# ISOTROPIC AND KÄHLER IMMERSIONS

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**1. Introduction.** Let $M^d$ and $\bar{M}^e$ be Riemannian manifolds. We shall say that an isometric immersion $\phi: M^d \rightarrow \bar{M}^e$ is *isotropic* provided that all its normal curvature vectors have the same length. The class of such immersions is closed under compositions and Cartesian products. Umbilic immersions (e.g. $S^d \subset R^{d+1}$) are isotropic, but the converse does not hold. If $M$ and $\bar{M}$ are Kähler manifolds of constant holomorphic curvature, then any Kähler immersion of $M$ in $\bar{M}$ is automatically isotropic (Lemma 6). We shall find the smallest co-dimension for which there exist non-trivial immersions of this type, and obtain similar results in the real constant-curvature case.

If $T$ is the second fundamental form tensor **(1**; **2)** of an isometric immersion $\phi: M \rightarrow \bar{M}$, then for each unit vector $x$ tangent to $M$, $T_x(x)$ is the normal curvature vector of $\phi$ in the $x$-direction.

**2. Isotropy at one point.** As in **(2)**, we abstract the second fundamental form at one point to a symmetric bilinear function $(x, y) \rightarrow T_x(y)$ on $R^d$ to $R^k$. We adopt for $T$ the usual terminology of isometric immersions; in particular, we say that $T$ is $\lambda$-*isotropic* provided that $||T_x(x)|| = \lambda$ for all unit vectors $x$ in $R^d$. The main invariant of $T$ is its *discriminant* $\Delta$, the real-valued function on planes (through $O$) in $R^d$ such that if $x$ and $y$ span $\Pi$, then

$$\Delta_{xy} = \Delta(\pi) = \frac{\langle T_x(x), T_y(y)\rangle - ||T_x(y)||^2}{||x \wedge y||^2}.$$

(For an isometric immersion $\phi: M \rightarrow \bar{M}$, the Gauss equation asserts that $K(\Pi) = \Delta(\Pi) + \bar{K}(d\phi(\Pi))$, where $K$ and $\bar{K}$ are the sectional curvatures of $M$ and $\bar{M}$, and $\Pi$ is any plane tangent to $M$.)

LEMMA 1. *$T$ is isotropic if and only if $\langle T_x(x), T_x(y)\rangle = 0$ for all orthogonal vectors $x$, $y$ in $R^d$.*

*Proof.* Let $f$ be the (differentiable) real-valued function on the unit sphere $\Sigma$ in $R^d$ such that $f(x) = ||T_x(x)||^2$. Thus $T$ is isotropic if and only if $f$ is constant. But if $y$ is a vector tangent to $\Sigma$ at $x$ (hence $x \perp y$),

$$y(f) = 4\langle T_x(x), T_x(y)\rangle.$$

LEMMA 2. *Suppose that $T$ is $\lambda$-isotropic on $R^d$, and let $x$, $y$, $u$, $v$ be orthogonal vectors in $R^d$. Then*

(1) $\langle T_x(x), T_y(y)\rangle + 2||T_x(y)||^2 = \lambda^2$ if $||x|| = ||y|| = 1$,

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(2) $\langle T_x(x), T_u(v)\rangle + 2\langle T_x(u), T_x(v)\rangle = 0$,
(3) $\langle T_x(y), T_u(v)\rangle + \langle T_x(u), T_y(v)\rangle + \langle T_x(v), T_y(u)\rangle = 0$.

