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Additive Riemann–Hilbert Problem in Line Bundles Over $\mathbb{CP}^1$

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Abstract. In this note we consider $\overline{\partial}$-problem in line bundles over complex projective space $\mathbb{CP}^1$ and prove that the equation can be solved for $(0,1)$ forms with compact support. As a consequence, any Cauchy-Riemann function on a compact real hypersurface in such line bundles is a jump of two holomorphic functions defined on the sides of the hypersurface. In particular, the results can be applied to $\mathbb{CP}^2$ since by removing a point from it we get a line bundle over $\mathbb{CP}^1$.

1 Introduction and Definitions

Let $\mathbb{CP}^1$ be the one-dimensional complex projective space. It is well known (*e.g.*, [GH, Gu]) that all line bundles over $\mathbb{CP}^1$ are E_k, $k = 0, \pm 1, 2, \dots$, where the transition functions are $z_2 = z_1^{-1}$ and $w_2 = z_1^k w_1$. We use the standard notation: $\mathcal{O}$, the sheaf of germs of holomorphic functions, $H^1(E_k, \mathcal{O})$, the first cohomology group of E_k with coefficients in $\mathcal{O}$, and $H^1_c(E_k, \mathcal{O})$, the cohomology group with compact support.

The main results of this note are the following:

Theorem A (Theorem 4.1) *In any E_k, $k = \pm 0, 1, 2, \dots$, the equation $\overline{\partial} u = \omega$ can be solved for any closed $(0,1)$ form ω with compact support. Additionally, if $k = 1, 2, \dots$, the solution can be chosen to have compact support; if $k = 0, 1, 2, \dots$, the solution exists even if the support of ω is not compact.*

Theorem B (Theorem 3.1)

(a) $H^1_c(E_k, \mathcal{O}) = H^1(E_k, \mathcal{O}) = 0$ *for* $k = 1, 2, \dots;$

(b) $H^1_c(E_0, \mathcal{O}) \neq 0$ *and* $H^1(E_0, \mathcal{O}) = 0;$

(c) $H^1_c(E_k, \mathcal{O}) \neq 0$ *and* $H^1(E_k, \mathcal{O}) \neq 0$ *for* $k = -1, -2, \dots.$

A smooth function defined on a real hypersurface in E_k is Cauchy–Riemann (CR) if it satisfies the tangential CR equations. As a consequence of Theorem A we have

Corollary (Corollary 4.3) *Let $M = \partial U$ be the boundary (connected) of a relatively compact domain U, $U = U^+ \Subset E = E_k$, $k = 0, \pm 1, \pm 2, \dots.$ Then any smooth CR function f on M can be represented as*

$$f = u^+ - u^-, \tag{1}$$

where u^+ (resp. u^-) is a holomorphic function in U^+ (resp. $U^- = E_k \setminus \overline{U}^+$) smooth on the closure $\overline{U}^+$ (resp. $\overline{U}^-$).

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In particular, the Corollary can be applied to $\mathbb{CP}^2$, since by removing a point from it we get a line bundle over $\mathbb{CP}^1$. In the $\mathbb{CP}^2$ a stronger result can be obtained, namely that one of the functions u^+ or u^- in (1) is constant under some weak hypothesis on M (global minimality); see [DM, S1, S2].

The $\overline{\partial}$-equation is intimately related to the first cohomology groups and also to the Hartogs and Hartogs–Bochner phenomena (see, for instance, [L]). Because of that we need the following definitions.

Let X be a connected complex manifold. By a domain U we always mean an open, connected, relatively compact set with smooth connected boundary. By *smooth* we mean C^∞, however the differentiability class in the results of this note can be relaxed significantly.

Definition 1.1 The *Hartogs phenomenon* ($\mathcal{H}$ in short) holds in a complex manifold X if for any compact set K such that $X \setminus K$ is connected, any holomorphic function defined on $X \setminus K$ can be holomorphically extended to X.

