MONOTONY OF THE OSCULATING CIRCLES OF ARCS OF CYCLIC ORDER THREE

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(received November 1, 1963)

1. Introduction. It is well-known in elementary calculus that if a differentiable function has a monotone increasing curvature, then its curvature is continuous and the circles of curvature at distinct points have no points in common. In particular, two one-sided osculating circles at distinct points of an arc A_3 of cyclic order three have no points in common; cf. [1], [2], [3]. The conformal proof given here that any two general osculating circles at distinct points of A_3 are disjoint (Theorem 1), may be of interest. We also prove that all but a countable number of points of A_3 are strongly conformally differentiable (Theorem 2).

2. The notations and definitions used in this discussion are the same as in [4] and [5]. For the convenience of the reader, we list some of the results which are needed here.

An arc A in the conformal plane is the continuous image of a real interval. P, Q, ... denote points in the conformal plane, and p, s, q, ... denote points of arcs. C denotes an oriented circle, with the interior C_* and exterior C^*, the latter region lying at its right.

An arc A is called once conformally differentiable at p if it satisfies the following:

CONDITION I. There exists a point $Q \neq p$ such that if s is sufficiently close to p on A, then the circle C(p, s, Q) exists. It converges if s converges to p [4; Theorem 1].

Canad. Math. Bull. vol. 7, no. 2, April 1964

https://doi.org/10.4153/CMB-1964-024-0 Published online by Cambridge University Press

We denote the limit tangent circle by $C(\tau;Q)$.

If Condition I holds for a single point $Q \neq p$, then it holds for all such points, and the closed set $\tau = \tau(p)$, of all the tangent circles of A at p is a parabolic pencil, i.e., any two circles of τ meet at p and nowhere else.

We call A conformally differentiable at p if it satisfies

CONDITION II. If $s \neq p$, then $\lim_{s \to p} C(\tau;s)$ exists.

The limit osculating circle is denoted by $C(p)$.

We call C a general tangent circle of an arc A at p, if there exists a sequence of triples of mutually distinct points t_n, u_n, Q_n, such that t_n and u_n converge on A to p, and $\lim C(t_n, u_n, Q_n) = C$. If, in addition, $Q_n \in A$ also converges to p, then we call C a general osculating circle of A at p.

A_3 denotes an arc of cyclic order three; thus no circle meets A_3 more than three times. Here, p is counted twice on any general tangent circle of A at p which is not a general osculating circle. On a general osculating circle, and, in particular, on $C(p)$, p is counted three times; cf. [5; Section 3].

Each point of A_3 has the property that if $Q, R \neq p$, $Q \to R$ and two distinct points u and v converge on A_3 to p, then $C(u,v,Q)$ always converges [5; Theorem 2].

If p is an end-point of A_3, then $C(t,u,v)$ converges if the three mutually distinct points t,u,v converge on A to p [5; Theorem 3].

3. Let $p \in A_3$. Let B_3 denote the open subarc of A_3 bounded by p and an end-point of A_3. Let C be any general osculating circle of A_3 at p, and let $C(p)$ be the (unique) osculating circle of B_3 at p.

https://doi.org/10.4153/CMB-1964-024-0 Published online by Cambridge University Press

If p is an end-point of A_3, the strong differentiability of A_3 at p implies that $C = C(p)$ (cf. [5], Theorem 3).

Suppose, next, that p is an interior point of A_3. Then C and $C(p)$ both intersect A_3 at p (cf. [5], Section 3.3). By [5; Theorem 2], the general tangent circles of A_3 at p form a pencil τ; thus, $C \in \tau$, $C(p) \in \tau$.

LEMMA. _If_ $C^* \subset C(p)^*$, _then_ $B_3 \subset C(p)_*$.

Proof. By [5; Sections 3.32 and 3.33], $B_3 \cap C = B_3 \cap C(p) = p$. Suppose that $B_3 \subset C(p)^*$. Then $B_3 \subset C(p)^* \cap C_*$; otherwise, $C(\tau; s)$ could not converge to $C(p)$ as s tends to p on B_3. This implies, however, that $C(p)$ and C cannot both intersect A_3 at p.