*Proof.* Let $F$ be the quadrilinear function on $R^d$ to $R$ such that

$$F(x, y, u, v) = \langle T_x(y), T_u(v)\rangle - \lambda^2\langle x, y\rangle\langle u, v\rangle$$

for *any* four vectors $x, y, u, v$ in $R^d$. Because $T$ is symmetric, $F(x, y, u, v)$ is symmetric in $x$ and $y$, and also in $u$ and $v$. Also $F$ is symmetric by pairs: $F(x, y, u, v) = F(u, v, x, y)$. Since $T$ is $\lambda$-isotropic $B(x) = F(x, x, x, x) = 0$ for all $x$ in $R^d$. Expansion of $B(x + y) + B(x - y) = 0$ leads to the result

$$(a) \qquad F(x, x, y, y) + 2F(x, y, x, y) = 0.$$

If we replace $y$ by $x + y$ in $(a)$, we obtain

$$(b) \qquad F(x, y, y, y) = 0.$$

If we replace $y$ by $u + v$ in $(a)$, we obtain

$$(c) \qquad F(x, x, u, v) + 2F(x, u, x, v) = 0.$$

Finally, replacing $x$ by $x + y$ in $(c)$ yields

$$(d) \qquad F(x, y, u, v) + F(x, u, y, v) + F(y, u, x, v) = 0.$$

Now assuming that the vectors $x, y, u, v$ are orthogonal, the identities $(a)$, $(c)$, $(d)$ imply the assertions in the lemma.

If the vectors $x, y$ in $R^d$ are orthonormal, the formula for $\Delta_{xy}$ reduces to

$$\Delta_{xy} = \langle T_x(x), T_y(y)\rangle - ||T_x(y)||^2.$$

Thus assertion (1) in the preceding lemma yields the following result.

LEMMA 3. *If $T$ is $\lambda$-isotropic, then for orthonormal vectors $x, y$ in $R^d$*
(1) $\Delta_{xy} + 3||T_x(y)||^2 = \lambda^2$,
(2) $2\Delta_{xy} + \lambda^2 = 3\langle T_x(x), T_y(y)\rangle$.

We deduce some consequences of this lemma. First, the following three conditions are equivalent:

$$\Delta_{xy} = \lambda^2, \qquad T_x(y) = 0, \qquad T_x(x) = T_y(y).$$

This means that $T$ is *umbilic* on the plane $\Pi$ spanned by $x$ and $y$, that is, $T_u(u)$ is the same for all unit vectors $u$ in $\Pi$. Similarly, the following are equivalent:

$$\Delta_{xy} = -2\lambda^2, \qquad ||T_x(y)|| = \lambda, \qquad T_x(x) + T_y(y) = 0$$

(hypotheses as in the lemma). In this case, we say that $T$ is *minimal* on the plane spanned by $x$ and $y$. Because $T$ is $\lambda$-isotropic, $|\langle T_x(x), T_y(y)\rangle| \leqslant \lambda^2$. Therefore we obtain the following corollary.

COROLLARY. *If $T$ is $\lambda$-isotropic, then the discriminant $\Delta$ of $T$ satisfies $-2\lambda^2 \leqslant \Delta \leqslant \lambda^2$. Furthermore, if $\Pi$ is a plane in $R^d$, then $\Delta(\Pi) = \lambda^2$ if and only if $T$ is umbilic on $\Pi$; $\Delta(\Pi) = -2\lambda^2$ if and only if $T$ is minimal on $\Pi$.*

*Remark* 1. Let $e_1, \ldots, e_d$ be an orthonormal basis for $R^d$. Let $A_{ij}$ $(1 \leqslant i \leqslant j \leqslant d)$ be any $d(d+1)/2$ vectors in $R^k$. Then:

1. There is a unique symmetric bilinear function $T$ on $R^d$ to $R^k$ such that $T_{ei}(e_j) = A_{ij}$,
2. If the vectors $A_{ij}$ satisfy (in an obvious sense) the identities in Lemmas 1 and 2, then $T$ is $\lambda$-isotropic,
3. The discriminant $\Delta$ of $T$ is constant if and only if $\Delta$ is constant on the planes spanned by $e_i$, $e_j$, and, for $i, j, k, l$ all different, $\langle A_{ij}, A_{kl}\rangle = \langle A_{il}, A_{kj}\rangle$ and $\langle A_{ii}, A_{jk}\rangle = \langle A_{ij}, A_{ik}\rangle$.

The *first normal space* $\mathfrak{N}$ of $T$ (on $R^d$ to $R^k$) is the subspace of $R^k$ spanned by all vectors $T_x(y)$ $(x, y \in R^d)$. Much of the scant information available about isometric immersions depends on knowledge of the effect on the dimension of $\mathfrak{N}$ produced by conditions on $\Delta$. Our aim is to investigate this matter when $T$ is isotropic.

