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ASYMPTOTICS FOR SEMILINEAR ELLIPTIC SYSTEMS

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ABSTRACT. A class of weakly coupled systems of semilinear elliptic partial differential equations is considered in an exterior domain in $\mathbb{R}^N$, $N \geq 3$. Necessary and sufficient conditions are given for the existence of a positive solution (componentwise) with the asymptotic decay $u(x) = O(|x|^{2-N})$ as $|x| \longrightarrow \infty$. Additional results concern the existence and structure of positive solutions u with finite energy in a neighbourhood of infinity.

Our objective is to establish necessary and sufficient conditions for the existence of two types of positive solutions (componentwise) of the semilinear elliptic system

$$-\Delta u_i = f_i(x, \mathbf{u}), \quad x \in \Omega, \quad i = 1, \ldots, M \tag{1}$$

in an exterior domain $\Omega \subset \mathbb{R}^N$, $N \geq 3$, where $x = (x_1, \ldots, x_N)$, $\mathbf{u} = (u_1, \ldots, u_M)$. It is not required that (1) be either a potential system or radially symmetric. The two types of positive solutions are:

(I) *Minimal positive solutions* $\mathbf{u}$, i.e., $|x|^{N-2}u_i(x)$ is bounded above and below by positive constants in some exterior domain Ω, $i = 1, \ldots, M$.

(II) *Solutions* $\mathbf{u}$ *with finite energy in a neighbourhood of infinity,* i.e., $\psi u_i \in D_0^{1,2}(\mathbb{R}^N)$, $i = 1, \ldots, M$, for some nonnegative radial function $\psi \in C^1(\mathbb{R}^N)$ with $\psi(x) \equiv 1$ for sufficiently large $|x|$.

As usual, $D_0^{1,2}(\mathbb{R}^N)$ denotes the completion of $C_0^\infty(\mathbb{R}^N)$ in the norm $\|\phi\| = \|\nabla\phi\|_{L^2(\mathbb{R}^N)}$. We also use the notation $\| \ \|_{q,B}$ for the norm in $L^q(B)$, where $B \subset \mathbb{R}^N$. Vector inequalities are to be interpreted componentwise; in particular $\mathbf{u} > 0$ means that each $u_i > 0$. For a multi-index $\gamma = (\gamma_1, \ldots, \gamma_M) > 0$ we use the notation

$$|\gamma| = \sum_{i=1}^M \gamma_i, \quad \mathbf{u}^\gamma = \prod_{i=1}^M (u_i)^{\gamma_i} \text{ for } \mathbf{u} \geq 0.$$

Assumptions for (1).

(A_1) There exists an exterior domain Ω_0 and $\theta \in (0, 1)$ such that $f_i \in C^\theta_{\mathrm{loc}}(\Omega_0 \times \mathbb{R}_+^M, \mathbb{R}_+)$, $i = 1, \ldots, M$, where $\mathbb{R}_+ = [0, \infty)$.

(A_2) $f_i(x, \mathbf{u})$ is continuously differentiable with respect to the components of $\mathbf{u}$ at each $x \in \Omega_0$, $\mathbf{u} \in \mathbb{R}_+^M$.

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(A_3) There exist positive constants A, R_0, a positive interval $I_0 = (0, \delta_0)$, multi-indices $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{iM}) > 0$ with $\gamma_{ii} > 1$, and locally Hölder continuous functions $g_i: [R_0, \infty) \longrightarrow (0, \infty)$ such that

$$g_i(|x|)\mathbf{u}^{\gamma_i} \le f_i(x, \mathbf{u}) \le Ag_i(|x|)\mathbf{u}^{\gamma_i}, \quad i = 1, \ldots, M \tag{2}$$

for all $|x| \ge R_0$, $\mathbf{u} \in I_0^M$.

THEOREM 1. *The system (1) has a minimal positive solution in some exterior subdomain of Ω_0 if and only if*

$$\int^{\infty} g_i(r) r^{N-1-|\gamma_i|(N-2)}\, dr < \infty \qquad \textit{for each } i = 1, \ldots, M. \tag{3}$$

PROOF. If $\mathbf{u}(x)$ is a minimal positive solution of (1) in an exterior domain, there exist positive constants C and R such that $u_i(x) \ge C|x|^{2-N}$ for all $|x| \ge R$, $i = 1, \ldots, M$. Then (1) and (2) show that u_i satisfies the inequality

$$-\Delta u_i(x) \ge C^{M-1} p_i(|x|)[u_i(x)]^{\gamma_{ii}}, \quad |x| \ge R, \quad i = 1, \ldots, M, \tag{4}$$

where

$$p_i(r) = g_i(r) r^{-(|\gamma_i| - \gamma_{ii})(N-2)}.$$

However, it is known [6, Theorem 12; 10, Theorem 1] that a necessary condition for a scalar inequality of type (4) to have a positive solution in an exterior domain in $\mathbb{R}^N$ is

$$\int^{\infty} p_i(r) r^{N-1-\gamma_{ii}(N-2)}\, dr < \infty, \quad i = 1, \ldots, M,$$

which is equivalent to (3). (The proof in [6] for $-\Delta u = f$ applies *verbatim* to $-\Delta u \ge f$).