COROLLARY. _If_ p _is an interior point of_ A_3, _then any general osculating circle of_ A_3 _at_ p _lies between the two one-sided osculating circles of_ A_3 _at_ p _in the pencil_ $\tau(p)$ (cf. [5], 3.42).

4. THEOREM 1. _Two general osculating circles at distinct points of_ A_3 _have no points in common._

Proof. On account of the above Corollary, we may now assume that A_3 is an open arc with the end-points p and q. Thus, A_3 has uniquely defined osculating circles $C(p)$ and $C(q)$ at p and q, respectively. We may assume that neither $C(p)$ nor $C(q)$ is a point-circle. Let τ and τ_q denote the families of tangent circles at p and q, respectively.

If t, u, v lie on A_3 in that order, we may assign to $C(t, u, v)$ the orientation associated with the order of the points t, u, v on $C(t, u, v)$.

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Thus, the arc A_3 induces a natural and continuous orientation on all the circles which meet $p \cup A_3 \cup q$ three times (cf. [5], Section 3.51).

We may assume that $A_3 \subset C(p)_*$. By considering the circles $C(\tau : s)$ and $C(p, s, q)$, and letting s move from p to q on A_3, we readily verify that

$$(1) \quad \begin{aligned} & A_3 \subset C(p)_* \cap C(\tau ;q)^* \cap C(p;\tau_q)_* \cap C(q)^* , \\ & C(\tau ;q)_* \subset C(p)_*, \text{ and } C(p;\tau_q)^* \subset C(q)^* . \end{aligned}$$

Since $C(p;\tau_q) \neq C(\tau ;q)$, $C(p;\tau_q)$ intersects $C(\tau ;q)$ at p and q. Hence $C(p;\tau_q)$ also intersects $C(p)$ at p and at another point. Since $C(\tau ;q)$ intersects $C(p;\tau_q)$ at q, $C(\tau ;q)$ also intersects $C(q)$ at q. Thus $C(\tau ;q)$ and $C(q)$ intersect at another point R. The points q and R decompose $C(q)$ into two arcs C' and C'', such that $C' \subset C(p;\tau_q)_* \cap C(\tau ;q)_*$, while $C'' \subset C(p;\tau_q)_* \cap C(\tau ;q)^*$. Since $C(\tau ;q)_* \subset C(p)_*$, we obtain $C' \subset C(p)_*$.

Suppose that C'' meets $C(p)$; thus C'' meets $C(p) \cap C(p;\tau_q)_*$. Then C'' decomposes the region

$$C(p)_* \cap C(p;\tau_q)_* \cap C(\tau ;q)^*$$

into three disjoint regions. Two of these lie in

$$(2) \qquad C(p;\tau_q)_* \cap C(q)^* \cap C(p)_* ,$$

and their boundaries have at most a single point in common which lies in $C(p)$. The region of (2) whose boundary includes an arc of $C(\tau ;q)$ [$C(p;\tau_q)$] contains points of A_3 close to p [q]. But then the continuity of A_3 and Relation (1) imply that these two regions are connected. Hence $C'' \subset C(p)_*$, and the whole of $C(q) = C' \cup C'' \cup \{q, R\}$ lies in $C(p)_*$.

Remark. The following alternative method of proving that $C'' \subset C(p)_*$ is shorter and direct, but it requires the full Jordan curve theorem.

As above, $C'' \subset C(p;\tau_q)_* \cap C(\tau;q)^*$. Since $C(q)$ does not meet A_3, C'' even lies in the region in $C(p;\tau_q)_*$ bounded by A_3 and $C(\tau;q)$. Hence $C'' \subset C(p)_*$.

5. THEOREM 2. All but a countable number of points of A_3 are strongly conformally differentiable; cf. [6].

Proof. Let p and q be the end-points of A_3. We may assume that $C(p) \neq p$, and $A_3 \subset C(p)_*$. By choosing a suitable co-ordinate system we may even assume that $C(p)$ is a circle of area 1.

Let $s \in A_3$ be a point at which A_3 is not strongly conformally differentiable; then A_3 does not satisfy Condition II at s; cf. 3, Corollary. Let $C(s)$ and $C'(s)$ be the one-sided osculating circles of A_3 at s. We may assume that $C(s)_* \subset C'(s)_*$. Let $f(s)$ be the area between $C(s)$ and $C'(s)$. By Theorem 1, the regions $C(s)^* \cap C'(s)_*$ and $C(t)^* \cap C'(t)_*$ are disjoint if $s \neq t$, and they lie in $C(p)_*$.