Definition 1.2 The *Hartogs–Bochner phenomenon for a domain* $U \Subset X$, ($\mathcal{HB}$-U in short), holds if any smooth CR function on ∂U can be holomorphically extended to U and smoothly up to the boundary.

Definition 1.3 The *Hartogs–Bochner phenomenon* ($\mathcal{HB}$ in short) holds in a complex manifold X if $\mathcal{HB}$-U holds for any domain $U \Subset X$.

2 Cohomology Groups and the Hartogs–Bochner Phenomenon

Let X be a complex manifold. We introduce the q-th cohomology group with compact support. Let $\mathcal{F}$ be a sheaf of abelian groups over X, and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a locally finite covering of X by relatively compact open sets U_i that are homeomorphic to a ball. In what follows, we always consider such coverings. We define the q-th compact cochain group of $\mathcal{F}$ with respect to $\mathcal{U}$, denoted by $C_c^q(\mathcal{U},\mathcal{F})$, as a collection

$$\Gamma(U_{i_0}\cap\cdots\cap U_{i_q},\mathcal{F}) \ni \{\xi_{i_0,\dots,i_q}\}, \quad \xi_{i_0,\dots,i_q} \not\equiv 0 \text{ for finitely many } i_0,\dots,i_q \in I.$$

We say that the q-cochain $\{\xi_{i_0,\dots,i_q}\}$ as above has compact support. Obviously we have the standard coboundary operators:

$$\delta\colon C_c^q(\mathcal{U},\mathcal{F}) \to C_c^{q+1}(\mathcal{U},\mathcal{F}).$$

Consequently we can define the group of compact q-cocycles and q-coboundaries

$$\mathcal{Z}_c^q(\mathcal{U},\mathcal{F}) := \operatorname{Ker}[C_c^q(\mathcal{U},\mathcal{F}) \to C_c^{q+1}(\mathcal{U},\mathcal{F})]$$
$$\mathcal{B}_c^q(\mathcal{U},\mathcal{F}) := \operatorname{Im}[C_c^{q-1}(\mathcal{U},\mathcal{F}) \to C_c^q(\mathcal{U},\mathcal{F})]$$

and the q-th cohomology group with respect to the covering $\mathcal{U}$, namely

$$H_c^q(\mathcal{U},\mathcal{F}) := Z_c^q(\mathcal{U},\mathcal{F})/B_c^q(\mathcal{U},\mathcal{F}).$$

The inductive limit

$$H^q_c(X,\mathcal{F}) = \lim_{\vec{\mathcal{U}}} H^q_c(\mathcal{U},\mathcal{F})$$

is called the q-th compact cohomology group of X with coefficients in $\mathcal{F}$. Later on we will work with $H^1_c(X,\mathcal{O})$ or $H^1(X,\mathcal{O})$, where $\mathcal{O}$ is the sheaf of germs of holomorphic functions.

Proposition 2.1 *Let X be a complex manifold.*

(a) $H^1_c(X,\mathcal{O}) = 0$ *if and only if for any smooth closed* $(0,1)$ *form* ω *on* X *with compact support there exists a compactly supported solution* u *of* $\overline{\partial}u = \omega$.

(b) *The compact cohomology group* $H^1_c(X,\mathcal{O})$ *is naturally mapped into the standard cohomology group* $H^1(X,\mathcal{O})$. *If* $\mathcal{H}$ *holds for* X *and moreover* X *has one end, then the mapping is injective.*

(c) *Let* X *be a noncompact complex manifold. If* $H^1_c(X,\mathcal{O}) = 0$, *then* $\mathcal{HB}$ *holds in* X.

(d) *Let* X *be a noncompact complex manifold with one end. We suppose that* $\mathcal{H}$ *holds for* X *and that the* $\overline{\partial}$*-problem has always a solution. Then* $H^1_c(X,\mathcal{O}) = 0$ *and* $\mathcal{HB}$ *holds in* X.