**3. The constant-discriminant case.** In this section we assume that $T$ is $\lambda$-isotropic and its discriminant $\Delta$ is constant. We shall show that the dimension of the first normal space is determined by the ratio $\Delta : \lambda^2$.

In general, $T$ is *minimal* provided that for one, hence every, frame $e_1, \ldots, e_d$ in $R^d$ we have $\sum_{i=1}^{d} T_{e_i}(e_i) = 0$. Also $T$ is *umbilic* provided $T_u(u)$ has the same value for every unit vector $u$ in $R^d$.

THEOREM 1. *Let $T$ be a symmetric bilinear function on $R^d$ to $R^k$ $(d \geqslant 2)$. Assume that $T$ is $\lambda$-isotropic $(\lambda > 0)$ and that its discriminant $\Delta$ is constant. Let $m_d = d(d+1)/2$, and $h_d = (d+2)/2(d-1)$. Then*

$$-h_d \lambda^2 \leqslant \Delta \leqslant \lambda^2.$$

*Furthermore, if $\mathfrak{N}$ is the first normal space of $T$, then*

(1) $\Delta = \lambda^2 \Leftrightarrow T$ *is umbilic* $\Leftrightarrow$ *dim* $\mathfrak{N} = 1$,

(2) $\Delta = -h_d \lambda^2 \Leftrightarrow T$ *is minimal* $\Leftrightarrow$ dim $\mathfrak{N} = m_d - 1$,

(3) $-h_d \lambda^2 < \Delta < \lambda^2 \Leftrightarrow$ dim $\mathfrak{N} = m_d$.

*Proof.* The principal effect on $T$ of the constancy of $\Delta$ is given in (**2**, Lemma 4), which implies that if $x, y, u, v$ are orthogonal vectors in $R^d$, then both $\langle T_x(y), T_u(v)\rangle$ and $\langle T_x(x), T_u(v)\rangle$ are unchanged by permutations of $x, y, u, v$. Thus Lemma 2 implies that $\langle T_x(y), T_u(v)\rangle$, $\langle T_x(x), T_u(v)\rangle$, and $\langle T_x(u), T_x(v)\rangle$ are zero when the arguments are orthogonal.

Fix an orthonormal basis $e_1, \ldots, e_d$ for $R^d$, and let $z_i = T_{e_i}(e_i)$ for $1 \leqslant i \leqslant d$. Now the $d(d-1)/2$ vectors $T_{e_i}(e_j)$ $(i < j)$ are orthogonal, and each is orthogonal to the subspace $Z$ spanned by $z_1, \ldots, z_d$. Assertion 1 follows immediately from the remarks in the preceding section. Henceforth we exclude the minimal case $\Delta = \lambda^2$. Then Lemma 3 shows that the vectors $T_{e_i}(e_j)$ $(i < j)$ all have the same non-zero length. Thus we have

$$\dim \mathfrak{N} = d(d-1)/2 + \dim Z.$$

Lemma 3 further shows that the inner products $\langle z_i, z_j \rangle$ $(i \neq j)$ are all equal; we write

$$\langle z_i, z_j \rangle = \lambda^2 \cos \theta.$$

By Euclidean geometry $\cos \theta \leqslant -1/(d-1)$, and equality holds if and only if the vectors $z_1, \ldots, z_d$ are linearly dependent. The vectors $z_1, \ldots, z_d$ then describe the vertices of an equilateral Euclidean simplex centred at the origin of the subspace $Z$. From Lemma 3, Formula (2), we obtain $\lambda^2(3\cos\theta - 1) = 2\Delta$. Thus it is easy to see that, for $m_d$ and $h_d$ as defined above, the following assertions are equivalent:

$$\Delta = -h_d\lambda^2, \qquad \cos\theta = -1/(d-1), \qquad \dim Z = d-1,$$

$$\dim \mathfrak{N} = m_d - 1, \qquad z_1 + \ldots + z_d = 0, \qquad T \text{ is minimal.}$$

Similarly, the following are equivalent:

$$\Delta > -h_d \lambda^2, \qquad \cos\theta < -1/(d-1), \qquad \dim Z = d, \qquad \dim \mathfrak{N} = m_d.$$

But, by Lemma 3, $\Delta \leqslant \lambda^2$ always holds, so, since the case $\Delta = \lambda^2$ was excluded earlier, the proof is complete.

