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SUPPORTS OF BOREL MEASURES

SUSUMU OKADA

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Abstract

We present a new class of topological spaces called SL-spaces, on which every Borel measure has a Lindelöf support. The class contains all metacompact spaces. However, a θ-refinable space is not necessarily an SL-space.

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1. Introduction

Let X be a topological space and $\mathscr{B}(X)$ be the *Borel field* of X, that is, the smallest σ-algebra generated by all open subsets of X. By a *Borel measure* we mean a totally finite, non-negative, countably additive set-function on $\mathscr{B}(X)$.

A Borel measure μ is a *regular* (resp. *Radon*) measure if it satisfies, for every Borel set B

$$\mu(B) = \sup\{\mu(F);\ B \supset F \text{ and } F \text{ is closed (resp. compact)}\}.$$

A Borel measure μ is *τ-smooth* if for every increasing net $\{U_\alpha\}$ of open subsets we have $\sup_\alpha \mu(U_\alpha) = \mu(\bigcup_\alpha U_\alpha)$. If μ is a regular measure on a topological space X, then it is necessary and sufficient for μ to be τ-smooth that $\sup_\alpha \mu(U_\alpha) = \mu(X)$ for every increasing net $\{U_\alpha\}$ of open subsets satisfying $\bigcup_\alpha U_\alpha = X$. Moreover if X is a regular topological space, then the τ-smoothness implies the regularity (see Gardner (1975), Theorem 5.4).

We define the *support* S_μ of μ as follows:

$$S_\mu = \{x \in X;\ \mu(U) > 0 \text{ for every open } U \text{ containing } x\}.$$

Moreover, if there exists the smallest closed subset S_μ^* satisfying $\mu(S_\mu^*) = \mu(X)$, then S_μ^* is said to be the *strong support* of μ. It is evident that S_μ^* equals S_μ if S_μ^* exists. Note that S_μ is equal to $\bigcap \{F;\ F \text{ is closed and } \mu(F) = \mu(X)\}$ and that S_μ may be empty.

From the definitions, a τ-smooth measure has a strong support. Since a Radon measure is τ-smooth, a Radon measure has a strong support. It still seems to be unsolved whether the τ-smoothness or the regularity is implied by the existence of the strong support. But the existence of a non-empty support S_μ does not necessarily imply even the regularity of μ, since the Dieudonné measure has a non-empty support and is not regular (see Example 2.3).

The main purpose of this paper is to study a new class of topological spaces where every Borel measure has a Lindelöf support. These spaces are called SL-spaces. We show that every metacompact space is an SL-space. Furthermore, a θ-paracompact space is shown to be an SL-space (for the definition of θ-paracompact spaces, see Section 3). However, a θ-refinable space is not necessarily an SL-space. Moreover, we prove that every F_σ-subset of an SL-space is also an SL-space, while an open subset is not always an SL-space. The class of SL-spaces is not closed by the products though the product space of an SL-space and a σ-compact space is an SL-space.

All topological spaces considered in this paper are Hausdorff spaces.

The author expresses his thanks to Professor W. Moran for suggesting an example which is not θ-paracompact but metacompact (Example 3.9). The author is also indebted to the referee for suggesting improvements on the original paper, particularly, for an example which is not metacompact but θ-paracompact (Example 3.10).

2. Fundamental properties of supports

In general, even if a Borel measure has a non-empty support, it does not imply the existence of the strong support (see Example 2.3). However, we have the following theorem by Gardner (1975), Theorem 3.1. Recall that a Borel measure μ is said to be *locally measure zero* if for each x in X there exists an open neighbourhood U of x with $\mu(U) = 0$.