Example 2.2 The opposite implication in Proposition 2.1(c) is not true. Let $X = \mathbb{C}^2 \setminus \{0\}$. Then $\mathcal{HB}$ holds in X but $H^1_c(X,\mathcal{O}) \neq 0$.

Problem It would be interesting to prove or disprove whether there is equivalence between $\mathcal{HB}$ and vanishing of the first cohomology group with compact support, excluding some obvious cases (as in Example 2.2).

Proof of Proposition 2.1 (a) Assume that $H^1_c(X,\mathcal{O}) = 0$. We take a closed $(0,1)$ form ω on X with compact support. We choose a covering $\{U_i\}_{i\in I}$ of X (the covering is as described above). In each U_i we can solve the equation $\overline{\partial}u = \omega|_{U_i}$ and denote the solution by η_i. Moreover we choose $\eta_i \equiv 0$ if $U_i \cap \operatorname{supp}\omega = \varnothing$. We set

$$\xi_{ij} = \eta_j - \eta_i \quad \text{on } U_i \cap U_j, \quad i,j \in I.$$

Obviously ξ_{ij} are holomorphic functions and $\xi_{ij} \equiv 0$ except for a finite number of indices. Since $H^1_c(X,\mathcal{O}) = 0$, there exists a 0-cochain $\{\xi_i\}_{i\in I}$ with compact support of holomorphic functions such that $\xi_{ij} = \xi_j - \xi_i$. So we have $\eta_j - \eta_i = \xi_j - \xi_i$ or $\eta_j - \xi_j = \eta_i - \xi_i$ on $U_i \cap U_j$. Consequently, we can define a global function u

$$u = u_j = \eta_j - \xi_j \quad \text{on} \quad U_j, \quad j \in I.$$

Also we have $\overline{\partial}u = \overline{\partial}u_j = \overline{\partial}\eta_j = \omega$ on U_j. Moreover, the support of u is compact since $u_j \equiv 0$ except for a finite number of j's.

Now we prove the opposite implication. We assume that the equation $\overline{\partial}u = \omega$ can be solved as in (a) and we take a 1-cocycle $\{\xi_{ij}\}_{i,j\in I}$ with compact support of

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holomorphic functions. Let $\{\psi_i\}_{i\in I}$ be a partition of unity by smooth functions. We define

$$\eta_i = \sum_{l\in I} \psi_l\, \xi_{li}.$$

We note that $\psi_l\, \xi_{li}$ is a well-defined smooth function on U_i and $\{\eta_i\}_{i\in I}$ has compact support. The summation makes sense because the covering is locally finite. We have

$$\eta_j - \eta_i = \sum_{l\in I}(\psi_l\, \xi_{lj} - \psi_l\, \xi_{li}) = \sum_{l\in I} \psi_l(\xi_{lj} - \xi_{li}) = \sum_{l\in I} \psi_l\, \xi_{ij} = \xi_{ij}.$$

Since $\overline{\partial}\eta_j - \overline{\partial}\eta_i = \overline{\partial}\xi_{ij} = 0$ on U_{ij}, we have a globally defined $(0,1)$ form $\omega = \overline{\partial}\eta_j$ on M with compact support. By our assumption, the equation $\overline{\partial}u = \omega$ can be solved with compactly supported u. Now we take

$$\{\xi_i\}_{i\in I}, \quad \xi_i = \eta_i - u, \quad i \in I,$$

which also has compact support. Moreover, $\overline{\partial}\xi_i = \overline{\partial}\eta_i - \overline{\partial}u = 0$, and obviously $\xi_j - \xi_i = \xi_{ij}$. Part (a) is proved.

(b) We take an element $\xi \in H^1_c(X, \mathcal{O})$ which is represented by a 1-cocycle $\{\xi_{ij}\}$ with compact support. Obviously it determines an element in $H^1(X, \mathcal{O})$. Such mapping does not depend on the representing element chosen.

