, then $R \simeq S$.*

Proof. As T is finite, we can find a torsion free subgroup G_1 with $G \simeq T \times G_1$

$$R(T)G_1 \simeq RG \simeq SG \simeq S(T)G_1.$$

Case (1): RG semi-simple. Then RT is semi-simple and thus a finite sum of fields $RT = F_1 \oplus \ldots \oplus F_k$. By Theorem 5.12 there exist subrings, $S_1, S_2, \ldots, S_k$ of S and subgroups H_i, K_i of $G_1 (i = l, \ldots, k)$ such that

$$H_i \oplus K_i \simeq G_1, \quad H_i \simeq G_1, \quad S_i \simeq F_i(K_i) \quad \text{and}$$

$$S_1 \oplus \ldots \oplus S_k = S.$$

If G is finitely generated, then G_1 is free abelian of finite rank. Since rank (H_i) + rank (K_i) = rank (G_1) and rank (H_i) = rank (G_i), we have $K_i = \{1\}$, $i = 1, \ldots, k$. So $S_i \simeq F_i$ and $ST \simeq RT$. By Theorem 4.5, we now have $R \simeq S$.

If S, and hence S_i, is artinian, as $S_i \simeq F_i(K_i)$, K_i must be finite ([**8**]) and thus K_i is again $\{1\}$. i.e., $RT \simeq ST$. By Theorem 4.5 we have in either case $R \simeq S$.

Case (ii): RG is not semi-simple. Let $p_1, p_2, \ldots p_k$ be the distinct primes dividing $o(T)$. Let

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$$R'(p_i) = \{x \in R \mid \ p_i x = 0\} \quad \text{and}$$
$$S'(p_i) = \{x \in S \mid \ p_i x = 0\}.$$

$S'(p_i)$ is an ideal of S and $R'(p_i)G \simeq S'(p_i)G$. Let P_i be the p_i-Sylow subgroup of G. Since $o(P_i) < \infty$, we can write G in the form $G \simeq P_i \times G_i$. then, from (3.7)

$$R'(p_i)G_i \simeq \frac{S'(p_i)}{N(S'(p_i))}(G_i/P_i)$$

and from case (1) we conclude

$$R'(p_i) \simeq \frac{S'(p_i)}{N(S'(p_i))}.$$

If G is finitely generated, then $R'(p_i)G$ is noetherian. We have a surjective homomorphism

$$\frac{S'(p_i)}{N(S'(p_i))} G \to R'(p_i)G \to S'(p_i)G$$

with kernel $N(S'(p_i))G_i$. From Lemma 4.3, $N(S'(p_i)) = 0$.

Continue, as in the proof of Theorem 4.5, to conclude $R \simeq S$ in this case.

COROLLARY 6.6. *Let G be an abelian group with finite torsion group T. Let R and S be finite sums of fields. If $RG \simeq SG$ then $R \simeq S$.*

COROLLARY 6.7. *Let G be an abelian group with finite torsion group T. Suppose R is a finite sum of fields and S an artinian ring. If RG is semi-simple and $RG \simeq SG$ then $R \simeq S$.*

Proof. This has been shown in the proof of Theorem 6.5.

7. Integral group rings.

LEMMA 7.1. *Let G be an abelian group with torsion subgroup T and R a ring. Suppose $ZG \simeq RG$, then*

(i) $u(R) \cap \Theta(G) = \{1\}$;

(ii) *if $n \in \Theta(G)$, n is regular in RG;*

(iii) $\sigma(ZT) \subseteq RT$;

(iv) *if R is an integral domain and x is a torsion element in $U(R)$, then $x = \pm 1$.*

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Proof. (i) If $n \in U(R) \cap \Theta(G)$, then there exists an $r \in R$ with $nr = 1$. Then $\sigma^{-1}(nr) = n\sigma^{-1}(r) = 1$. Write $\sigma^{-1}(r) = \sum n_i g_i$, $n_i \in \mathbf{Z}$. So $1 = \sum nn_i g_i$. If $g_i = 1$, we have nn_1 and $nn_i = 0$ for $i \neq 1$. As $n \geq 1$, $n = n_1 = 1$.

(ii) Suppose n is not regular in RG. Then there is an $r \neq 0$ in RG with $nr = 0$. $\sigma^{-1}(nr) = n\sigma^{-1}(r) = 0$. If $\sigma^{-1}(r) = \sum n_i g_i$, $\sum nn_i g_i = 0$ and $nn_i = 0$ for all i. Thus $n_i = 0$ for all i and $\sigma^{-1}(r) = 0$, or $r = 0$, a contradiction.

(iii) Let $t \in T$ with $t^n = 1$. From (ii) n is regular in RG. Write $\sigma(t) = \alpha$, so that $\alpha^n = 1$. From [**17**], Proposition 5, $\alpha \in RT$, and $\sigma(T) \leqq RT$. Hence $\sigma(ZT) \leqq RT$.