THEOREM 2.1. *Let X be a topological space; then the following statements are equivalent:*

(1) *every non-zero regular Borel measure has a non-empty support;*

(2) *every non-zero regular Borel measure has a strong support;*

(3) *every regular Borel measure is τ-smooth;*

(4) *every regular Borel measure which is locally measure zero is identically zero.*

For continuous mappings and supports, we have

THEOREM 2.2. *Let X and Y be two topological spaces and f be a continuous mapping of X to Y. Then, for a Borel measure μ on X, the relation $\overline{f(S_\mu)} \subset S_{f(\mu)}$ holds, where $f(\mu)$ is the image measure of μ by f, that is, $f(\mu)(B) = \mu(f^{-1}(B))$ for each Borel set B in $\mathscr{B}(Y)$.*

PROOF. We may assume that S_μ is non-empty. For every x in S_μ and each open neighbourhood V of $f(x)$, we have

$$f(\mu)(V) = \mu(f^{-1}(V)) > 0,$$

which implies $f(S_\mu) \subset S_{f(\mu)}$. Since $S_{f(\mu)}$ is closed, we have $\overline{f(S_\mu)} \subset S_{f(\mu)}$.

The following example shows that $\overline{f(S_\mu)}$ is not equal to $S_{f(\mu)}$ in general.

EXAMPLE 2.3. Let Ω be the first uncountable ordinal and $[0, \Omega]$ be the set of ordinals less than or equal to Ω. We put $[0, \Omega]$ the usual interval topology. We consider the Dieudonné measure μ on $[0, \Omega]$, that is, $\mu(B) = 1$ or 0 according as B does or does not contain an unbounded closed subset of $[0, \Omega]$ for each Borel subset B of $[0, \Omega]$ (see Halmos (1950), Section 52 (10), or Schwartz (1973), p. 45). Hence we have $\mu(\{\Omega\}) = 0$ and $\mu(U) = 1$ for every open set U containing Ω, so that S_μ is equal to $\{\Omega\}$ and μ is not a regular measure. By ν we denote the restriction of μ to $[0, \Omega) = [0, \Omega] - \{\Omega\}$, then ν is a regular measure by Gruenhage and Pfeffer (1978), Example 5, and we can easily show that S_ν is empty. So we have

$$\emptyset = \overline{\iota(S_\nu)} \subsetneq S_{\iota(\nu)} = S_\mu = \{\Omega\},$$

where ι is the natural injection of $[0, \Omega)$ to $[0, \Omega]$.

Nevertheless, for strong supports the equality holds:

THEOREM 2.4 (Rajput and Vakhania (1977), Lemma 1). *Let X and Y be two topological spaces and μ be a Borel measure on X with $S_\mu^* \neq \emptyset$. For a continuous mapping f of X to Y, $S_{f(\mu)}^*$ exists and we have $S_{f(\mu)}^* = \overline{f(S_\mu^*)}$.*

THEOREM 2.5. *Let X be a topological space and A be a subset of X. Then, for a Borel measure μ on A we have $S_{\iota(\mu)} \cap A = S_\mu$, where ι is the natural injection of A into X.*

PROOF. We may assume that S_μ is not empty. For every x in S_μ and every open set U containing x, we have $\iota(\mu)(U) = \mu(U \cap A) > 0$, which implies $S_\mu \subset S_{\iota(\mu)}$.

Conversely, for each x in $S_{\iota(\mu)} \cap A$ and any open neighbourhood U of x, we have $\mu(U \cap A) = \iota(\mu)(U) > 0$, which shows $S_{\iota(\mu)} \cap A \subset S_{\mu}$. This completes the proof.

As to strong supports, we have $S^*_{\iota(\mu)} \cap A = S^*_{\mu}$ by Theorem 2.4.

