8

Heisenberg and Carnot Groups

For an introduction to Heisenberg and Carnot groups, see, for example, [87] or [71]. Nice surveys related to rectifiability are given by Serapioni [395] and by Serra Cassano [396]. Both Euclidean spaces and Heisenberg groups are special cases of Carnot groups. Except for the Euclidean case, they are non-Abelian and have a structure similar to Heisenberg groups, but instead of two levels there can be any finite number of levels with different dilation exponents. I do not discuss them explicitly, but I make some comments about them along the way.

8.1 The Heisenberg Group $\mathbb{H}^n$

Heisenberg group $\mathbb{H}^n$ is $\mathbb{R}^{2n+1}$ as a set but with a different metric and non-Abelian group structure. We denote the points of $\mathbb{H}^n$ by $p = (z, t) = (x, y, t)$, $z = (x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $t \in \mathbb{R}$, and define the non-Abelian group operation by

$$p \cdot p' = (z + z', t + t' + \omega(p, p')),$$

where

$$\omega(p, p') = \omega(z, z') = -2\sum_{i=1}^{n}(x_i y_i' - y_i x_i').$$

We shall use the Koranyi metric d given by

$$d(p, p') = \left(|z - z'|^4 + (t - t' - \omega(z, z'))^2\right)^{\frac{1}{4}}. \tag{8.1}$$

Then the ball of radius r at the origin

$$B(0, r) = \left\{p\colon (|z|^4 + t^2)^{\frac{1}{4}} \le r\right\}$$

is like a cylinder of width $2r$ and height $2r^2$, so $\mathcal{L}^{2n+1}(B(0, r)) \sim r^{2n+2}$. The ball $B(p, r)$ is the image of $B(0, r)$ under the left translation τ_p; $\tau_p(q) = p \cdot q$. The metric is left invariant, that is, τ_p is an isometry. Moreover, $\mathcal{L}^{2n+1} = c(n)\mathcal{H}_d^{2n+2}$ is a Haar measure of the group, so $\mathcal{L}^{2n+1}(B(p,r)) = c'(n)r^{2n+2}$. In particular, we see that the Hausdorff dimension of $\mathbb{H}^n$ is $2n+2$. We also define the dilations $\delta_r, r > 0$, by

$$\delta_r(z,t) = \left(rz, r^2 t\right).$$

Then $d(\delta_r(p), \delta_r(q)) = rd(p,q)$.

In $\mathbb{H}^n$ the metric on all *horizontal lines* $L_e := \{te, 0) \colon t \in \mathbb{R}\}, e \in S^{2n-1}$, and their left translates is Euclidean, so their subsets are 1-rectifiable. There are many other 1-rectifiable sets; any two points of $\mathbb{H}^n$ can be joined with a rectifiable curve. This fact also leads to the geodesic metric which is equivalent to the one we have chosen.

When $n > 1$, there are many m-rectifiable sets for $m = 1, \ldots, n$ in the same way. Define the horizontal plane $\mathbb{H} = \{t = 0\}$ identified with $\mathbb{R}^{2n}$ and define the space of *horizontal subgroups* $V \subset \mathbb{H}$:

$$G_h(2n,m) = \{V \in G(2n,m) \colon \omega(p,q) = 0 \text{ for all } p,q \in V\}.$$

The metric on each $V \in G_h(2n,m)$ is Euclidean, so they are nice m-rectifiable sets. The unitary transformations act transitively on $G_h(2n,m)$ and lead to an invariant measure $\mu_{n,m}$. Not all linear subspaces of $\mathbb{H}$ of dimension $1 < m \le n$ are subgroups, and no linear subspaces of dimension bigger than n are subgroups.

The *vertical subgroups* of linear dimension $m-1, 1 \le m-1 \le 2n$ and Hausdorff dimension m are the vertical planes $V \times \mathbb{R}, V \in G(2n, m-2)$. They, together with the horizontal subgroups, are the only non-trivial homogeneous subgroups of $\mathbb{H}^n$, that is, they are closed and invariant under the dilations. But the narrower collection of complementary subgroups will be more relevant for us.

The subgroups $V \in G_h(2n,m)$ and $W = V^\perp$ are complementary: $V \cap W = \{0\}$, and they span $\mathbb{H}^n$ in the sense that $V \cdot W = \mathbb{H}^n$. In particular, any $p \in \mathbb{H}^n$ has a unique decomposition

$$p = p_V \cdot p_W, p_V \in V, p_W \in W. \tag{8.2}$$

Define $\mathcal{G}(\mathbb{H}^n, m)$ as the set of $V \in G_h(2n,m)$, when $1 \le m \le n$, and $V \times \mathbb{R}, V \in G(2n, m-2), V^\perp \in G_h(2n, 2n+2-m)$, when $n+2 \le m \le 2n+1$. They are the homogeneous subgroups of Hausdorff dimension m which admit a complement in the sense of (8.2).

Also the vertical subgroups in $\mathcal{G}(\mathbb{H}^n, m)$ will be m-rectifiable, but we have to change the definition, as we shall soon do.

