Relations between the Integrals of the Hypergeometric Equation.

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(*Read 9th May 1919. Received 22nd May 1919.*)

The following discussion of the analytical continuation of the hypergeometric function is believed by the writer to possess the advantages of brevity and simplicity.

Denote the integrals of Gauss's Equation

$$z(1-z)w''+\{\gamma-(\alpha+\beta+1)z\}w'-\alpha\beta w=0$$

which are regular near 0, 1, ∞, respectively by

$$W_1^{(0)},\ W_2^{(0)};\ W_1^{(1)},\ W_2^{(1)};\ W_1^{(\infty)},\ W_2^{(\infty)}.$$

Part I.—*The Four Forms of the Integrals.*

Consider the integral

$$I\equiv\int^{(1+,\,0+,\,1-,\,0-)}\zeta^{\beta-1}(1-\zeta)^{\gamma-\beta-1}(1-z\zeta)^{-\alpha}\,d\zeta,$$

where the initial point lies on the real axis between 0 and 1, and amp (ζ) and amp $(1-\zeta)$ are initially zero; that value of $(1-z\zeta)^{-\alpha}$ is considered which has the value $+1$ when $z=0$. Expand $(1-z\zeta)^{-\alpha}$ in powers of z, and integrate term by term; then

$$I=(1-e^{2\pi i\beta})\{1-e^{2\pi i(\gamma-\beta)}\}\,B(\beta,\gamma-\beta)\,F(\alpha,\beta,\gamma,z).$$

In the integral put $\zeta=1-t$; then

$$I=-\int^{(0+,\,1+,\,0-,\,1-)}t^{\gamma-\beta-1}(1-t)^{\beta-1}(1-z+zt)^{-\alpha}\,dt.$$

Now this path is the previous path described in the opposite direction ; hence

$$I=(1-z)^{-\alpha}\int^{(1+,0+,1-,0-)} t^{\gamma-\beta-1}(1-t)^{\beta-1}\left(1-\frac{z}{z-1}t\right)^{-\alpha}dt$$

$$=(1-z)^{-\alpha}\{1-e^{2\pi i(\gamma-\beta)}\}(1-e^{2\pi i\beta})$$

$$B(\beta,\gamma-\beta)\,F\left(\alpha,\gamma-\beta,\gamma,\frac{z}{z-1}\right).$$

Accordingly,

$$F(\alpha,\beta,\gamma,z)=(1-z)^{-\alpha}\,F\left(\alpha,\gamma-\beta,\gamma,\frac{z}{z-1}\right).$$

In this equation interchange α and β ; then

$$F(\alpha,\beta,\gamma,z)=(1-z)^{-\beta}\,F\left(\beta,\gamma-\alpha,\gamma,\frac{z}{z-1}\right).$$

It follows that

$$F\left(\alpha,\gamma-\beta,\gamma,\frac{z}{z-1}\right)=(1-z)^{\alpha-\beta}\,F\left(\beta,\gamma-\alpha,\gamma,\frac{z}{z-1}\right).$$

In this equation replace α, β, γ, z by $\alpha, \gamma-\beta, \gamma, z/(z-1)$; thus

$$F(\alpha,\beta,\gamma,z)=(1-z)^{\gamma-\alpha-\beta}\,F(\gamma-\alpha,\gamma-\beta,\gamma,z).$$

Hence

$$\begin{aligned} W_1^{(0)} &= F(\alpha,\beta,\gamma,z)\\ &=(1-z)^{\gamma-\alpha-\beta}\,F(\gamma-\alpha,\gamma-\beta,\gamma,z)\\ &=(1-z)^{-\alpha}\,F\left(\alpha,\gamma-\beta,\gamma,\frac{z}{z-1}\right)\\ &=(1-z)^{-\beta}\,F\left(\beta,\gamma-\alpha,\gamma,\frac{z}{z-1}\right). \end{aligned}$$

From the four forms for $W_1^{(0)}$ the four forms for the other five integrals can be deduced.

PART II.—*Relations between the Integrals.*

Consider the integrals

$$A\equiv\int^{(1+,z+,1-,z-)} f(z,\zeta)\,d\zeta,\quad B\equiv\int^{(1+,0+,1-,0-)} f(z,\zeta)\,d\zeta,$$

$$C\equiv\int^{(0+,z+,0-,z-)} f(z,\zeta)\,d\zeta,$$

https://doi.org/10.1017/S0013091500035446 Published online by Cambridge University Press

where $f(z, \zeta) = \zeta^{\alpha-\gamma}(1-\zeta)^{\gamma-\beta-1}(z-\zeta)^{-\alpha}$; the initial point is taken on the real axis between 0 and 1, and that value of $(1 - \zeta/z)^{-\alpha}$ is taken which tends to $+1$ when $z \rightarrow \infty$.