To prove that the mapping is injective, it is enough to prove that only the zero element of $H^1_c(X, \mathcal{O})$ is mapped at the zero element of $H^1(X, \mathcal{O})$. Assume that there exists $\xi \in H^1_c(X, \mathcal{O})$, $\xi \neq 0$, which is mapped at zero in $H^1(X, \mathcal{O})$. Let $\{\xi_{ij}\}$ be a compactly supported cocycle which represents ξ and which determines the zero element in $H^1(X, \mathcal{O})$. This means that there exists a 0-cochain $\{\eta_i\}_{i\in I}$ of holomorphic functions such that

$$\xi_{ij} = \eta_j - \eta_i, \quad i, j \in I.$$

Since $\xi_{ij} \equiv 0$ except for a finite number of indices, we have that $\eta_i = \eta_j$ for almost all $i, j \in I$. It means that we have a function f defined on $X \setminus K$, where K is a compact set. Since X has one end, there is only one unbounded component of $X \setminus K$. This and the assumption that $\mathcal{H}$ holds in X gives that the function f can be holomorphically extended to a function F on X. Now replacing $\{\eta_i\}$ by $\{\eta_i - F\}$ we obtain that $\{\xi_{ij}\}$ determines the zero element in $H^1_c(X, \mathcal{O})$, which contradicts $\xi \neq 0$. Part (b) is proved.

(c) This part is very well known in the literature (see, for instance, [AH]). Let f be a CR function defined on the boundary ∂U of a domain U. We can extend f smoothly to $\tilde{f}$ on X in such a way that $\overline{\partial}\tilde{f}$ vanishes to infinite order on ∂U. We consider the $(0,1)$ form

$$\omega = \begin{cases} \overline{\partial}\tilde{f} & \text{on } \overline{U}, \\ 0 & \text{on } X \setminus \overline{U}. \end{cases}$$

The form ω is smooth. By our assumption, there is a smooth, compactly supported function u such that $\overline{\partial}u = \omega$. The function u is holomorphic on $X \setminus \overline{U}$, and since u has compact support, it must be zero on $X \setminus \overline{U}$ because it is connected. The function

$F = \tilde{f} - u$ is holomorphic on U and $F|_{\partial U} = f|_{\partial U}$. This means that the Hartogs–Bochner phenomenon holds. Part (c) is proved.

(d) Let ω be a $\overline{\partial}$-closed $(0,1)$ form with compact support on X. By the assumption, there exists a solution u of $\overline{\partial} u = \omega$, and obviously u is holomorphic on $X \setminus \operatorname{supp} \omega$. Since X has one end, there is only one unbounded component of $X \setminus \operatorname{supp} \omega$. Because the Hartogs phenomenon holds, there exists a holomorphic extension u_0 of $u|_{X \setminus \operatorname{supp} \omega}$ to X. So we have that $\overline{\partial}(u - u_0) = \overline{\partial} u = \omega$ and $\operatorname{supp}(u - u_0)$ is compact. Consequently $H^1_c(X, \mathcal{O}) = 0$ and, from (c), $\mathcal{HB}$ holds in X. This proves part (d) and completes the proof of the proposition. ■

3 Some Cohomology Groups of E_k

Now a few words about line bundles over $\mathbb{CP}^1$. Any line bundle E over $\mathbb{CP}^1$ can be identified with an element of $H^1(\mathbb{CP}^1, \mathcal{O}^*) \simeq \mathbb{Z}$ (see [GH, Gu]). In practice, this means that we can choose the atlas $(U_1, (z_1, w_1)), (U_2, (z_2, w_2))$ of E such that

$$\begin{matrix} U_1 \simeq \mathbb{C} \times \mathbb{C}, (z_1, w_1), \\ U_2 \simeq \mathbb{C} \times \mathbb{C}, (z_2, w_2), \end{matrix} \quad z_2 = \frac{1}{z_1}, \quad w_2 = z_1^k w_1. \tag{2}$$

We denote by E_k the line bundle over $\mathbb{CP}^1$ which is determined by the transition function $\xi_{21}(z_1) = z_1^k$, $k = \pm 0, 1, 2, \ldots$, *i.e.*, an element of $H^1\big(\{U_1, U_2\}, \mathcal{O}^*\big)$.