8.2 Some Analytic Tools in Heisenberg and Carnot Groups

The analytic structure of $\mathbb{H}^n$ is generated by the following vector fields:

$$X_i = \partial_{x_i} + 2y_i\partial_t, Y_i = \partial_{y_i} - 2x_i\partial_t, i = 1, \dots n, \nabla_H = (X_1, \dots, X_n, Y_1, \dots, Y_n).$$

These vector fields at the origin span the horizontal plane $\mathbb{H}$ and at p the plane $\tau_p(\mathbb{H})$. The tangent vectors of rectifiable curves are spanned by them. We say that a continuous function u on an open subset U of $\mathbb{H}^n$ belongs to C^1_H if $\nabla_H u$ is continuous.

There is a very general Rademacher theorem due to Pansu [379]. It says that Lipschitz maps f between Carnot groups are almost everywhere differentiable. Now the derivative $d_H f(p)$ is a homogeneous (it commutes with the dilations) homomorphism between the groups such that

$$\lim_{q \to p} \frac{d\left(f(p)^{-1} \cdot f(q), d_H f(p)(p^{-1} \cdot q)\right)}{d(p,q)} = 0.$$

Franchi, Serapioni and Serra Cassano proved an implicit function theorem for real-valued functions on subsets of $\mathbb{H}^n$ and a Whitney extension theorem for Euclidean-valued (or values in the horizontal subbundle) functions on subsets of $\mathbb{H}^n$, see [212, 213]. But a Whitney extension theorem for $\mathbb{H}^n$ valued functions is missing, and hence also approximation of Lipschitz maps with differentiable maps.

There also are several versions of the area and coarea formulas, see [24, 268, 430] and the references given there.

8.3 Definitions of Rectifiability

There are several natural ways to define rectifiable sets in Heisenberg and Carnot groups. Some of them are known to be equivalent in some cases, for some the relations are unknown. We begin with the definition we have already used in metric spaces:

Definition 8.1 Let $1 \leq m \leq n$. A set $E \subset \mathbb{H}^n$ is m-rectifiable if there are Lipschitz maps $f_i \colon A_i \to \mathbb{H}^n, A_i \subset \mathbb{R}^m, i = 1, 2, \dots$ such that

$$\mathcal{H}^m_d\left(E \setminus \bigcup_{i=1}^{\infty} f_i(A_i)\right) = 0.$$

The corresponding notion for measures is defined as in 4.2. I only gave this definition for low-dimensional sets for a good reason: there are no non-trivial m-rectifiable subsets of $\mathbb{H}^n$ for $m > n$ by the following result. In $\mathbb{H}^1$ it was proved by Ambrosio and Kirchheim in [16]. A very general statement in Carnot groups is due to Magnani [300].

Theorem 8.2 *If* $m = n+1, \ldots, 2n+2$ *and* $f\colon A \to \mathbb{H}^n, A \subset \mathbb{R}^m$ *is Lipschitz, then* $\mathcal{H}_d^m(f(A)) = 0$*. In particular,* $\mathbb{H}^n$ *is purely* $(2n+2)$*-unrectifiable.*

To get an idea why this is true, suppose that $n = 1$ and $f = (f_1, f_2)$ maps $\mathbb{R}^3$ into the vertical plane $W = \{y = 0\}$. The metric in W is given by the 'norm' $(|x|^4 + t^2)^{1/4}$. Then $|f_2(u) - f_2(v)| \lesssim |u - v|^2$, so f_2 is constant and $\mathcal{H}_d^3(f(\mathbb{R}^3)) = 0$ follows. Of course, f need not map into a vertical plane, but its Pansu differentials $d_H f(u)$ must because they are group homomorphisms and thus $d_H f(u)(\mathbb{R}^3)$ is an Abelian subgroup. From this one can argue similarly.

Are there any non-trivial m-rectifiable subsets when $m > n$? The answer is no with our present definition, as we saw above. But there are many with an alternate definition. Recall that Euclidean rectifiable sets can be defined using level sets of regular functions. Based on this we first define *regular surfaces* S. If $1 \le m \le n$, this means that S is locally the image of an open subset of $\mathbb{R}^m$ under an injective continuously differentiable (in Pansu's sense) map with injective derivative. If $n+2 \le m \le 2n+1$, we say that S is regular if for every $p \in S$ there are an open set U with $p \in U$ and a function $u\colon U \to \mathbb{R}^{2n+2-m}$, whose coordinate functions belong to C_H^1 such that $S \cap U = \{q \in U\colon u(q) = 0\}$ and for $q \in U$ the Pansu derivative $d_H u(q)$ is surjective. These surfaces are also regular in the sense that they have tangent subgroups, in the first case via Pansu derivative and in the second via the kernel of $d_H f(p)$. Notice that when $m \ge n+2$, the topological dimension of S is $m-1$.

Definition 8.3 Let $m = 1, \ldots, n, n+2, \ldots, 2n+1$. A set $E \subset \mathbb{H}^n$ is $(m, \mathbb{H})$-rectifiable if there are m-regular surfaces $S_i, i = 1, 2, \ldots$ such that $\mathcal{H}_d^m(E \setminus \bigcup_{i=1}^\infty S_i) = 0$.

This definition in codimension 1 is due to Franchi, Serapioni and Serra Cassano [212]. They introduced this concept and used it to develop De Giorgi's theory of sets of finite perimeter in Heisenberg groups. We shall return to this in Section 12.4. For general dimensions and Carnot groups, see [215, 301].