*Remark* 2. Given an integer $d \geqslant 2$, and numbers $\Delta, \lambda \geqslant 0$ such that $-h_d \lambda^2 \leqslant \Delta \leqslant \lambda^2$, there exists a $\lambda$-isotropic $T$ on $R^d$ to $R^{m_d}$ whose discriminant has the constant value $\Delta$. To construct $T$, use Remark 1, arranging the vectors $A_{ij}$ as dictated by the preceding proof.

**4. Isotropic immersions.** Theorem 1 has the following basic consequence, which shows that, in the case of constant $\Delta$, large co-dimensions are required if an isotropic immersion is not umbilic.

COROLLARY. *Let $\phi: M^d \to \bar{M}^e$ be an isotropic immersion with $\Delta = K - \bar{K} \circ d\phi$ constant. If $e < e_d - 1$, where $e_d = d(d+3)/2$, then $\phi$ is umbilic.*

In the case of constant curvature we get

THEOREM 2. *Let $\phi: M^d \to \bar{M}^e$ be an isotropic immersion of manifolds of constant curvature $C$ and $\bar{C}$. Let $e_d = d(d+3)/2$.*

(1) *If $C > \bar{C}$ and $e < e_d$, then $\phi$ is umbilic.*

(2) *If $C = \bar{C}$ and $e < e_d$, then $\phi$ is totally geodesic.*

(3) *If $C < \bar{C}$, then $e \geqslant e_d - 1$. Furthermore, if $e = e_d - 1$, then $\phi$ is minimal.*

An isometric immersion is *minimal* provided its second fundamental form tensor is minimal at each point. Evidently this generalizes the classical definition of minimal surface in $R^3$.

*Proof.* If $e < e_d - 1$, then the co-dimension $e - d$ is strictly less than $e_d - d - 1 = m_d - 1$. Hence, by Theorem 1, $\phi$ is umbilic. But this is impossible if $C < \bar{C}$, and implies that $\phi$ is totally geodesic if $C = \bar{C}$. Now suppose that $e = e_d - 1$, so the co-dimension is $m_d - 1$. If $C > \bar{C}$, so $\Delta > 0$, then by Theorem 1, $\phi$ cannot be minimal, and hence it is again umbilic. If $C = \bar{C}$, we similarly deduce that $\phi$ is totally geodesic.

We now obtain examples of isotropic, non-umbilic immersions which show that the above dimensional restrictions cannot be improved. It is noteworthy that global examples can be obtained from a second fundamental form tensor $T$ at one point. In fact, suppose that $T$ is symmetric, bilinear on $R^{d+1}$ to $R^{k+1}$. For each element $x$ of the unit sphere $\Sigma$ in $R^{d+1}$, let $\phi(x) = T_x(x)$. Since $\phi(-x) = \phi(x)$, we obtain a differentiable map $\phi$ of real projective space $P^d$ into $R^{k+1}$. Now suppose that $T$ is $\lambda$-isotropic and has constant discriminant. Then if $x$ and $u$ are orthonormal in $R^{d+1}$, $T_x(u)$ has the constant value $\mu$ such that $\mu^2 = (\lambda^2 - \Delta)/3$. Excluding the umbilic case ($\mu = 0$), we alter $\phi$ by a scalar and define the *associated mapping* $\psi$ of $T$ to be the differentiable function

$$\psi : P^d(1) \to S^k(\lambda/2\mu)$$

such that $\psi(\{x, -x\}) = T_x(x)/2\mu$. (Here $P^d$ and $S^k$ have the canonical Riemannian structures appropriate to their radii.)