Theorem 3.1 *Let E_k, $k = \pm 0, 1, 2, \ldots$, be a line bundle over $\mathbb{CP}^1$. Then*

(a) $H^1_c(E_k, \mathcal{O}) = H^1(E_k, \mathcal{O}) = 0$ *for* $k = 1, 2, \ldots$;
(b) $H^1_c(E_0, \mathcal{O}) \neq 0$ *and* $H^1(E_0, \mathcal{O}) = 0$;
(c) $H^1_c(E_k, \mathcal{O}) \neq 0$ *and* $H^1(E_k, \mathcal{O}) \neq 0$ *for* $k = -1, -2, \ldots$.

Corollary 3.2 *Let ω be a $\overline{\partial}$-closed $(0,1)$ form on E_k. Then*

(a) *The equation $\overline{\partial} u = \omega$ has a solution in E_k for $k \geq 1$. Moreover, if $\operatorname{supp} \omega$ is compact, then u can be chosen with compact support. Consequently, $\mathcal{HB}$ holds in E_k.*
(b) *The equation $\overline{\partial} u = \omega$ has a solution in E_0, but $\mathcal{HB}$ does not hold in E_0.*

Proof of Theorem 3.1 First we prove that $H^1(E_k, \mathcal{O}) = 0$ for $k = 0, 1, 2, \ldots$. Since the covering $\{U_1, U_2\}$ is a Leray covering, we have the equality

$$H^1(E_k, \mathcal{O}) = H^1(\{U_1, U_2\}, \mathcal{O}),$$

so it is enough to show that the latter cohomology group is zero. To prove this, we take a holomorphic function $g_{21}(z_1, w_1)$ defined on $\mathbb{C}_* \times \mathbb{C}$ and write its Laurent

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expansion:

$$\begin{aligned} g_{21}(z_1,w_1) &= \sum_{\alpha=-\infty}^{\infty}\sum_{\beta=0}^{\infty} c_{\alpha\beta}\, z_1^{\alpha} w_1^{\beta} = \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} c_{\alpha\beta}\, z_1^{\alpha} w_1^{\beta} + \sum_{\alpha=-\infty}^{-1}\sum_{\beta=0}^{\infty} c_{\alpha\beta}\, z_1^{\alpha} w_1^{\beta} \\ &= \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} c_{\alpha\beta}\, z_1^{\alpha} w_1^{\beta} + \sum_{\alpha=1}^{\infty}\sum_{\beta=0}^{\infty} c_{(-\alpha)\beta}\, z_2^{\alpha+k\beta} w_2^{\beta} \\ &= g_1(z,w) - g_2(z_2,w_2), \end{aligned}$$

where the functions g_1, g_2 are holomorphic on U_1, U_2, respectively, which gives that $H^1\big(\{U_1,U_2\},\mathcal{O}\big) = 0$.