Let μ be a Borel measure on a topological space X and A be a subset of X. Then there exists a Borel subset B in $\mathscr{B}(X)$ such that $\mu^*(A) = \mu(B)$, where μ^* is the outer measure derived from μ. By μ_B we mean the restriction of μ to the Borel subset B. Since A is μ_B-thick in B, we can consider the restriction $(\mu_B)_A$ of μ_B to A (see Halmos (1950), Section 17, Theorem A). It is easy to prove that the restriction measure $(\mu_B)_A$ is independent of the choice of B satisfying $\mu^*(A) = \mu(B)$, so that we can denote it by μ_A. Remark that $\mu_A(C) = \mu^*(C)$ for every C in $\mathscr{B}(A) = A \cap \mathscr{B}(X)$.

THEOREM 2.6. *Let μ be a regular Borel measure on a topological space X. Then the following statements are equivalent:*

(1) *μ is a τ-smooth measure;*
(2) *for each closed subset F with $\mu_F > 0$, the support S_{μ_F} is non-empty;*
(3) *for each closed subset F with $\mu_F > 0$, the strong support $S^*_{\mu_F}$ exists;*
(4) *for each subset A with $\mu_A > 0$, the support S_{μ_A} is non-empty;*
(5) *for each subset A with $\mu_A > 0$, the strong support S^*_{μ} exists.*

PROOF. If μ is τ-smooth measure, then so is μ_A by Amemiya, Okada and Okazaki (1978), Section 5. Hence (1) implies (5). So it is sufficient to show that (2) implies (1). Let $\{U_\alpha\}$ be an increasing net of open subsets of X such that $\bigcup_\alpha U_\alpha = X$ Suppose $a = \sup \mu(U_\alpha) < \mu(X)$, then we can choose an increasing sequence $\{U_{\alpha_n}\}$ from $\{U_\alpha\}$ such that $\sup_n \mu(U_{\alpha_n}) = a$. If we put $F = (\bigcup_{n=1}^{\infty} U_{\alpha_n})^c$, then we have $\mu_F > 0$, so that S_{μ_F} is not empty. For an element x in S_{μ_F}, there exists a $U_{\alpha(x)}$ containing x, which implies $\mu(F \cap U_{\alpha(x)}) = \mu_F(F \cap U_{\alpha(x)}) > 0$. Then we have

$$a = \mu\left(\left(\bigcup_{n=1}^{\infty} U_{\alpha_n}\right) \cup U_{\alpha(x)}\right) = \mu\left(\bigcup_{n=1}^{\infty} U_{\alpha_n}\right) + \mu(U_{\alpha(x)} \cap F) > \mu\left(\bigcup_{n=1}^{\infty} U_{\alpha_n}\right) = a,$$

which is a contradiction. Therefore we have $\sup_\alpha \mu(U_\alpha) = \mu(X)$, which completes the proof.

REMARK 2.7. (1) For each subset A of X, we have $S_{\mu_A} \subset S_\mu \cap A$.

(2) If A is a μ-thick subset or an open subset, then we have $S_{\mu_A} = S_\mu \cap A$.

(3) We consider the restriction to the support. It holds $S_{(\mu_{S_\mu})} \subset S_\mu$. If S^*_μ exists, then $S^*_{(\mu_{S^*_\mu})}$ exists and we have $S^*_{(\mu_{S^*_\mu})} = S^*_\mu$.

In general S_{μ_A} is not necessarily equal to $S_\mu \cap A$ even if both μ and μ_A are Radon measures and A is a closed G_δ-subset.

EXAMPLE 2.8. Let a, b, c be real numbers such that $a<b<c$. Then there is a non-negative valued continuous function f such that (i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$; (ii) $[a,b] = f^{-1}(0)$, where dx is the Lebesgue measure. If we put $d\mu = f\,dx$, then μ is a Radon measure and the restriction $\mu_{[a,c]}$ is a non-zero Radon measure. It is evident that $a \in S_\mu$ and $a \notin S_{\mu_{[a,c]}}$, which implies $S_{\mu_{[a,c]}} \subsetneq S_\mu \cap [a,c]$.