In A put $\zeta = 1 - t$; then $z - \zeta = -(\overline{1-z} - t)$.

This equation can be written

$$z(1 - \zeta/z) = -(1-z)\{1 - t/(1-z)\}.$$

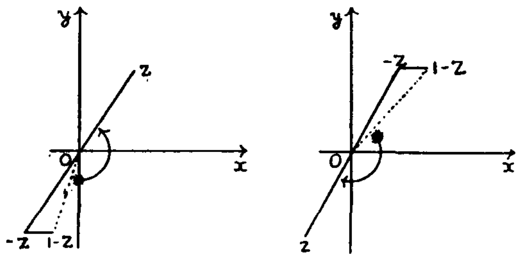

Fig. 1.

Now it is clear from Fig. 1. that, as $z \rightarrow \infty$, amp $\{(z/1-z)\}$ tends to $\pm\pi$ according as $I(z) \gtrless 0$. Also amp$(1-\zeta/z)$ and amp $\{1 - t/(1-z)\}$ tend to zero. Hence

$$z - \zeta = e^{\pm\pi i}(\overline{1-z} - t),$$

according as $I(z) \gtrless 0$.

It follows that

$$A = -e^{\mp\pi i a}\int^{(0+,\, \overline{1-z}+,\, 0-,\, \overline{1-z}-)} t^{\gamma-\beta-1}(1-t)^{\alpha-\gamma}(\overline{1-z} - t)^{-\alpha}\, dt,$$

Again, in this integral put $t = (1-z)Z$; thus

$$A = -e^{\mp\pi i a}(1-z)^{\gamma-\alpha-\beta}$$

$$\int^{(0+,\,1+,\,0-,\,1-)} Z^{\gamma-\beta-1}(1-Z)^{-\alpha}(1 - \overline{1-z}\,Z)^{\alpha-\gamma}\, dZ$$

$$= e^{\mp\pi i a}\{1 - e^{2\pi i(\gamma-\beta)}\}(1 - e^{-2\pi i a})\, B(\gamma-\beta,\, 1-\alpha) \times W_2^{(1)}.$$

In B expand $(z-\zeta)^{-\alpha}$ in descending powers of z; then

$$B = \{1 - e^{2\pi i(\alpha-\gamma)}\}\{1 - e^{2\pi i(\gamma-\beta)}\}\, B(\alpha-\gamma+1,\, \gamma-\beta) \times W^{(\infty)}.$$

In C put $\zeta = zZ$, and expand in powers of z; thus

$$C = -\{1 - e^{2\pi i(\alpha-\gamma)}\}(1 - e^{-2\pi i a})\, B(\alpha-\gamma+1,\, 1-\alpha) \times W_2^{(0)}.$$

Again, let

$$L \equiv \int^{(0+)} f(z, \zeta)\, d\zeta,\ M = \int^{(1+)} f(z, \zeta)\, d\zeta,\ N = \int^{(z+)} f(z, \zeta)\, d\zeta\ ;$$

then

$$A = M(1 - e^{-2\pi i\alpha}) - N\{1 - e^{2\pi i(\gamma-\beta)}\},$$
$$B = M\{1 - e^{2\pi i(\alpha-\gamma)}\} - L\{1 - e^{2\pi i(\gamma-\beta)}\},$$
$$C = L(1 - e^{-2\pi i\alpha}) - N\{1 - e^{2\pi i(\alpha-\gamma)}\}.$$

Hence

$$A\{1 - e^{2\pi i(\alpha-\gamma)}\} - B(1 - e^{-2\pi i\alpha}) - C\{1 - e^{2\pi i(\gamma-\beta)}\} = 0.$$

In this equation replace A, B, C, by the values found above; thus

$$\text{(i)}\quad e^{\mp\pi i\alpha} B(\gamma-\beta, 1-\alpha) \times W_2^{(1)} - B(\alpha-\gamma+1, \gamma-\beta) \times W_1^{(\infty)}$$
$$+ B(\alpha-\gamma+1, 1-\alpha) \times W_2^{(0)}. = 0.$$

In this equation interchange α and β; then

$$\text{(ii)}\quad e^{\mp\pi i\beta} B(\gamma-\alpha, 1-\beta) \times W_2^{(1)} - B(\beta-\gamma+1, \gamma-\alpha) \times W_2^{(\infty)}$$
$$+ B(\beta-\gamma+1, 1-\beta) \times W_2^{(0)} = 0.$$