(iv) Suppose $x^n = 1$. If $\sigma^{-1}(x) = \alpha$, then $\alpha \in \mathbf{Z}(T)$ by Theorem 5.1. Since $\alpha \in \mathbf{Z}T$, $\alpha^n = 1$, we have that $\alpha = \pm t$ for some $t \in T$ (see, e.g. [**12**]). Suppose, $\sigma^{-1}(a) = t$. Then $t^n = 1$ implies

$$(t-1)(1 + t + t^2 + \ldots + t^{n-1}) = 0$$

with $1 + t + t^2 + \ldots + t^{n-1} \neq 0$. Similarly

$$(1 - a)(1 + a + a^2 + \ldots + a^{n-1}) = 0$$

with either $1 - a = 0$ or $1 + a + a^2 + \ldots + a^{n-1} = 0$ (R is an integral domain). But $1 + t + t^2 + \ldots + t^{n-1} \neq 0$ implies

$$(1 + t + t^2 + \ldots + t^{n-1}) = 1 + a + a^2 + \ldots + a^{n-1} \neq 0$$

guaranteeing $a = 1$. Similarly if $\sigma^{-1}(a) = -t$, then $\sigma^{-1}(-a) = t$ and $-a = 1$ or $a = -1$. Hence $t(U(R)) = \pm 1$.

THEOREM 7.2. *Let G be a torsion group, and R a ring. Then $ZG \simeq RG$ if and only if there exist subgroups H, K of G with*

(i) $H \simeq G$
(ii) $H \oplus K \simeq G$
(iii) $R \simeq ZK$.

Proof. If subgroups H, K exist satisfying (i), (ii), (iii), then $ZG \simeq RG$ as in Corollary 5.9.

Conversely, suppose $\sigma:ZG \rightarrow RG$ is the given isomorphism. If $x \in \pm G$, $\sigma^{-1}(x) \in \pm G$. Note that we cannot have $\sigma^{-1}(g_1) = h$ and $\sigma^{-1}(g_2) = -h$ for $g_1, g_2 \in G$. So let

$$H = \{h \in G | \quad \sigma^{-1}(g) = \pm h \text{ for some } g \in G\}.$$

Then H is a subgroup of G and $H \simeq G$, since $\sigma(ZH) \subseteq ZG$ and $\sigma^{-1}(ZG) \subseteq ZH$ implying $\sigma|_{ZH}:ZH \to ZG$ is an isomorphism. By [**25**], Corollary 2.10, $G \simeq H$. This shows (i).

Let

$$K = \{g \in G| \quad \sigma(g) \in R\}.$$

K is a subgroup of G and $\sigma(ZK) \subseteq R$.

We prove (ii) by showing that G is the internal direct sum of H and K. Clearly $H \cap K = \{1\}$. Let

$$L = \{g \in G| \quad \sigma(g) = uh \text{ for some } h \in G, u \in t(U(R))\}.$$

L is a subgroup of G and H, K are subgroups of L. Let $g \in L$. Then $\sigma(g) = uh$ for some $h \in G$ and $u \in t(U(R))$, and

$$\sigma^{-1}(uh) = \sigma^{-1}(u)\sigma^{-1}(h) = g.$$

But

$$\sigma^{-1}(u) = \pm k, k \in K \quad \text{and}$$

$$\sigma^{-1}(h) = \pm h_1, h_1 \in H$$

implying $\sigma^{-1}(u) = k$ and $\sigma^{-1}(h) = h_1$ or $\sigma^{-1}(u) = -k$ and $\sigma^{-1}(h) = -h$ and $g = kh_1$. Thus $L = HK$ (direct sum), and we must check that $L = G$.

Let S_p denote a p-Sylow subgroup of G. Define

$$\text{Sup } G = \{p \in Z| \quad p \text{ a prime and } S_p \neq 1\},$$

$$\text{Inv } R = \{p \in Z| \quad p \text{ a prime and } p \in U(R)\}$$

and

$$\text{Zd } R = \{p \in Z| \quad p \text{ a prime and } p \text{ is a zero divisor in } R\}.$$

From Lemma 1.1

$$\text{Sup } G \cap \text{Inv } R = \emptyset \quad \text{and} \quad \text{Sup } G \cap \text{Zd } R = \emptyset.$$

Thus from [**17**], p. 494, $S^p = V_p$ where V_p denotes the p component of $U(RG)$.

Let $g \in G$. Then $\sigma(g) = u \cdot \alpha_1$ with $u \in U(R)$, $\alpha_1 \in U(RG)$ ([**17**]) and $\alpha_1^n = 1$ for some n. Then

$$\alpha_1 \in V_{p_1} \times \ldots \times V_{p_k} = S_{p_1} \times \ldots \times S_{p_k} \subset G$$

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for some finite k. i.e., $\alpha_1 \in G$. Thus $\sigma(g) \in U(R) \cdot G$. As g is of finite order $\sigma(g) = u\bar{g}$, $u \in U(R)$, $\bar{g} \in G$, then $u \in t(U(R))$. Thus $g \in L$. This shows $L = G$ and establishes (ii).

Finally

$$RG = R(KH) \simeq \sigma(ZKH) \simeq \sigma(ZK)\sigma(H) \simeq \sigma(ZK)G \subseteq RG$$

and so $RG = \sigma(ZK)G$. Thus $R = \sigma(ZK)$.

Modifications of Theorem 7.2 can be given if G is not torsion.

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