As an example, consider the vertical plane $W = \{y = 0\} \subset \mathbb{H}^1$. Then $u(x, y, t) = y$ belongs to C_H^1 with $Y_1 u \neq 0$ on W. Hence W is a regular surface and a $(3, \mathbb{H})$-rectifiable set. As another example, the horizontal plane $\mathbb{H}$ in $\mathbb{H}^n$ is not a regular surface because it has a singularity, a characteristic point, at 0. But it is regular outside 0, and hence a $(2n+1, \mathbb{H})$-rectifiable set.

In fact, all C^1 smooth Euclidean m-dimensional, $n+1 \le m \le 2n$, surfaces are $(m+1, \mathbb{H})$-rectifiable. They have positive and locally finite $\mathcal{H}_d^{m+1}$ measure, and they are regular outside the set of characteristic points, which has $\mathcal{H}_d^{m+1}$ measure zero by results of Balogh [53] for $m = 2n$ and Magnani [301] for general m. A point p is a characteristic point of a hypersurface S if the tangent space (in Heisenberg sense) at p is spanned by the horizontal vector fields X_i, Y_j. More generally, the Euclidean m-rectifiable sets, $n + 1 \le m \le 2n$, are $(m+1, \mathbb{H})$-rectifiable since $\mathcal{H}^m(A) = 0$ implies $\mathcal{H}_d^{m+1}(A) = 0$ for $A \subset \mathbb{H}^n$. The converse is false because $(2n+1, \mathbb{H})$-rectifiable sets can have Euclidean Hausdorff dimension bigger than $2n$, see [278] for an example in $\mathbb{H}^1$ of Hausdorff dimension 2.5. General comparisons of Euclidean and Carnot Hausdorff measures can be found in [55].

When $1 \le m \le n$, clearly $(m, \mathbb{H})$-rectifiable sets are m-rectifiable, but it is not known if the converse holds. Although Lipschitz maps are almost everywhere differentiable by Pansu's theorem, one would need something like Whitney's extension theorem to go to C_H^1 from Lipschitz. This is not known for Heisenberg-valued maps.

Intrinsic differentiable graphs and intrinsic Lipschitz graphs have recently been investigated intensively. They provide another definition for rectifiability. In Euclidean spaces, cones were used to characterize rectifiability in terms of the approximate tangent planes, and they are directly connected to Lipschitz maps as in the argument preceding Theorem 3.3. In Heisenberg groups, the situation is more complicated but we can define a class of Lipschitz maps geometrically in terms of cones. This was done by Franchi, Serapioni and Serra Cassano in [214] in Heisenberg groups and in [211] in general Carnot groups.

For a homogeneous subgroup G, define the cone

$$\begin{aligned} X(p, G, s) &= \left\{ q \in \mathbb{H}^n : d(p^{-1} \cdot q, G) < s d(p, q) \right\} \\ &= p \left\{ q \in \mathbb{H}^n : d(q, G) < s d(q, 0) \right\}. \end{aligned} \tag{8.3}$$

Geometrically these cones look rather different from the Euclidean cones.

Let V and W be complementary subgroups of $\mathbb{H}^n$; $V \cap W = \{0\}$ and $V \cdot W = \mathbb{H}^n$, with V horizontal and W vertical. For much that follows, they could also be complementary homogeneous subgroups of a general Carnot group. We say that $S \subset \mathbb{H}^n$ is a (vertical) *intrinsic Lipschitz graph* if there is $s > 0$ such that for all $p \in S$,

$$S \cap X(p, V, s) = \emptyset.$$

We say that a function $f : A \to V, A \subset W$ is (vertical) *intrinsic Lipschitz* if $gr(f) := \{p \cdot f(p) : p \in A\}$ is an intrinsic Lipschitz graph. In Euclidean spaces, this just means that f is Lipschitz. But now the intrinsic Lipschitz functions need not be Lipschitz in the metric sense, and vice versa, see [27, Example 3.21].

Changing the roles of V and W, we get horizontal intrinsic Lipschitz graphs and functions. Arena and Serapioni proved in [27, Proposition 3.20] that a horizontal function f is intrinsic Lipschitz if and only if $p \mapsto p \cdot f(p)$ is metric Lipschitz.

Intrinsic Lipschitz graphs over the whole planes have positive and locally finite Hausdorff measure (in the appropriate dimension); they are even AD-regular. This holds in general Carnot groups, see [211, Theorem 3.9].

Definition 8.4 Let $m = 1, \dots, n, n+2, \dots, 2n+1$. A set $E \subset \mathbb{H}^n$ is $(m, \mathbb{H}_{intL})$-rectifiable if there are m-dimensional (in terms of Hausdorff dimension) intrinsic Lipschitz graphs $S_i, i = 1, 2, \dots$ such that $\mathcal{H}^m_d(E \setminus \bigcup_{i=1}^{\infty} S_i) = 0$.