LEMMA 4. *Let $T$ be a symmetric bilinear function on $R^{d+1}$ to $R^{k+1}$ ($d \geqslant 2$). Suppose that $T$ is $\lambda$-isotropic and has constant discriminant $\Delta \neq \lambda^2$. Then the associated mapping $\psi$: $P^d(1) \to S^k(\lambda/2\mu)$ is an isotropic imbedding with*

$$\lambda_*^2 = \frac{4}{3}\left(\frac{\Delta}{\lambda^2} + 2\right) \quad \textit{and} \quad \Delta_* = \frac{1}{3}\left(4\frac{\Delta}{\lambda^2} - 1\right).$$

*Proof.* If $u$ is a unit tangent vector to the unit sphere $\Sigma \subset R^{d+1}$ at the point $x$, then let $\sigma$ be the geodesic

$$\sigma(t) = (\cos t)x + (\sin t)u.$$

Then $\psi \circ \sigma = T_\sigma(\sigma)/2\mu$. But $(\psi \circ \sigma)'(0) = T_x(u)/\mu$, so $||d\psi(u)|| = 1$, and thus $\psi$ is an isometric immersion. By Lemma 3, $\Delta + 3\mu^2 = \lambda^2$. The manifolds involved have curvatures $C = 1$ and $\bar{C} = 4\mu^2/\lambda^2$. Hence we obtain the required result for $\Delta_* = C - \bar{C}$.

We briefly outline the proof that $\psi$ is $\lambda_*$-isotropic. Now the *Euclidean* acceleration of the curve $\psi \circ \sigma$ is given by

$$(\psi \circ \sigma)''(0) = (T_u(u) - T_x(x))/\mu.$$

The unit normal to $S^k(\lambda/2\mu)$ at $\psi(x)$ is $T_x(x)/\lambda$. Let $\tau$ be the second fundamental form tensor of the immersion $\psi$. Subtracting from $(\psi \circ \sigma)''(0)$ its component orthogonal to $S^k(\lambda/2\mu)$, we obtain

$$\tau_u(u) = T_u(u)/\mu - \left(\frac{\Delta}{\mu} + \mu\right) T_x(x)/\lambda^2.$$

A straightforward computation yields

$$\lambda_*^2 = \|\tau_u(u)\|^2 = \frac{4}{3}\left(\frac{\Delta}{\lambda^2} + 2\right).$$

To show that $\psi$ is one–one, we observe that, by Lemma 3, $T_u(v)$ is never zero if $u$ and $v$ are non-zero and orthogonal. Suppose that $x$ and $y$ are unit vectors in $R^d$ such that $y \neq \pm x$. Then

$$0 \neq T_{x-y}(x + y) = T_x(x) - T_y(y);$$

hence $\psi(\{x, -x\}) \neq \psi(\{y, -y\})$.

Corollary 1. *For each $d \geqslant 2$, there exists a non-umbilic isotropic imbedding $\psi: P^d(1) \to S^{e_d-1}(r_d)$ where $e_d = d(d + 3)/2$ and $(r_d)^2 = d/2(d + 1)$. Furthermore $\psi$ is minimal.*

Corollary 2. *Let $M(C)$ denote a complete Riemannian manifold of constant curvature $C$. If $C \leqslant 2(d + 1)/d$, then there exists a non-umbilic isotropic immersion $\psi: P^d(1) \to M^{e_d}(C)$, where $e_d = d(d + 3)/2$.*

*Proof.* For Corollary 1, choose $T$ (by Remark 2) so that $\lambda = 1$ and $\Delta = -h_{d+1}$. The associated map $\psi$ is isotropic by the preceding Lemma, and since $\Delta_*/\lambda_*^2$ turns out to be $-h_d$, Theorem 1 asserts that $\psi$ is minimal. Also

$$(r_d)^2 = (\lambda/2\mu)^2 = 3/4(1 + h_{d+1}) = d/2(d + 1).$$

If $\mathfrak{N}$ is the first normal space of $T$, then the values of $\psi$ actually lie in the great sphere $\mathfrak{N} \cap S^k(r_d)$. By Theorem 1, dim $\mathfrak{N} = m_{d+1} - 1$; hence the values of $\psi$ are in the sphere $S^{e_d-1}$, where $e_d = m_{d+1} - 1 = d(d + 3)/2$.