In general, the equality $S_{(\mu_{S_\mu})} = S_\mu$ does not hold even if $\mu_{S_\mu} > 0$. In fact we have

EXAMPLE 2.9. Let μ be the Dieudonné measure on $[0,\Omega]$ and a be in $[0,\Omega)$. We put $\nu = \delta_a + \mu$, where δ_a is the Dirac measure at a. Then we have $S_\nu = \{a, \Omega\}$ and $\nu_{S_\nu} > 0$. But it follows that $\Omega \notin S_{(\nu_{S_\nu})}$ and $\Omega \in S_\nu$, which implies $S_{(\nu_{S_\nu})} \subsetneq S_\nu$.

PROBLEM 2.10. Let μ be a regular Borel measure on a topological space. Then, is $S_{(\mu_{S_\mu})}$ equal to S_μ?

In a product space, we have the following theorem.

THEOREM 2.11. *Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ be a product space and μ be a Borel measure on X, then we have*

$$S_\mu \subset \prod_{\lambda \in \Lambda} S_{\mu_\lambda},$$

where μ_λ is the image measure of μ by the projection p_λ of X to X_λ.

PROOF. For each $x = (x_\lambda) \in S_\mu$, we have

$$\mu_\lambda(V_\lambda) = \mu(V_\lambda \times \prod_{\lambda' \neq \lambda} X_{\lambda'}) > 0$$

for every neighbourhood V_λ of x_λ in X_λ, which implies $x \in \prod_{\lambda \in \Lambda} S_{\mu_\lambda}$. Therefore we have $S_\mu \subset \prod_{\lambda \in \Lambda} S_{\mu_\lambda}$.

S_μ is not necessarily identical to $\prod_{\lambda \in \Delta} S_{\mu_\lambda}$, in general.

EXAMPLE 2.12. In the space $\mathbf{R}$ of real numbers, we put $d\mu = (1/2\pi)\exp(-x^2/2)\,dx$. There is a non-negative continuous function f on $\mathbf{R}^2$ vanishing outside a compact set such that $\operatorname{supp} f \subsetneq p_1(\operatorname{supp} f) \times p_2(\operatorname{supp} f)$, where p_i is the projection $(i = 1, 2)$. If we put $d\nu = f\,d(\mu \times \mu)$, then it is easily verified that $S_\nu = \operatorname{supp} f$ and $S_{p_i(\nu)} = p_i(\operatorname{supp} f)$ $(i = 1, 2)$. Thus we have

$$S_\nu = \operatorname{supp} f \subsetneq p_1(\operatorname{supp} f) \times p_2(\operatorname{supp} f) = S_{p_1(\nu)} \times S_{p_2(\nu)}.$$

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3. Lindelöf supports

We show that every Borel measure on a metacompact space has a Lindelöf support. A topological space is called metacompact if every open cover has an open point finite refinement. A paracompact space is metacompact. By using the idea of Moran (1970) we have

THEOREM 3.1. *Let X be a metacompact space and μ be a Borel measure on X. Then the support S_μ of μ is a Lindelöf space.*

PROOF. Assume that S_μ is non-empty. Let $\mathscr{U}_0$ be a family of open subsets of X which covers S_μ. Then there is an open point finite refinement $\mathscr{U}_1$ of $\mathscr{U}_0 \cup \{X - S_\mu\}$. We put $\mathscr{U}_2 = \{U \in \mathscr{U}_1;\ U \cap S_\mu \neq \varnothing\}$ and $\mathscr{U}_2^n = \{U \in \mathscr{U}_2;\ \mu(U) \geqslant 1/n\ (n = 1, 2, \ldots)\}$. Then we have $\mathscr{U}_2 = \bigcup_{n=1}^\infty \mathscr{U}_2^n$. Suppose that the cardinal of $\mathscr{U}_2$ is uncountable. Then there exists an n such that the cardinal of $\mathscr{U}_2^n$ is uncountable, so that we can take a countable sequence $\{P_m\}$ in $\mathscr{U}_2^n$. If we put $P = \overline{\lim}_m P_m$, then we have