Next we prove that $H^1(E_k,\mathcal{O}) \neq 0$ for $k = -1,-2,\dots$. To see this, we take as $g_{21}(z_1,w_1)$ the function

$$g_{21}(z_1,w_1) = z_1^{2k+1} w_1^2.$$

We assume that $g_{21}(z_1,w_1) = g_1(z_1,w_1) - g_2(z_2,w_2)$, *i.e.*,

$$\begin{aligned} z_1^{2k+1} w_1^2 &= \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} a_{\alpha\beta} z_1^{\alpha} w_1^{\beta} - \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} b_{\alpha\beta} z_2^{\alpha} w_2^{\beta} \\ &= \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} a_{\alpha\beta} z_1^{\alpha} w_1^{\beta} - \sum_{\alpha=0}^{\infty}\sum_{\beta=0}^{\infty} b_{\alpha\beta} z_1^{k\beta-\alpha} w_1^{\beta}. \end{aligned}$$

Consequently we have

$$z_1^{2k+1} = \sum_{\alpha=0}^{\infty} a_{\alpha 2} z_1^{\alpha} - \sum_{\alpha=0}^{\infty} b_{\alpha 2} z_1^{2k-\alpha},$$

and in both sums there are no powers of z_1 of order $2k < \alpha < 0$, which gives a contradiction. The claim is proved.

Finally, let k be arbitrary and consider the *cohomology groups with compact support.* We take an arbitrary holomorphic function u defined in a "ring neighborhood" of the zero section of E_k, which in local coordinates on U_1 and U_2 can be written as

$$u_1(z_1,w_1) = \sum_{\alpha=-\infty}^{+\infty} a_{1\alpha}(z_1)\, w_1^{\alpha} \quad \text{on } \{(z_1,w_1)\,;\, r_1(z_1) < |w_1| < R_1(z_1)\},$$

$$u_2(z_2,w_2) = \sum_{\alpha=-\infty}^{+\infty} a_{2,\alpha}(z_2)\, w_2^{\alpha} \quad \text{on } \{(z_2,w_2)\,;\, r_2(z_2) < |w_2| < R_2(z_2)\}$$

for some smooth functions $R_1(z_1) > r_1(z_1) > 0$ and $R_2(z_2) > r_2(z_2) > 0$. Because these two functions u_1 and u_2 should be the same in the common domain, therefore we have

$$a_{2\alpha}(1/z_1) = z_1^{-k\alpha} a_{1\alpha}(z_1), \quad \alpha = \pm 0, 1, 2, \dots.$$

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We have three cases with respect to k:

If $k \geq 1$, then $a_{1\alpha} \equiv a_{2\alpha} \equiv 0$ for $\alpha = -1, -2, \dots$ and

$$u_1(z_1, w_1) = \sum_{\alpha=0}^{+\infty} a_{1\alpha}(z_1)\, w_1^{\alpha}, \quad u_2(z_2, w_2) = \sum_{\alpha=0}^{+\infty} a_{2\alpha}(z_2)\, w_2^{\alpha}.$$

This means that the Hartogs phenomenon holds on E_k, $k \geq 1$. Combining this information with $H^1(E_k, \mathcal{O}) = 0$, $k \geq 1$, and Proposition 2.1(b) we get that $H_c^1(E_k, \mathcal{O}) = 0$ for $k = 1, 2, \dots$.

If $k = 0$, then $a_{1\alpha} = a_{2\alpha} \equiv \text{const}$ for $\alpha = \pm 0, 1, 2, \dots$ and

$$u_1(z_1, w_1) = \sum_{\alpha=-\infty}^{+\infty} a_{1\alpha}\, w_1^{\alpha}, \qquad u_2(z_2, w_2) = \sum_{\alpha=-\infty}^{+\infty} a_{2\alpha}\, w_2^{\alpha}.$$

From the form of the function u we see that the Hartogs–Bochner phenomenon does not hold in this case. Thus, applying Proposition 2.1(c) we get $H_c^1(E_0, \mathcal{O}) \neq 0$.

If $k \leq -1$, then $a_{1\alpha} \equiv a_{2\alpha} \equiv 0$ for $\alpha = 1, 2, \dots$ and

$$u_1(z_1, w_1) = \sum_{\alpha=-\infty}^{0} a_{1\alpha}(z_1)\, w_1^{\alpha}, \qquad u_2(z_2, w_2) = \sum_{\alpha=-\infty}^{0} a_{2\alpha}(z_2)\, w_2^{\alpha}.$$

Again it is obvious that the Hartogs–Bochner phenomenon does not hold in this case. Consequently, we have that $H_c^1(E_k, \mathcal{O}) \neq 0$ for $k \leq -1$.