There is a slightly more complicated notion of intrinsic differentiable functions and graphs, see [27, 214]. Arena and Serapioni proved in [27, Theorem 4.2] that regular surfaces and intrinsic differentiable graphs are the same locally. So the rectifiable sets defined in terms of intrinsic differentiable graphs are the same as $(m, \mathbb{H})$-rectifiable sets. To get to intrinsic Lipschitz, one would need a Rademacher-type theorem for intrinsic Lipschitz functions. Such a theorem was proved by Franchi, Serapioni and Serra Cassano [216] in the codimension 1 case ($m = 2n + 1$) in Heisenberg groups. For this they used their results on sets of finite perimeter. Franchi, Marchi and Serapioni [210] extended this to some Carnot groups. In [430], Vittone proved that intrinsic Lipschitz functions $f: A \to V, A \subset W, W$ vertical, are almost everywhere intrinsic differentiable. From this using Whitney-type arguments he further showed that intrinsic Lipschitz graphs can be approximated in measure by regular surfaces. This leads to part (2) of the following theorem, see [430, Corollary 7.4]. Part (1) follows from the above-mentioned result [27, Proposition 3.20] together with the fact the m-rectifiable sets have approximate tangent subgroups (Theorem 8.6) and positive lower density (Theorem 7.7).

Theorem 8.5 *Let $E \subset \mathbb{H}^n$ be $\mathcal{H}^m_d$ measurable with $\mathcal{H}^m_d(E) < \infty$. Then*

(1) *If $1 \leq m \leq n$, E is $(m, \mathbb{H}_{intL})$-rectifiable if and only if it is m-rectifiable.*
(2) *If $n + 2 \leq m \leq 2n + 1$, E is $(m, \mathbb{H}_{intL})$-rectifiable if and only if it is $(m, \mathbb{H})$-rectifiable.*

There are several ingredients of independent interest in Vittone's proof. He introduced an alternative equivalent definition. According to that, S is a vertical intrinsic Lipschitz W-graph if and only if there exist $\delta > 0$ and a Lipschitz map $g: \mathbb{H}^n \to W^{\perp}$ such that $g(x) = 0$ on S and g satisfies the ellipticity-type condition $(g(p \cdot v) - g(p)) \cdot v \geq \delta |v|^2$ for $v \in W^{\perp}, p \in \mathbb{H}^n$ (the second $\cdot$ is the inner product). In the main part of the argument, he used currents (in a

Heisenberg sense) to show that the blow-ups of S converge to a vertical plane locally uniformly.

Counterexamples to Rademacher's theorem for intrinsic Lipschitz graphs in some Carnot groups were given by Julia, Nicolussi Golo and Vittone in [269]. In [270], they proved the almost everywhere tangential differentiability of Euclidean-valued functions on $C^1_{\mathbb{H}}$ submanifolds of $\mathbb{H}^n$ yielding on $\mathbb{H}$-rectifiable sets a Lusin-type approximation of Lipschitz functions and a coarea formula.

As mentioned previously, there are no Lipschitz maps from Euclidean spaces to parametrize higher-dimensional non-trivial sets. But perhaps one could consider Lipschitz maps from other spaces. For instance, in $\mathbb{H}^1$ could three-dimensional rectifiable sets be defined using Lipschitz maps from a fixed vertical plane? Although an intrinsic Lipschitz map need not be metric Lipschitz, maybe some other Lipschitz map could be used to parametrize an intrinsic Lipschitz graph? Such a definition of rectifiability and its consequences, in the generality of Carnot groups, was studied by Pauls [380] and by Cole and Pauls [121]. Let (G_1, d_1) and (G_2, d_2) be Carnot groups and G a subgroup of G_1 with Hausdorff dimension m. Let us say that $E \subset G_2$ is *G-rectifiable* if up to $\mathcal{H}^m_{d_2}$ measure zero it can be covered with countably many Lipschitz images of subsets of G. Except for the cases where G_1 is Euclidean, not much is known about the relation of G-rectifiability to other concepts of rectifiability, but some partial information exists.

Fix a vertical subgroup $W_n \in \mathcal{G}(\mathbb{H}^n, 2n+1)$. It does not matter which – they all are isometric. Recall that C^1 Euclidean hypersurfaces in $\mathbb{H}^n$ are $(2n+1, \mathbb{H})$-rectifiable. Cole and Pauls proved in [121] that the C^1 hypersurfaces in $\mathbb{H}^1$ are W_1-rectifiable too. Di Donato, Fässler and Orponen [177] proved that the $C^{1,\alpha}, \alpha > 0$, hypersurfaces in $\mathbb{H}^n$ are W_n-rectifiable; moreover, they have big pieces of bi-Lipschitz images of W_n, recall Section 5.2. Earlier the rectifiability was proved by Antonelli and Le Donne [22] for C^∞ surfaces.

Antonelli and Le Donne showed in [22] that there exists a Carnot group containing a C^∞ hypersurface without characteristic points which is not G-rectifiable for any Carnot group G, but it is rectifiable in the sense of Franchi, Serapioni and Serra Cassano.

The papers [177] and [22] show more than rectifiability. Instead of Lipschitz maps, they use bi-Lipschitz maps. Antonelli and Le Donne discuss more generally definitions of rectifiability based on bi-Lipschitz maps. Bigolin and Vittone [65] give a counterexample concerning bi-Lipschitz parametrizations. Orponen [373] shows that bi-Lipschitz images of W_1 in $\mathbb{H}^1$ admit a corona decomposition by intrinsic bi-Lipschitz graphs.

In [286], Le Donne and Young proved that a sub-Riemannian manifold that has a Carnot group G as a constant Gromov–Hausdorff limit, see Section 7.7, is G-rectifiable with bi-Lipschitz parametrizations. The converse also holds by Pansu's Rademacher theorem.

A possibly weaker notion of rectifiability with cones was proposed by Don, Le Donne, Moisala and Vittone in [179]. They applied it to finite perimeter sets in general Carnot groups.