$$\mu(P) = \lim_m \mu\left(\bigcup_{k=m}^{\infty} P_k\right) \geqslant 1/n,$$

which means that P is not empty. For an element x in P, there is an increasing sequence $\{k_s\}$ such that $x \in \bigcap_{s=1}^\infty P_{k_s}$, which contradicts the point finiteness of $\{P_k\}$. Hence the cardinal of $\mathscr{U}_2$ is countable. For each U in $\mathscr{U}_2$, there is a V_U in $\mathscr{U}_0$ such that $U \subset V_U$, therefore we have

$$S_\mu \subset \bigcup_{U \in \mathscr{U}_2} U \subset \bigcup_{U \in \mathscr{U}_2} V_U,$$

so that S_μ is a Lindelöf space. This completes the proof.

COROLLARY 3.2. *Every Borel measure on a paracompact space has a Lindelöf support.*

COROLLARY 3.3. *Assume that a regular topological space X is not paracompact but metacompact. Then there is no Borel measure of which support is identical to X.*

COROLLARY 3.4 (Rajput and Vakhania (1977), Lemma 2). *Let X be a metric space and μ be a Borel measure on X. The support S_μ is separable. Particularly if S_μ^* exists, then S_μ^* is separable.*

COROLLARY 3.5. *Let X be an inseparable Banach space. Then there is no Borel measure μ such that the linear hull of the support S_μ is equal to X, in particular S_μ is not equal to the unit ball.*

Note that Corollary 3.6 implies Corollary 1 of Ionescu Tulcea (1973).

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Now we introduce a new class of topological spaces. Recall that a family $\{A_\alpha\}$ of subsets is called *locally finite* at x if there exists a neighbourhood V of x such that $\{\alpha;\ A_\alpha \cap V \neq \emptyset\}$ is finite. We call a topological space X *θ-paracompact* if every open cover $\mathscr{U}$ of X has an open refinement $\mathscr{W} = \bigcup_{n=1}^{\infty} \mathscr{W}_n$ satisfying that for every x in X there is an $n(x)$ such that $x \in \bigcup_{U \in \mathscr{W}_{n(x)}} U$ and $\mathscr{W}_{n(x)}$ is locally finite at x.

THEOREM 3.6. *Let μ be a Borel measure on a θ-paracompact space X. Then the support S_μ of μ is a Lindelöf space.*

PROOF. Suppose that S_μ is not empty. Let $\mathscr{U}$ be a family of open subsets of X which covers S_μ; then there is an open refinement $\mathscr{W} = \bigcup_{n=1}^{\infty} \mathscr{W}_n$ of $\mathscr{U}$ which satisfies the condition preceding Theorem 3.6. Put

$$X_n = \{x \in X;\ x \in \bigcup_{U \in \mathscr{W}_n} U,\ \mathscr{W}_n \text{ is locally finite at } x\},$$

then we have $X = \bigcup_{n=1}^{\infty} X_n$. Without difficulty it is shown that X_n is an open subset of X for every n. We put

$$\mathscr{U}_n = \{U \cap X_n;\ U \in \mathscr{W}_n \text{ and } U \cap X_n \cap S_\mu \neq \emptyset\},$$

then we have $\mu(V) > 0$ for every V in $\mathscr{U}_n$. We can show that the cardinality of $\mathscr{U}_n$ is countable for every n from the definition of X_n by the same idea as in the proof of Theorem 3.4. For every $V = U \cap X_n$ in $\mathscr{U}_n$, there is a W_V^n in $\mathscr{U}$ containing U. Thus we have

$$S_\mu = \bigcup_{n=1}^{\infty} (S_\mu \cap X_n) \subset \bigcup_{n=1}^{\infty} \bigcup_{V \in \mathscr{U}_n} V \subset \bigcup_{n=1}^{\infty} \bigcup_{V \in \mathscr{U}_n} W_V^n,$$

which implies that S_μ is a Lindelöf space. This completes the proof.