Thus there are not more than 2^n members in the class of points s for which

$$1/2^{n-1} > f(s) \geq 1/2^n \qquad (n = 1, 2, 3, \ldots).$$

Since every point $s \in A_3$ with $f(s) > 0$ is included in exactly one of these classes, there is only a countable set of points s with $f(s) > 0$.

https://doi.org/10.4153/CMB-1964-024-0 Published online by Cambridge University Press

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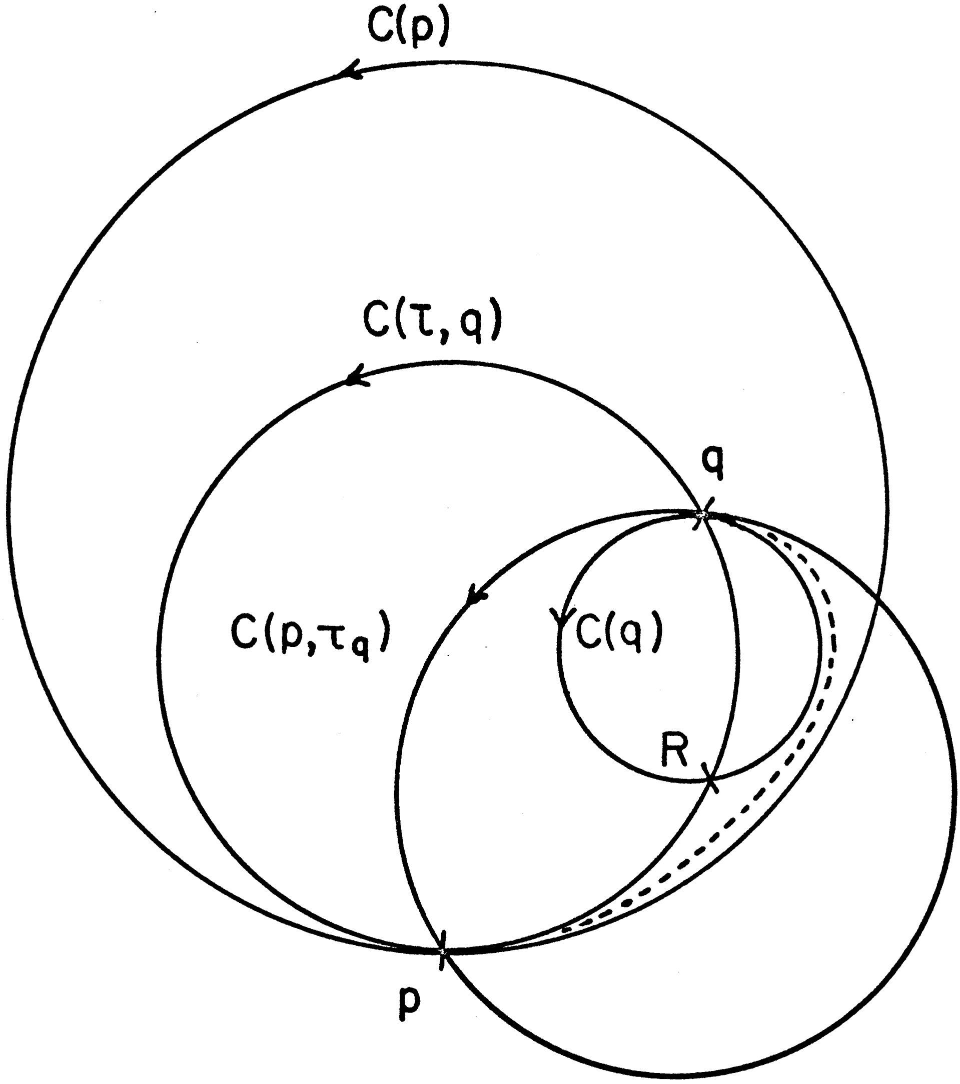
https://doi.org/10.4153/CMB-1964-024-0 Published online by Cambridge University Press



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