Julia, Nicolussi Golo and Vittone [268] investigated a very general notion of rectifiability covering with level sets of C^1_H maps between Carnot groups.

For the $\mathcal{P}$-rectifiability of Antonelli and Merlo, see Section 8.5.

8.4 Rectifiable Sets and Tangent Subgroups

Next we shall give a characterization of rectifiability in terms of approximate tangent subgroups and tangent measures. They are defined as in the Euclidean case but using the Heisenberg structure. Recall the definition of the cone $X(p, G, s)$ from (8.3).

We say that $G \in \mathcal{G}(\mathbb{H}^n, m)$ is an *approximate tangent subgroup* of $E \subset \mathbb{H}^n$ at a point $p \in \mathbb{H}^n$ if $\Theta^{*m}(E, p) > 0$ and for all $0 < s < 1$,

$$\lim_{r\to 0} r^{-m}\mathcal{H}^m_d(E \cap B(p, r) \setminus X(p, G, s)) = 0.$$

To define the tangent measures, we now set

$$T_{a,r}(p) = \delta_{1/r}(a^{-1} \cdot p),\ \ p, a \in \mathbb{H}^n, r > 0.$$

Then, as before, if μ is a Radon measure on $\mathbb{H}^n$, a non-zero Radon measure ν is called a *tangent measure* of μ at $a \in \mathbb{H}^n$ if there are sequences (c_i) and (r_i) of positive numbers such that $r_i \to 0$ and $c_i T_{a,r_i\#}\mu \to \nu$ weakly. We denote the set of tangent measures of μ at a by $\mathrm{Tan}(\mu, a)$. The following theorems were proven in [330]:

Theorem 8.6 *Let $1 \le m \le n$ and let $E \subset \mathbb{H}^n$ be $\mathcal{H}^m_d$ measurable with $\mathcal{H}^m_d(E) < \infty$. Then the following are equivalent:*

(1) *E is m-rectifiable.*
(2) *E has an approximate tangent subgroup $G_p \in \mathcal{G}(\mathbb{H}^n, m)$ at $\mathcal{H}^m_d$ almost all $p \in E$.*
(3) *For $\mathcal{H}^m_d$ almost all $p \in E$ there is $G_p \in \mathcal{G}(\mathbb{H}^n, m)$ such that*

$$\mathrm{Tan}\left(\mathcal{H}^m_d \llcorner E, p\right) = \left\{c\mathcal{H}^m_d \llcorner G_p \colon 0 < c < \infty\right\}.$$

(4) *For $\mathcal{H}^m_d$ almost all $p \in E$ $\mathcal{H}^m_d \llcorner E$ has a unique tangent measure at p.*

Theorem 8.7 *Let $n+2 \leq m \leq 2n+1$ and let $E \subset \mathbb{H}^n$ be $\mathcal{H}_d^m$ measurable with $\mathcal{H}_d^m(E) < \infty$. Suppose also that $\Theta_*^m(E,p) > 0$ for $\mathcal{H}_d^m$ almost all $p \in E$. Then the following are equivalent:*

(1) *E is $(m, \mathbb{H})$-rectifiable.*
(2) *E has an approximate tangent subgroup $G_p \in \mathcal{G}(\mathbb{H}^n, m)$ at $\mathcal{H}_d^m$ almost all $p \in E$.*
(3) *For $\mathcal{H}_d^m$ almost all $p \in E$ there is $G_p \in \mathcal{G}(\mathbb{H}^n, m)$ such that*

$$\operatorname{Tan}\left(\mathcal{H}_d^m \llcorner E, p\right) = \left\{c\mathcal{H}_d^m \llcorner G_p \colon 0 < c < \infty\right\}.$$

(4) *For $\mathcal{H}_d^m$ almost all $p \in E$ $\mathcal{H}_d^m \llcorner E$ has a unique tangent measure at p.*

That (4) implies (3) follows from Theorem 4.7. There the uniqueness means uniqueness up to multiplication by positive constants. By results of Antonelli and Merlo, see [25, Theorem 1.1], the assumption on positive lower density can be relaxed; it is only needed to derive from (2) the other conditions. Probably it is not really needed there either, but this seems to be unknown.

It is not known if we can replace m-rectifiable by $(m, \mathbb{H})$-rectifiable in Theorem 8.6(1). As mentioned before, the problem here is the lack of Whitney's extension theorem for $\mathbb{H}^n$ valued functions.

A few words about the proofs. The case $m \leq n$ is proved by arguments similar to those used in the Euclidean case. But the cones now are harder to deal with if the lower density is 0. This is overcome by proving first that the existence of tangent subgroups implies positive lower density. When $m \geq n+2$, the implication (1) $\Longrightarrow$ (2) follows from [215]. The key for proving that approximate tangents imply rectifiability is the Whitney extension theorem for $\mathbb{R}^k$ valued maps of Franchi, Serapioni and Serra Cassano [212].

Theorem 8.6 was generalized by Idu, Magnani and Maiale [248] to general homogeneous groups, which need not be Carnot groups. For the extensions by Antonelli and Merlo of both theorems to general Carnot groups, see the discussion at the end of the next section.

8.5 Densities and Tangent Measures

For low-dimensional sets ($m \leq n$) we can apply Kirchheim's Theorem 7.7 to conclude that m-rectifiable subsets of $\mathbb{H}^n$ with finite measure have density 1 almost everywhere. The converse is not known, except for $m = 1$, see Theorem 8.12.