COROLLARY 3.7. *Let X be a metacompact or θ-paracompact space and μ be a regular Borel measure on X. Then μ is τ-smooth if and only if its strong support S_μ^* exists and S_μ^* is a Lindelöf space.*

REMARK 3.8. (1) The 'if' part of Corollary 3.8 always holds even if X is neither metacompact nor θ-paracompact.

(2) The statement does not hold if we replace S_μ^* with S_μ. In fact, consider a Borel measure $\nu = \mu_{[0,\Omega)} + \delta_a$ on $[0, \Omega)$, where $a \in [0, \Omega)$ and μ is the Dieudonné measure on $[0, \Omega]$. Then S_μ^* does not exist but $S_\mu = \{a\}$. So ν is a regular measure of which support is compact. Nevertheless ν is not a τ-smooth measure since

$$1 = \sup_\alpha \nu([0, \alpha)) < \nu([0, \Omega)) = 2.$$

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The following example is originally given by Bing (1951) and Michael (1955). Moran (1968), IV.2 has treated it again in his thesis and has shown the author that a modification of the proof there gives a proof that G is not θ-paracompact.

EXAMPLE 3.9. Let P be an uncountable set, Q be the power set of P and F be the power set of Q, that is, the set of all two-valued functions on Q. For $p \in P$, we put

$$f_p(A) = \begin{cases} 1 & \text{if } p \in A, \\ 0 & \text{if } p \notin A \end{cases}$$

for each A in Q and put $K = \{f_p; p \in P\}$. We define a topology on F as follows: $\{f\}$ is a neighbourhood of f if $f \in F - K$; for $f_p \in K$ and a finite subset Λ of Q, $N(f_p; \Lambda) = \{f \in F; f(A) = f_p(A)$ for each $A \in \Lambda\}$ is a neighbourhood of f_p. Let G be the union of K and $\{f \in F; f(A) = 0$ for all except finitely many $A \in Q\}$. Then G is a normal, countably paracompact, metacompact space. But G is not θ-paracompact.

EXAMPLE 3.10. Consider 'pointed extension' of the real number field (69 in Steen and Seebach, Jr. (1970)). Let X be the set of real numbers and Q be the set of rationals. We define a topology on X generated by all sets $\{x\} \cup (Q \cap U)$, where $x \in U$, U is open for the Euclidean topology. Then X is a θ-paracompact Hausdorff space. But X is not a metacompact space. This result is quoted by the referee.

4. SL-spaces

We define SL-spaces.

DEFINITION 4.1. A topological space X is called an SL-*space* if every Borel measure μ on X has a Lindelöf support S_μ (S_μ may be empty).

From Section 3 we have

THEOREM 4.2. *A metacompact or θ-paracompact space is an* SL-*space.*

The class of θ-refinable spaces is introduced by Worrell, Jr. and Wicke (1965). Recall that a topological space is said to be *θ-refinable* if each open cover of X has an open refinement $\mathscr{U} = \bigcup_{n=1}^{\infty} \mathscr{U}_n$ such that every $\mathscr{U}_n$ covers X and for each x in X there exists an $n(x)$ satisfying that $\mathscr{U}_{n(x)}$ is point finite at x, that is, the cardinal of $\{U \in \mathscr{U}_{n(x)}; x \in U\}$ is finite. From the definition a metacompact or θ-paracompact space is θ-refinable.

Now we present a θ-refinable space which is not an SL-space. Let S be the Sorgenfrey line, that is, the real line with the right half-open interval topology. S is Hausdorff and hereditarily Lindelöf, particularly paracompact. The product space $X = S \times S$ is separable since the set of rational numbers is dense in X and X is not Lindelöf (see Steen and Seebach Jr. (1970), 51 and 84). But X is θ-refinable by Burke (1970), Theorem 1.6 and Lutzer (1972), Proposition 3.1. As the following example shows, X is not an SL-space, that is to say, a θ-refinable space is not always an SL-space.