To prove Corollary 2, recall that any simply connected, constant-curvature manifold $X^d(K)$ may be imbedded as an *umbilic* Riemannian submanifold in $X^{d+1}(\bar{K})$ if $K \geqslant \bar{K}$. Since $C \leqslant 2(d + 1)/d$, the sphere $S^{e_d-1}(r_d)$ may be imbedded as an umbilic hypersurface in the simply connected covering manifold of $M^{e_d}(C)$. Then we derive the required immersion from the imbedding in Corollary 1.

Evidently these corollaries show that the dimensional restrictions in Theorem 2 cannot be improved.

**5. Kähler immersions.** We use the definition of *Kähler manifold $M$* under which $M$ is a Riemannian manifold furnished with an almost complex structure $J$ such that $\langle JX, JY\rangle = \langle X, Y\rangle$ and $\nabla_X(JY) = J(\nabla_X Y)$ for all vector fields $X$, $Y$ on $M$. A *Kähler immersion* $\phi: M \to \bar{M}$ (of Kähler manifolds) is an isometric immersion which is almost complex, that is, the differential map of $\phi$ commutes with the almost complex structures on $M$ and $\bar{M}$.

Lemma 5. *If $\phi: M \to \bar{M}$ is a Kähler immersion, then its second fundamental form tensor $T$ is almost complex, that is, $T_x(Jy) = J(T_x y)$ for $x$, $y$ tangent to $M$.*

*Proof.* If $Y$ is a vector field on $M$, then (locally) there is a $\phi$-related vector field $\bar{Y}$ on $\bar{M}$ and $J\bar{Y}$ is $\phi$-related to $JY$. If $x$ is tangent to $M$, then

$$\nabla_{d\phi(x)}(\bar{Y}) = d\phi(\nabla_x Y) + T_x(Y).$$

Hence

$$J(T_x Y) = J(\nabla_{d\phi(x)}\bar{Y}) - J(d\phi(\nabla_x Y) \\ = \nabla_{d\phi(x)}(J\bar{Y}) - d\phi(\nabla_x(JY)) = T_x(JY).$$

The *holomorphic curvature* $K_{\text{hol}}$ of a Kähler manifold $M$ is the function on unit tangent vectors $x$ such that $K_{\text{hol}}(x)$ is the sectional curvature $K(\Pi_{x,Jx})$ of the holomorphic section through $x$. A Kähler immersion preserves holomorphic planes, and, corresponding to the function $\Delta = K - \bar{K} \circ d\phi$ on all tangent planes to $M$, we have the *holomorphic difference*

$$\Delta_{\text{hol}} = K_{\text{hol}} - \bar{K}_{\text{hol}} \circ d\phi.$$

LEMMA 6. *If $\phi$: $M \to \bar{M}$ is a Kähler immersion, then $\Delta_{\text{hol}} \leqslant 0$, and $\Delta_{\text{hol}} = 0$ if and only if $\phi$ is totally geodesic. Furthermore, $\phi$ is $\lambda$-isotropic if and only if $\Delta_{\text{hol}}$ has the constant value $-2\lambda^2$.*

*Proof.* The first assertion is well known. However, both assertions are proved by observing that the symmetry of $T$ and the fact that $T$ is almost complex imply $T_{Jx}(Jx) = -T_x(x)$. For then

$$\Delta_{\text{hol}}(x) = \Delta(\Pi_{x,Jx}) = \langle T_x(x), T_{Jx}(Jx)\rangle - ||T_x(Jx)||^2 = -2||T_x(x)||^2.$$

In particular, a Kähler immersion of manifolds of constant holomorphic curvature is isotropic.

