CORRIGENDUM AND ADDENDUM

to the paper

AN INFINITE CONSTRUCTION IN RING THEORY

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1. Point (3) of the main theorem of our paper [**3**, Theorem 1.1] is incorrect: this note corrects the main and consequential errors, and shows that (after minor adjustments) almost all the other results of [**3**], including the remaining seven points of Theorem 1.1, remain correct.

2. The theme of [**3**] was a family of functors $G_t(-)$, defined on the category of rings with unity for each cardinal t. For $t = 0$, 1, the results of [**3**] are unchanged, but, for $2 \leqslant t < \infty$, major, and, for t infinite, less major, corrections are necessary; *we therefore assume* $2 \leqslant t$. Terminology and notation are standard or as in [**3**], and I would like to thank A. W. Chatters and an anonymous referee for comments which prompted this correction.

3. In [**3**, p. 350] we defined the (functorial) ring extensions $R \to G_t(R)$ using an index set B such that $\text{card}(B) = b \geqslant t$ is an infinite cardinal, and a family T_1 of injective mappings $\sigma : B \to B$ such that:

(a) $\forall \sigma \in T_1$, $\text{card}(B \setminus \text{Im}(\sigma)) = b$;

(b) $\forall \sigma, \tau \in T_1$, $(\text{Im}(\sigma) \cap \text{Im}(\tau) \neq \varnothing) \Rightarrow \sigma = \tau$. To these axioms we must now add the stipulation (incorrectly treated as optional in [**3**]) that:

(c) if $t \geqslant 2, B = \bigcup_{\sigma \in T_1} \text{Im}(\sigma)$, i.e. $\{\text{Im}(\sigma) : \sigma \in T_1\}$ is a complete set of equivalence classes (each of cardinality b) in B.

4. As in [**3**], let H_1 be the unital monoid of injective mappings $B \to B$ generated by $T_1, H_2 = \{\text{Im}(\tau) : \tau \in H_1\}$. Additionally, for $n \geqslant 0$, let X_n be the subset of mappings comprising the products of exactly n elements of T_1 and $Y_n = \{\text{Im}(\tau) : \tau \in X_n\}$ for $n \geqslant 0$. Suppose that M is as in [**3**], that, for each $i \in B$ and each pair $I, J \in H_2$, $x_i \in M$, $E_{IJ} \in \text{End}(M_R)$ are also as in [**3**], and that the ring $G = G_t(R)$ is (still) the subring of $\text{End}(M_R)$ generated by $\mathbb{E} = \{E_{IJ} : I, J \in H_2\}$ plus the left multiplications by the elements of R.

5. From [**3**],

(a) each E_{IJ} centralizes R;

(b) $\mathbb{E} \cup \{0\}$ is multiplicatively closed;

(c) $E_{BB} = 1_G = \text{id}_M$;

(d) for all $I, J \in \Omega$, $E_{BB} = E_{BI} E_{IJ} E_{JB}$;

(e) for all I, J, K, $L \in \Omega$, $E_{IJ} E_{KL} = 0$ if and only if $J \cap K = \varnothing$;

(f) if $t \geqslant 2$ then for every $n \geqslant 1$ and every $I \in X_n$ there exists $J \in X_n$ such that $I \cap J = \varnothing$.

6. The error in Theorem 1.1(3) of [**3**] is the claim that $G = G_t(R)$ is left and right free on $\mathbb{E}$. To get a counter-example, take $t = 2$, $T_1 = \{\sigma, \tau\}$, and $I = \mathrm{Im}(\sigma)$, $J = \mathrm{Im}(\tau)$, so that $B = I \cup J$ and $I \cap J = \varnothing$. Then E_{II}, E_{JJ} are orthogonal idempotents and $E_{II} + E_{JJ} = E_{BB} = \mathrm{id}_M$.

7. The purported proof of 'freeness' is at [**3**, 2.3], but depends crucially on [**3**, 2.2(d)], which claims without specific proof that a form of partial cancellation holds in $\mathbb{E}$. The following is an easy counter-example: for any $t \neq 0$, if $J \in \Omega$, $J \neq B$, then $E_{BJ}E_{JJ} = E_{BJ}E_{BB} = E_{BJ} \neq 0$.

8. For infinite t, the new assumption (c) of Section 3 makes no practical difference, but for $2 \leqslant t < \infty$, it ensures that the idempotent $1 - \sum_{I \in Y_n} E_{II}$ is zero for every $n \geqslant 0$.