EXAMPLE 4.3. (1) Since X is separable, there exists a countable dense subset $\{x_n\}$. If we put $\mu = \sum_{n=1}^{\infty} 1/2^n \, \delta_{x_n}$, then the support S_μ of μ is equal to X. Since X is not a Lindelöf space, X is not an SL-space.

(2) All Borel subsets $\mathscr{B}(X)$ of X are Lebesgue measurable by Vitali's covering theorem (for example, see Saks (1937), Chap. 4 (3.1)). Let ν be the restriction of the Lebesgue measure to $\mathscr{B}(X)$. Since ν is still σ-finite, there exists a totally finite Borel measure ν_1 such that ν is absolutely continuous with respect to ν_1 and ν_1 is absolutely continuous with respect to ν. So the support S_{ν_1} is identical to X, which also shows that X is not an SL-space.

A closed subset of an SL-space is also an SL-space. In general, we have

THEOREM 4.4. *Let X be an* SL-*space. Then every F_σ-subset L of X is also an* SL-*space.*

PROOF. We can write $L = \bigcup_{n=1}^{\infty} F_n$, where F_n is a closed subset of X for every n. Let μ be a Borel measure with a non-empty support S_μ. Then

$$S_\mu = S_{\iota(\mu)} \cap L = \bigcup_{n=1}^{\infty} (S_{\iota(\mu)} \cap F_n)$$

by Theorem 2.6. Since $S_{\iota(\mu)} \cap F_n$ is a Lindelöf space for every n, so is S_μ, which completes the proof.

An open subset of an SL-space is not necessarily an SL-space.

EXAMPLE 4.5. Let X be the real line with the rational sequence topology (see Steen and Seebach, Jr. (1970), 65). Then X is a locally compact separable space, but X is not Lindelöf. Let $\check{X}$ be the one-point compactification of X; then $\check{X}$ is an SL-space since $\check{X}$ is compact. X is an open subset of $\check{X}$ but X is not an SL-space as (1) in Example 4.3.

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Next we treat the products of SL-spaces. Note that the class of SL-spaces is not closed for the products. In fact, although the Sorgenfrey line S is an SL-space, the product $S \times S$ is not an SL-space as we have shown in Example 4.3. Still we have

THEOREM 4.6. *Let X be an SL-space and Y be a σ-compact space; then the product space $X \times Y$ is also an SL-space.*

PROOF. Let μ be a Borel measure on $X \times Y$ of which support S_μ is not empty. By Theorem 2.11 we have

$$S_\mu \subset S_{p_X(\mu)} \times S_{p_Y(\mu)},$$

where p_X (resp. p_Y) is the projection to X (resp. Y). From the assumption, $S_{p_X(\mu)}$ is Lindelöf and $S_{p_Y(\mu)}$ is σ-compact. Then $S_{p_X(\mu)} \times S_{p_Y(\mu)}$ is a Lindelöf space since the product of a compact space and a Lindelöf space is also Lindelöf in general. Thus S_μ is a Lindelöf space, which completes the proof.

Finally we show that SL-spaces are not transferred by a continuous map or an open map.

EXAMPLE 4.7. (1) We take a topological space Y which is not an SL-space and let X be a topological space which is equal to Y as a set and has the discrete topology. Then the identity map of X to Y is a continuous bijection. Y is not an *SL*-space though X is an *SL*-space.

(2) Let X be the Euclidean plane and Y be the product of the Sorgenfrey lines. Then the identity map of X to Y is an open bijective map. Y is not an SL-space, whereas X is an SL-space.

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Department of Mathematics I.A.S.
Australian National University
Canberra A.C.T. 2600
Australia

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