4 The $\overline{\partial}$-Equation in Line Bundles Over $\mathbb{CP}^1$

In this section we deal with the line bundles E_k over $\mathbb{CP}^1$ for $k = -1, -2, \dots$, unless otherwise stated. The main theorem of this section is:

Theorem 4.1 *Let ω be a smooth (C^∞), $\overline{\partial}$-closed $(0, 1)$ form on $E = E_k$ with compact support. Then there is a smooth solution of the equation*

$$\overline{\partial} u = \omega,$$

and moreover the function u is determined uniquely up to a constant.

Remark 4.2 In general, the solution u in the proposition does not have compact support.

Corollary 4.3 *Let M be the boundary of a domain U, $M = \partial U$ smooth, $U = U^+ \Subset E = E_k$, $k = 0, \pm 1, \pm 2, \dots$. Then any smooth CR function f on M can be represented as*

$$f = u^+ - u^-,$$

where u^+ (resp. u^-) is a holomorphic function in U^+ (resp. $U^- = E \setminus \overline{U}^+$) smooth on $\overline{U}^+$ (resp. $\overline{U}^-$).

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Proof of Corollary 4.3 We can extend f to a C^∞ function F in such a way that $\operatorname{supp} F$ lies in an arbitrarily small neighborhood of M and $\overline{\partial}F|_M = 0$ to the infinite order. We define

$$\omega = \begin{cases} \overline{\partial}F & \text{on } U^+, \\ 0 & \text{on } U^-. \end{cases}$$

The form ω is of class C^∞ and with compact support. From Theorem 4.1 we can solve the equation $\overline{\partial}u = \omega$ in E_k for $k \le -1$ and from Corollary 3.2 in E_k for $k \ge 0$, and u is of class C^∞. So we have

$$\overline{\partial}(F-u) = 0 \text{ on } U^+ \quad \text{i.e., } F-u \text{ is holomorphic on } U^+,$$

$$\overline{\partial}u = 0 \text{ on } U^- \quad \text{i.e., } u \text{ is holomorphic on } U^-,$$

and

$$F = (F-u) - (-u), \quad F|_M = f.$$

Obviously the components of the decomposition are of class C^∞. The corollary is proved.

4.1 Properties of Compactly Supported $(0,1)$ Forms on E_k

Any $(0,1)$ form on E_k can be written

$$\omega = \begin{cases} \omega(z_1,w_1) = a_1(z_1,w_1)\,d\overline{z}_1 + b_1(z_1,w_1)\,d\overline{w}_1 & \text{in } U_1, \\ \omega(z_2,w_2) = a_2(z_2,w_2)\,d\overline{z}_2 + b_2(z_2,w_2)\,d\overline{w}_2 & \text{in } U_2. \end{cases}$$

Since

$$d\overline{z}_2 = -\overline{z}_1^{-2}\,d\overline{z}_1, \quad d\overline{w}_2 = k\overline{z}_1^{k-1}\,\overline{w}_1\,d\overline{z}_1 + \overline{z}_1^k\,d\overline{w}_1\,,$$

we have

$$\begin{aligned} a_1(z_1,w_1)\,d\overline{z}_1 &+ b_1(z_1,w_1)\,d\overline{w}_1 \\ &= -a_2(z_1^{-1}, z_1^k w_1)\overline{z}_1^{-2}d\overline{z}_1 + b_2(z_1^{-1}, z_1^k w_1)[k\overline{z}_1^{k-1}\,\overline{w}_1\,d\overline{z}_1 + \overline{z}_1^k\,d\overline{w}_1], \end{aligned}$$

which gives

$$\begin{aligned} a_1(z_1,w_1) &= -\overline{z}_1^{-2}\,a_2(z_1^{-1}, z_1^k w_1) + k\overline{z}_1^{k-1}\overline{w}_1\,b_2(z_1^{-1}, z_1^k w_1) \\ b_1(z_1,w_1) &= \overline{z}_1^k\,b_2(z_1^{-1}, z_1^k w_1). \end{aligned} \tag{3}$$