Recently there has been remarkable progress on densities by Merlo in two papers [342, 343]. He proved Preiss's density characterization of rectifiability for codimension 1 subsets of $\mathbb{H}^n$, recall Theorems 4.10 and 4.11:

Theorem 8.8 *Let $\mu \in \mathcal{M}(\mathbb{H}^n)$ be such that the positive and finite limit $\lim_{r\to 0} r^{-2n-1}\mu(B(p,r))$ exists for μ almost all $p \in \mathbb{H}^n$. Then μ is $(2n+1,\mathbb{H})$-rectifiable.*

As a corollary, we have

Theorem 8.9 *Let $E \subset \mathbb{H}^n$ be $\mathcal{H}_d^{2n+1}$ measurable with $\mathcal{H}_d^{2n+1}(E) < \infty$. Then E is $(2n+1,\mathbb{H})$-rectifiable if and only if $\Theta^{2n+1}(E,p)$ exists for $\mathcal{H}_d^{2n+1}$ almost all $p \in E$.*

The existence of density essentially follows from [215], and in much more general settings from the results of Julia, Nicolussi Golo and Vittone [268, Corollary 3.6] and Antonelli and Merlo [24, Theorem 1.3].

Often the density of high-dimensional Heisenberg rectifiable sets is strictly less than 1, but for many metrics the density of the spherical Hausdorff measure is 1 almost everywhere, while for some others it also is strictly less than 1, see [302].

As in the Euclidean case, the proof of Theorem 8.8 splits into two parts. In the first part [343], Merlo proves that the tangent measures are flat:

Theorem 8.10 *Let $\mu \in \mathcal{M}(\mathbb{H}^n)$ be such that the positive and finite limit $\lim_{r\to 0} r^{-2n-1}\mu(B(p,r))$ exists for μ almost all $p \in \mathbb{H}^n$. Then for μ almost all $p \in \mathbb{H}^n$,*

$$\mathrm{Tan}(\mu,p) \subset \left\{ c\mathcal{H}_d^{2n+1} \llcorner V : V \in \mathcal{G}(\mathbb{H}^n, 2n+1), 0 < c < \infty \right\}.$$

Again an easy argument shows that the tangent measures ν are uniform;

$$\nu(B(p,r)) = cr^{2n+1} \text{ for } p \in \operatorname{spt}\nu,\ r > 0.$$

Using moments in the spirit of [382] and [281], Merlo proves that their supports are contained in quadratic conical surfaces. Then disconnectedness is verified by saying that vertical flat uniform measures are separated from the others. As in the Euclidean case, this implies that typically only vertical flat uniform tangent measures exist. Although many of the arguments run as in the Euclidean case, many essential changes have to be made, in particular, replacing algebraic computations by geometric arguments. I find it surprising that Preiss's proof can be followed at all, since it heavily used the fact that the metric is given by an inner product. Merlo overcomes this by cleverly applying the polarization

$$V(p,q) = \left(\|p\|^4 + \|q\|^4 - \|p^{-1}\cdot q\|^4 \right)/2.$$

It can be written, with a rather complicated formula, in terms of the inner product in $\mathbb{R}^{2n}$ involving for $p = (w, s), q = (z, t)$ all the coordinates w, z, s, t in various combinations.

In the second paper [342], Merlo proves that the flatness of the tangent measures implies rectifiability. This is the case $m = 2n + 1$ in the following theorem:

Theorem 8.11 *Let $\mu \in \mathcal{M}(\mathbb{H}^n)$ be such that for μ almost all $p \in \mathbb{H}^n$, $0 < \Theta^m_*(\mu, p) \le \Theta^{*m}(\mu, p) < \infty$ and*

$$\operatorname{Tan}(\mu, p) \subset \left\{ c\mathcal{H}^m_d \llcorner V : V \in \mathcal{G}(\mathbb{H}^n, m), 0 < c < \infty \right\}.$$

Then μ is $(2n + 1, \mathbb{H})$-rectifiable, if $m = 2n + 1$, and m-rectifiable, if $1 \le m \le n$.

The proof for $m = 2n + 1$ is very complicated. It is given in general Carnot groups. As for Theorem 4.9 one of the ideas is to use big projections implied, but not easily, by the assumptions. For this, some ideas of David and Semmes [147] in uniform rectifiability help. But now projections mean the splitting projections $p \mapsto p_W$, recall (8.2), which are much more complicated to handle than the Euclidean projections. They are not even Lipschitz. Big projection on W improves the information given by the assumption: the approximating planes have to be rather close to W. Then one can proceed to show that some intrinsic Lipschitz graph intersects $\operatorname{spt}\mu$ in positive measure.

The proof for low dimensions was given by Antonelli and Merlo in [24]. It follows similar patterns, but now the projections are Lipschitz, which helps. On the other hand, serious difficulties are caused by the fact that they prove this in quite general Carnot settings (see below); it is assumed that the tangent subgroups admit at least one normal complementary subgroup.

There is also interesting information about the uniform measures in $\mathbb{H}^1$: they are just the flat measures on horizontal lines, the t-axis and vertical planes. The first two cases are due to Chousionis, Magnani and Tyson [104], the third to Merlo [343], using also some results from [104]. It seems to be open whether there can be non-flat uniform measures in higher-dimensional Heisenberg groups.