**6. Constant holomorphic discriminant.** We examine the second fundamental form (at one point) of a Kähler immersion with $\Delta_{\text{hol}}$ constant. Thus we assume that $T$ is a symmetric bilinear form on $R^{2d}$ to $R^{2k}$ such that $T$ is isotropic and almost complex (relative to natural almost complex operators $J$ on $R^{2d}$ and $R^{2n}$).

Of course one gets a large number of identities by inserting $J$ in Lemmas 1 and 2. We shall need

LEMMA 7. *Let $T$ be isotropic and almost complex.*

(1) *If $x$, $Jx$, $u$, $v$ are orthogonal vectors in $R^{2d}$, then*

$$\langle T_x(u), T_x(v)\rangle = \langle T_x(x), T_u(v)\rangle = \langle T_x(x), T_u(u)\rangle = 0.$$

(2) *If $H$ and $H'$ are orthogonal holomorphic planes in $R^{2d}$, then*

$$\langle T_x(y), T_u(v)\rangle = 0$$

*for all $x$, $y \in H$ and $u$, $v \in H'$.*

*Proof.* We may suppose that $x$ and $u$ are unit vectors. Applying the first identity in Lemma 2 to $Jx$ and $u$, we obtain

$$-\langle T_x(x), T_u(u)\rangle + 2||T_x(u)||^2 = \lambda^2.$$

It follows that $\langle T_x(x), T_u(u)\rangle = 0$. Replacing $x$ by $Jx$ in the second identity yields the remaining assertions in (1).

For $x, y, u, v$ as in (2), consider the orthogonal vectors $x + y$, $J(x + y)$, $u + v$, $J(u + v)$. Then (1) implies that $\langle T_{x+y}(x + y), T_{u+v}(u + v)\rangle = 0$. Expansion of this inner product yields $\langle T_x(y), T_u(v)\rangle = 0$.

It is now easy to give a complete description of $T$.

LEMMA 8. *Let $T$ be $\lambda$-isotropic and almost complex on $R^{2d}$ to $R^{2k}$. If $e_1, \ldots, e_d$, $Je_1, \ldots, Je_d$ is an orthnormal basis for $R^{2d}$, then*

(1) $$||T_{e_i}(e_j)||^2 = \begin{cases} \lambda^2 & \text{if } i = j, \\ \lambda^2/2 & \text{if } i \neq j. \end{cases}$$

(2) *The $d(d + 1)$ vectors $T_{e_i}(e_j)$, $JT_{e_i}(e_j)$ $(1 \leqslant i \leqslant j \leqslant d)$ are orthogonal.*

*Proof.* The norm in the case $i \neq j$ follows from the first two sentences in the proof of Lemma 7. To prove the orthogonality assertion, let $H_{ij}(1 \leqslant i \leqslant j \leqslant d)$ be the (holomorphic) plane in $R^{2k}$ spanned by $T_{e_i}(e_j)$ and $JT_{e_i}(e_j)$. There are now three cases: $H_{ii} \perp H_{ij}(i \neq j)$, $H_{ij} \perp H_{ik}(i, j, k$ mutally distinct), $H_{ij} \perp H_{kl}(\{i, j\}$ and $\{k, l\}$ disjoint). All three follow immediately from Lemmas 1, 2, and 7.

**7. Dimensions for Kähler immersions.** We now obtain the Kähler analogues of the results in Section 4.

THEOREM 3. *Let $\phi: M^{2d} \to \bar{M}^{2e}$ be a Kähler immersion with $\Delta_{\text{hol}}$ constant. If $e < d(d + 3)/2$, then $\phi$ is totally geodesic.*

*Proof.* If $\phi$ is not totally geodesic, then the second fundamental form tensor $T$ of $\phi$ is $\lambda$-isotropic with $\lambda > 0$. Then by Lemma 8, the first normal space of $T$ (at each point) has dimension at least $d(d + 1)$. Hence $2e \geqslant 2d + d(d + 1)$, so $e \geqslant d(d + 3)/2$.

In the constant holomorphic case, the results (1) and (2) (below) are well known.