4.2 Proof of Theorem 4.1

Let ω be a $\overline{\partial}$-closed $(0,1)$ form with compact support. We define

$$u_1(z_1,w_1) = \frac{1}{2\pi i}\int_{\mathbb{C}_\xi} b_1(z_1,\xi)\frac{1}{\xi - w_1}\,d\xi \wedge d\overline{\xi} \quad \text{for } (z_1,w_1) \in U_1$$

and

$$u_2(z_2, w_2) = \frac{1}{2\pi i} \int_{\mathbb{C}_\xi} b_2(z_2, \xi) \frac{1}{\xi - w_2}\, d\xi \wedge d\overline{\xi} \quad \text{for } (z_2, w_2) \in U_2.$$

It is clear that the functions u_1 and u_2 are smooth in U_1 and U_2 respectively. Since for fixed z_1 the support of $w_1 \longrightarrow b_1(z_1, w_1)$ is compact, and the same with the second function b_2, therefore (see *e.g.*, [N]) we have

$$\text{(4)} \qquad \frac{\partial u_1}{\partial \overline{w}_1}(z_1, w_1) = b_1(z_1, w_1) \quad \text{and} \quad \frac{\partial u_2}{\partial \overline{w}_2}(z_2, w_2) = b_2(z_2, w_2)$$

Now, using (3) and (4), we show that the pair u_1, u_2 determines a global smooth function on E_k.

$$\begin{aligned} u_1(z_1, w_1) &= \frac{1}{2\pi i} \int_{\mathbb{C}_\xi} b_1(z_1, \xi) \frac{1}{\xi - w_1}\, d\xi \wedge d\overline{\xi} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}_\xi} \overline{z}_1^k b_2(1/z_1, z_1^k \xi) \frac{1}{\xi - w_1}\, d\xi \wedge d\overline{\xi} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}_\zeta} \overline{z}_1^k b_2(1/z_1, \zeta) \frac{1}{\frac{\zeta}{z_1^k} - w_1} z_1^{-k} \overline{z}_1^{-k}\, d\zeta \wedge d\overline{\zeta} \\ &= \frac{1}{2\pi i} \int_{\mathbb{C}_\zeta} b_2(z_2, \zeta) \frac{1}{\zeta - w_2}\, d\zeta \wedge d\overline{\zeta} \\ &= u_2(z_2, w_2). \end{aligned}$$

We note that the coefficients $b_1 = b_1(z_1, w_1)$ and $b_2 = b_2(z_2, w_2)$ uniquely determine the a_1 and a_2 coefficients of ω. To see this, assume that two forms have the same b's coefficients, so subtracting the forms we get a closed form on $E_k = U_1 \cup U_2$

$$c_1(z_1, w_1) d\overline{z}_1 = c_2(z_2, w_2) d\overline{z}_2$$

with

$$\frac{\partial c_1}{\partial \overline{w}_1} \equiv 0 \quad \text{and} \quad \frac{\partial c_2}{\partial \overline{w}_2} \equiv 0,$$

which means that the functions

$$w_1 \longrightarrow c_1(z_1, w_1) \quad \text{and} \quad w_2 \longrightarrow c_2(z_2, w_2)$$

are holomorphic. Since the supports of these functions are compact, they are identically zero. On the other hand, the form $\overline{\partial} u$ is closed and because of (4) we get $\overline{\partial} u = \omega$. The theorem is proved.

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