In (i) and (ii) replace α, β, γ, by $\alpha-\gamma+1$, $\beta-\gamma+1$, $2-\gamma$, and multiply by $z^{1-\gamma}$; thus

$$\text{(iii)}\quad e^{\mp\pi i(\alpha-\gamma+1)} B(\gamma-\alpha, 1-\beta) \times W_2^{(1)} - B(\alpha, 1-\beta) \times W_1^{(\infty)}$$
$$+ B(\gamma-\alpha, \alpha) \times W_1^{(0)} = 0.$$

$$\text{(iv)}\quad e^{\mp\pi i(\beta-\gamma+1)} B(\gamma-\beta, 1-\alpha) \times W_2^{(1)} - B(\beta, 1-\alpha) \times W_2^{(\infty)}$$
$$+ B(\gamma-\beta, \beta) \times W_1^{(0)} = 0.$$

In (i) and (ii) replace α, β, γ, by $1-\alpha$, $1-\beta$, $2-\gamma$, and multiply by $z^{1-\gamma}(1-z)^{\gamma-\alpha-\beta}$; then

$$\text{(v)}\quad e^{\mp\pi i(1-\alpha)} B(\alpha, \beta-\gamma+1) \times W_1^{(1)} - e^{\mp\pi i(\gamma-\alpha-\beta)} B(\gamma-\alpha, \beta-\gamma+1) \times W_2^{(\infty)}$$
$$+ B(\alpha, \gamma-\alpha) \times W_1^{(0)} = 0.$$

$$\text{(vi)}\quad e^{\mp\pi i(1-\beta)} B(\beta, \alpha-\gamma+1) \times W_1^{(1)} - e^{\mp\pi i(\gamma-\alpha-\beta)} B(\gamma-\beta, \alpha-\gamma+1) \times W_1^{(\infty)}$$
$$+ B(\beta, \gamma-\beta) \times W_1^{(0)} = 0.$$

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Note—Amp $\{(z-1)/(1-z)\} = \pm\pi$, according as $I(z) \gtrless 0$. (Fig. 2).

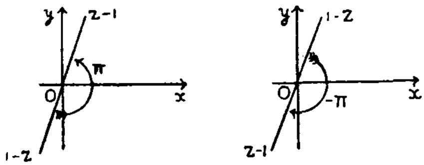

Fig. 2.

In (i) and (ii) replace α, β, γ, by $\gamma-\alpha, \gamma-\beta, \gamma$, respectively, and multiply by $(1-z)^{\gamma-\alpha-\beta}$; then

$$\text{(vii)}\quad e^{\mp\pi i(\gamma-\alpha)} B(\beta, \alpha-\gamma+1) \times W_1^{(1)} - e^{\mp\pi i(\gamma-\alpha-\beta)} B(1-\alpha, \beta) \times W_2^{(\infty)} + B(1-\alpha, \alpha-\gamma+1) \times W_2^{(0)} = 0.$$

$$\text{(viii)}\quad e^{\mp\pi i(\gamma-\beta)} B(\alpha, \beta-\gamma+1) \times W_1^{(1)} - e^{\mp\pi i(\gamma-\alpha-\beta)} B(1-\beta, \alpha) \times W_1^{(\infty)} + B(1-\beta, \beta-\gamma+1) \times W_2^{(0)} = 0.$$

By means of these eight equations any of the integrals can be expressed in terms of the two integrals at one of the other singularities.

For example, to express $W_1^{(0)}$ in terms of $W_1^{(1)}$ and $W_2^{(1)}$, multiply (iv) by $1/B(\beta, 1-\alpha)$ and (v) by $e^{\pm\pi i(\gamma-\alpha-\beta)}/B(\gamma-\alpha, \beta-\gamma+1)$ and subtract; then

$$-e^{\pm\pi i(\gamma-\beta)} \frac{\sin(\gamma-\alpha-\beta)\pi}{\pi} \frac{\Gamma(\alpha)\Gamma(1-\alpha+\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)} \times W_1^{(0)}$$

$$= -e^{\pm\pi i(\gamma-\beta)} \frac{\sin(\gamma-\alpha-\beta)\pi}{\pi} \frac{\Gamma(\alpha)\Gamma(1-\alpha+\beta)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)} \times W_1^{(1)}$$

$$+ e^{\pm\pi i(\gamma-\beta)} \frac{\sin(\alpha+\beta-\gamma)\pi}{\pi} \frac{\Gamma(\gamma-\beta)\Gamma(1-\alpha+\beta)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\beta)} \times W_2^{(1)}.$$

Therefore

$$W_1^{(0)} = \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \times W_1^{(1)} + \frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \times W_2^{(1)}.$$