Combining the case $m = 1$ with Theorem 8.11, we get the Besicovitch–Preiss theorem for one-dimensional measures in $\mathbb{H}^1$. More generally Bate has proved (recall Theorem 7.4):

Theorem 8.12 *Equip $\mathbb{H}^n$ with any homogeneous norm. Let $\mu \in \mathcal{M}(\mathbb{H}^n)$ be such that the positive and finite limit $\lim_{r\to 0} r^{-1}\mu(B(p, r))$ exists for μ almost all $p \in \mathbb{H}^n$. Then μ is 1-rectifiable.*

In [342], Merlo introduced $\mathcal{P}_m$*-rectifiable* measures on a Carnot group G. They were thoroughly investigated by Antonelli and Merlo in [23], [24] and

[25]. Let $0 < m < \dim G$ and let $\mathcal{G}(G,m)$ be the set of all homogeneous subgroups of Hausdorff dimension m which have a complementary subgroup, as in (8.2). The latter condition can sometimes be relaxed to get weaker results. A Radon measure μ on G is called $\mathcal{P}_m$-rectifiable if for μ almost all $p \in \mathbb{H}^n$, $0 < \Theta^m_*(\mu,p) \le \Theta^{*m}(\mu,p) < \infty$ and

$$\mathrm{Tan}(\mu,p) = \{c\mathcal{H}^m \llcorner V(p) \colon 0 < c < \infty\},\ V(p) \in \mathcal{G}(G,m). \tag{8.4}$$

μ is called $\mathcal{P}^*_m$-rectifiable if (8.4) is replaced by

$$\mathrm{Tan}(\mu,p) \subset \{c\mathcal{H}^m \llcorner V \colon V \in \mathcal{G}(G,m),\ 0 < c < \infty\}. \tag{8.5}$$

In $G = \mathbb{H}^n$ compare with Theorems 8.6, 8.7 and 8.11. By Theorem 4.7 the condition (8.4) can be replaced by

$$\mathrm{Tan}(\mu,p) = \{c\nu(p) \colon 0 < c < \infty\},\ \nu(p) \in \mathcal{M}(G).$$

So from this point of view this is a very general definition: it just requires that at almost all points the measure looks the same at all small scales without caring what it looks like.

Actually these authors use a somewhat different definition, but it is easy to see that they are equivalent. However, in [25, Theorem 1.1] there is a related deeper fact: they show that assuming only positive and finite upper density (8.4) implies that the lower density is positive almost everywhere and whence the measure is $\mathcal{P}_m$-rectifiable.

Antonelli and Merlo have many interesting results generalizing a lot of earlier rectifiability theory. I mention here a few of them. In [24], they proved that the $\mathcal{P}_m$-rectifiability is equivalent to covering with intrinsic differentiable graphs. There they also proved the rectifiability of the level sets of Lipschitz functions. In [25], they proved that the $\mathcal{P}_m$-rectifiability is equivalent to the $\mathcal{P}^*_m$-rectifiability, provided the tangents have normal complementary subgroups, which in $\mathbb{H}^n$ corresponds to the low-dimensional case, $m \le n$.

For the next discussion, recall Conjecture 7.8. Let us have a look at the full group $\mathbb{H}^n$ from the point of view of general metric spaces. We know that it has positive and locally finite $\mathcal{H}^{2n+2}_d$ measure and it is purely unrectifiable. The density of the spherical measure $\Theta^{2n+2}(\mathcal{S}^{2n+2}_d, p) = 1$ but $\Theta^{2n+2}(\mathcal{H}^{2n+2}_d, p) = c(n) < 1$ for every $p \in \mathbb{H}^n$, see [302]. This difference comes from the fact that balls are not isodiametric, that is, extremals for the isodiametric inequality (recall Section 1.2) in $\mathbb{H}^n$. In fact, if Q is the Hausdorff dimension of a Carnot group G, then the balls in G are isodiametric if and only if $\Theta^Q(G,p) = 1$ for all $p \in G$ by Proposition 4.1 in [267]. In particular, the Heisenberg group does not give a counterexample to the Hausdorff measure Conjecture 7.8.

Let G be a Carnot group, or somewhat more general as in [267]. Julia and

Merlo showed that there exists a homogeneous left invariant metric d on G such that Conjecture 7.8 holds in the metric space (G, d). They proved that if E has m-density one, then E is m-rectifiable. Isodiametric inequality again plays a central role. When G is not Euclidean, the metric is constructed so that the balls are far from being isodiametric. Then, by what was said above, E cannot be the full group and a proper choice of the metric makes it look Euclidean. In particular, in $\mathbb{H}^n$ we must have $m \leq n$. So for high-dimensional sets the conjecture holds in the sense that they are purely unrectifiable and density cannot be one.

8.6 Projections

Recall from (8.2) that if the subgroups $V \in G_h(2n, m)$ and $W = V^\perp$ are complementary, then any $p \in \mathbb{H}^n$ has a unique decomposition $p = p_V \cdot p_W, p_V \in V, p_W \in W$. So we have the projections $P_V(p) = p_V$ and $Q_W(p) = p_W$. Here V is horizontal and P_V is just the standard orthogonal projection onto V, but Q_W is more awkward, in particular, it is not Lipschitz. The effect of these projections on the Hausdorff dimension (Marstrand-type theorems) was studied in [54].