9. Whether t is finite or infinite, it is (now) easy to check that, for each $n \geqslant 0$, $E_{IJ}E_{KL} = E_{IL}\delta_{JK}$ for all $I, J, K, L \in Y_n$, i.e. each set $\mathbb{E}_n = \{E_{IJ} : I, J \in Y_n\}$ is a set of matrix units over R of degree t^n. Thus, using the condition mentioned in Section 8, we have the following result.

Proposition 1. *If* $2 \leqslant t < \infty$ *and* $n \geqslant 0$:

(a) *the bimodule* $G(n) = R\mathbb{E}_n = \mathbb{E}_n R$ *is a subring of* $G_t(R)$, *isomophic* (*over* R) *to* $M_{t^n}(R)$;

(b) $\mathbb{E}_n \subsetneq \mathbb{Z}\mathbb{E}_{n+1}$;

(c) *hence* $R = G(0) \subset G(1)$ $(\simeq M_t(R)) \subset \ldots \subset G(n)$ $(\simeq M_{t^n}(R)) \subset \ldots$ *is a strictly ascending chain of subrings of* $G = G_t(R)$, *with union* G.

10. It now follows easily that, for $t < \infty$, the other points of Theorem 1.1 of [**3**] are correct, as are the results of Section 4 and of Section 3 excluding 3(iv). The problems concern one-sided ideals, where our "proofs" made extensive but implicit use of freeness. Using the subrings $G(n)$ it follows that, if $A \subset B$ are right ideals of R then $AG \subset BG$, and hence (correcting Sections 3(iv) and 5 of [**3**]) we have the following theorem.

Theorem 2. (a) *If* R *is right primitive then so is* $G = G_t(R)$;

(b) $J(R) = R \cap J(G)$ *and* $J(G) = GJ(R) = J(R)G$.

We do not know if the converse to Theorem 2(a) is true. Clearly, however, every prime of R is an intersection of maximal, right primitive or quasi-primitive ideals (see [**4**] for the latter definition) if and only if the same holds in $G_t(R)$.

11. Using the rings $G(n)$ it is also possible for us to obtain information about the structure and isomorphism classes of the ring extensions $R \to G_t(R)$, $2 \leqslant t < \infty$. If $n \in \mathbb{N}$, let $\sqrt{n}$ denote the product of the distinct prime divisors of n.

Proposition 3. *If* $2 \leqslant t \leqslant \infty$ *then* $G_t(R)$ *and* $G_{\sqrt{t}}(R)$ *are isomorphic as* R*-algebras to the tensor product* (*over* R)

$$H = G_{p^{(1)}}(R) \otimes G_{p^{(2)}}(R) \otimes \ldots \otimes G_{p^{(r)}}(R),$$

where $p^{(1)}, p^{(2)}, \ldots, p^{(r)}$ *are the distinct prime divisors of* t.

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We observe that each extension $R \to G_{p(i)}(R)$ has no non-trivial tensor product decomposition (over R), that the decomposition as a product of such indecomposables in Proposition 3 is unique, and that ring extensions resembling these have been discussed at various places in the literature, e.g. [**1**, p. 341]. Finally, we note that (contrary to [**3**, p. 351]) if $2 \leq t(1)$, $t(2) < \infty$ then $G_{t(1)}(R)$ embeds over R in $G_{t(2)}(R)$ if and only if $\sqrt{t}(1)$ divides $\sqrt{t}(2)$.

12. Apart from the error over cancellation (see Section 7 above), most of the rest of [**3**] remains correct in the case that t is infinite. In particular, "freeness" (Theorem 1.1(3) of [**3**]) is correct, and a proof may be found in Theorem 2.3 of [**2**]. For infinite t, the only further errors in [**3**] concern embeddings (where $G_{t(1)}(R)$ does *not* embed over R in $G_{t(2)}(R)$ when $2 \leq t(1) < t(2)$ and $t(2)$ is infinite), and the discussion of the Jacobson radical in [**3**, 5.1, 5.2 and 5.3]. This discussion becomes correct if the assertions just cited are amended to stipulate that t is infinite (though a drafting error must also be eliminated: in [**3**, 5.1] the term "right quasi-inverse" should read "right inverse".

13. By ([**3**, Theorem 1.1(8)], if $2 \leq t$ each ring $G = G_t(R)$ has the pleasing property that every finitely generated one-sided G-module is cyclic. It is not difficult to establish the analogous property for bimodules: if $t(1)$, $t(2) > 1$ are cardinals, S, R are rings and $G = G_{t(1)}(S)$, $H = G_{t(2)}(R)$ then every finitely generated $H - G$ bimodule is principal (as a bimodule).

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