COROLLARY. *Let $M$ and $\bar{M}$ be Kähler manifolds of constant holomorphic curvature $C_h$ and $\bar{C}_h$.*

(1) *For $C_h > \bar{C}_h$, there exist no Kähler immersions of $M$ in $\bar{M}$.*

(2) *For $C_h = \bar{C}_h$, every Kähler immersion of $M$ in $\bar{M}$ is totally geodesic.*

(3) *For $C_h < \bar{C}_h$, there exist no Kähler immersions of $M^{2d}$ in $\bar{M}^{2e}$ if $e < d(d + 3)/2$.*

We now construct an example to show that this last dimensional restriction (hence that of Theorem 3) cannot be improved. Using Remark 1 and the proof of Lemma 8, it is easy to show that there exists, for each $\lambda > 0$, a $\lambda$-isotropic, almost complex $T$ on $R^{2d+2}$ to $R^{2k}$, where $k = (d+1)(d+2)/2$. As in Section 4, let $\Sigma$ be the unit sphere in $R^{2d+2}$, and let $\phi\colon \Sigma \to R^{2k}$ be the map such that $\phi(x) = T_x(x)/\lambda\sqrt{2}$.

If $x \in \Sigma$, the *holomorphic circle* $C(x)$ through $x$ is the intersection of $\Sigma$ and the holomorphic plane $H(x)$ through $x$. By the usual Euclidean identifications, the orthogonal complement of $H(x)$ corresponds to $C(x)^{\perp}$, the subspace of the tangent space $\Sigma_x$ consisting of vectors normal to $C(x)$. Thus the natural almost complex structure of $R^{2d+2}$ induces on $\Sigma$ a partial almost complex structure, defined only on the spaces $C(x)^{\perp}$. From previous identities, it follows that the differential map $d\phi$ of $\phi$ preserves both inner products and almost complex structure *on the spaces* $C(x)^{\perp}$.

Denote by $\mathbf{P}^d(1)$ the complex projective $d$-space obtained by identifying holomorphic circles in $\Sigma \subset R^{2d+2}$. Explicitly, the Kähler structure of $\mathbf{P}^d(1)$ is such that if $\pi\colon \Sigma \to \mathbf{P}^d(1)$ is the natural projection, then $d\pi$ preserves inner products and $J$-operators on each space $C(x)^{\perp}$.

THEOREM 4. *For each $d \geqslant 1$, there exists a Kähler imbedding*

$$\psi\colon \mathbf{P}^d(1) \to \mathbf{P}^e(1/\sqrt{2}),$$

*where $e = d(d+3)/2$.*

*Proof* (notation as above). The values of $\phi$ lie in the sphere $S^{2k-1}(1/\sqrt{2})$. Since $k = (d+1)(d+2)/2$, we have $k - 1 = d(d+3)/2$. A holomorphic circle in $\Sigma$ may be parametrized by a curve $\sigma$ such that $\sigma(t) = \mathrm{c}x + \mathrm{s}Jx$, where $\mathrm{c} = \cos t$, $\mathrm{s} = \sin t$. But

$$T_{\mathrm{c}x+\mathrm{s}Jx}(\mathrm{c}x + \mathrm{s}Jx) = (\mathrm{c}^2 - \mathrm{s}^2)T_x(x) + 2\mathrm{sc}J(T_x(x)).$$

Thus $\phi$ carries holomorphic circles in $\Sigma$ to holomorphic circles in $S^{2e+1}(1/\sqrt{2})$, where $e = d(d+3)/2$. Hence $\phi$ determines a differentiable map

$$\psi\colon \mathbf{P}^d(1) \to \mathbf{P}^e(1/\sqrt{2})$$

which commutes with the natural projections $\pi$. It follows immediately that $\psi$ is actually a Kähler immersion. Since $\phi$ carries holomorphic circles *onto* holomorphic circles, we can show that $\psi$ is one–one by essentially the same argument as in Lemma 4.

Note that for $\psi$, $\Delta_{\text{hol}} = 1 - 2 = -1$. This sequence of imbeddings is *reproductive* in the sense that $\psi_d$ is precisely the imbedding induced by the second fundamental form tensor (at any point) of $\psi_{d+1}$.

## REFERENCES


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