Very little is known about the relations between projections and rectifiability. There is a bit of that in the proof of Theorem 8.11, as briefly explained above. As mentioned in Section 4.5, Hovila, E. and M. Järvenpää and Ledrappier [242] proved the Besicovitch–Federer projection theorem for transversal families of linear maps $\mathbb{R}^n \to \mathbb{R}^m$. In [241], Hovila proved that the family $\{P_V : V \in G_h(2n, m)\}$ is transversal. Thus the Besicovitch–Federer projection theorem holds in $\mathbb{R}^{2n}$ for $G_h(2n, m)$, which is strictly lower dimensional than the whole $G(2n, m)$ when $m > 1$. This immediately leads to the following result in $\mathbb{H}^n$. There $\pi(z, t) = z$.

Theorem 8.13 *Let $1 \leq m \leq n$ and let $E \subset \mathbb{H}^n$ be a Borel set with $\mathcal{H}^m(\pi(E)) < \infty$. Then $\mathcal{H}^m(P_V(E)) = 0$ for $\mu_{n,m}$ almost all $V \in G_h(2n, m)$ if and only if $\pi(E)$ is purely m-unrectifiable.*

8.7 Uniform Rectifiability

The above results on rectifiability could give a starting point for uniform rectifiability. So far it is not nearly fully developed, but there are many interesting results. However, as far as I know, this is the case only in the cases of dimension 1 and codimension 1. Some of the terminology here is analogous to that in Chapter 5.

For one-dimensional sets the question again is about travelling salesman–type results. Define $\beta_E(x, r)$ as in (3.1), but the infimum is taken only over the horizontal lines, that is, the left translates of the lines through 0 in the horizontal plane.

Theorem 8.14 *Let G be a step-two Carnot group of Hausdorff dimension Q. If* $\Gamma \subset G$ *is a rectifiable curve, then*

$$\int_G \int_0^\infty r^{-Q} \beta_\Gamma(x, r)^4 \, dr \, dx \lesssim \mathcal{H}^1(\Gamma).$$

Theorem 8.15 *If* $p < 4$ *and* $E \subset \mathbb{H}^1$ *is compact and such that*

$$\beta_p(E) := \int_{\mathbb{H}^1} \int_0^\infty r^{-4} \beta_E(x, r)^p \, dr \, dx < \infty,$$

then E is contained in a rectifiable curve Γ *for which* $\mathcal{H}^1(\Gamma) \lesssim d(E) + \beta_p(E)$.

Theorem 8.14 was first proved in $\mathbb{H}^1$ by Li and Schul [293] and then extended by Chousionis, Li and Zimmerman [102], with a modified statement, to arbitrary Carnot groups. Theorem 8.15 was proved by Li and Schul in [294]. The exponent 4 of β comes from the geometry of $\mathbb{H}^1$.

It is not known if Theorem 8.15 holds with $p = 4$. However, Li [291] produced a Carnot group where there is a gap for the exponents. He then defined modified β numbers, based on the stratification of the group, and proved that they give a necessary and sufficient condition in arbitrary Carnot groups; the analogues of Theorems 8.14 and 8.15 hold with the exponent $2s$, where s is the step of the group.

As we have seen, intrinsic Lipschitz graphs play an important role in low-codimensional rectifiability, so they probably should be basic examples of uniformly rectifiable sets. Here are some results in this direction for codimension 1 subsets of $\mathbb{H}^n$. The β numbers are defined as before but restricting the approximation to vertical planes.

Chousionis, Fässler and Orponen proved in [97] that an AD-3-regular set in $\mathbb{H}^1$ has big pieces of intrinsic Lipschitz graphs if and only if it satisfies the weak geometric lemma and has big vertical projections. The projections here are the group projections Q_W onto vertical planes defined above.

Let $E \subset \mathbb{H}^n$ be Lebesgue measurable such that E and $\mathbb{H}^n \setminus E$ are AD-$(2n+2)$-regular and ∂E is AD-$(2n+1)$-regular. Then Naor and Young [360] established a corona decomposition with intrinsic Lipschitz graphs for ∂E. This led to an isoperimetric-type inequality, which was used to solve a fundamental combinatorial problem. See also [361]. By [198, Section 8] the set ∂E as above is a particular case of a Semmes surface, that is, a closed set F such that for every

$p \in F$ and $0 < r < r_0$, there are balls of radius cr in different components of $B(p,r) \setminus F$. Partially using techniques of [360], Fässler, Orponen and Rigot proved in [198] that Semmes surfaces have big pieces of intrinsic Lipschitz graphs. David [134] had earlier proved the corresponding Euclidean result for which this paper gives a new proof.

Di Donato, Fässler and Orponen proved in [177] that $C^{1,\alpha}, \alpha > 0$, codimension 1 intrinsic graphs have big pieces of bi-Lipschitz images of vertical hyperplanes in $\mathbb{H}^n$.

Chousionis, Li and Young proved in [101] that intrinsic Lipschitz graphs in $\mathbb{H}^n$ satisfy the geometric lemma when $n \geq 2$. They used a Dorronsoro inequality of Fässler and Orponen [196] and reduction to lower-dimensional groups by slicing. This slicing technique does not work when $n = 1$ and the problem is open.

We shall return to Heisenberg groups in connection with singular integrals.