Conversely, if (3) holds the scalar equation $-\Delta \phi_i = g_i(r)\phi_i^{|\gamma_i|}$ has a minimal positive solution $\phi_i(r)$ in some interval $[R, \infty)$ [9,10], and hence $\phi_j(r)/\phi_i(r)$ is bounded above and below in $[R, \infty)$ by positive constants, $i, j = 1, \ldots, M$. For a sufficiently small positive constant λ, it follows from (2) that the vector $\mathbf{v}$ with components $v_i = \lambda \phi_i$ satisfies

$$\begin{aligned} f_i(x, \mathbf{v}) &\le A\lambda^{|\gamma_i|} g_i(|x|)\phi_1^{\gamma_{i1}} \cdots \phi_M^{\gamma_{iM}}. \\ &\le (\text{Constant})\lambda^{|\gamma_i|} g_i(|x|)\phi_i^{|\gamma_i|} \\ &\le \lambda g_i(|x|)\phi_i^{|\gamma_i|} = -\Delta v_i, \quad |x| \ge R. \end{aligned}$$

Therefore $\mathbf{v}$ is a positive supersolution and $\mathbf{w} = 0$ is a subsolution of the boundary value problem

$$\begin{aligned} -\Delta u_i &= f_i(x, \mathbf{u}) \quad \text{for } |x| > R \\ u_i &= v_i \qquad\quad \text{on } |x| = R, \quad i = 1, \ldots, M. \end{aligned} \tag{5}$$

The method described by Kawano [3] and Kawano and Kusano [4] for systems in $\mathbb{R}^N$, and described in [7, p. 843] for exterior boundary value problems, shows that (5) has a

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nontrivial solution $\mathbf{u}$ such that $0 \leq u_i(x) \leq v_i(x) = \lambda\phi_i(x)$, $i = 1, \dots, M$. The proof by Sattinger's monotone iteration procedure is almost exactly as in [3, pp. 146–150] since (A_2) shows, for every bounded domain $B \subset \Omega_0$ and every $T > 0$, there exists a constant $K_i = K_i(B, T) > 0$ such that $f_i(x, \mathbf{u}) + K_i u_i$ is nondecreasing in $u_i \in [0, T]$ for all $x \in \bar{B}$, $\mathbf{u} \in T^M$, $i = 1, \dots, M$.

The strong maximum principle for $-\Delta u_i \geq 0$ implies that $u_i(x) > 0$ for $|x| \geq R$. Let $z(x) = A|x|^{2-N}$, where A is a positive constant satisfying $A < R^{N-2} \min_{|x|=R} u_i(x)$. Then

$$\begin{cases} -\Delta(u_i - z)(x) \geq 0 & \text{for } |x| > R \\ u_i(x) - z(x) > 0 & \text{on } |x| = R \\ u_i(x) - z(x) \longrightarrow 0 & \text{as } |x| \longrightarrow \infty, \end{cases}$$

and consequently $u_i(x) \geq z(x) = A|x|^{2-N}$ for all $|x| \geq R$ by the maximum principle. Hence $\mathbf{u}$ is the required minimal positive solution of (1).

COROLLARY 2. *Suppose that $g_i(r)$ in (2) is specialized to $g_i(r) = 0(r^{-b_i})$ as $r \longrightarrow \infty$ for a constant b_i satisfying $N - b_i < (N-2)|\gamma_i|$, $i = 1, \dots, M$. Then (1) has a positive solution with finite energy in a neighbourhood of infinity.*

PROOF. Since (3) holds, Theorem 1 shows that (1) has a positive solution $\mathbf{u}(x) = 0(|x|^{2-N})$ as $|x| \longrightarrow \infty$. By (2), each u_i can be regarded as a solution of Poisson's equation $-\Delta u_i = F_i$, where

$$F_i(x) = f_i\big(x, \mathbf{u}(x)\big) \leq C|x|^{-b_i-(N-2)|\gamma_i|} \leq C|x|^{-N}$$

for some positive constant C, $|x| \geq R \geq 1$. Then an *a priori* estimate [2, Theorem 3.9] for Poisson's equation in a ball $B_{r/2}(x)$ of centre x and radius $r/2$, $r = |x| \geq 2R$, yields

$$|(\nabla u_i)(x)| \leq C_1 \left[\frac{2}{r} \sup_{B_{r/2}} |u_i| + \frac{r}{2} \sup_{B_{r/2}} |F_i|\right] \leq C_2 r^{1-N}$$

for some constants C_1 and C_2, implying the conclusion of Corollary 2.

COROLLARY 3. *If $g_i(r)$ is bounded and $|\gamma_i| > N/(N-2)$, then (1) has a positive solution with finite energy in a neighbourhood of infinity.*

This follows by taking each $b_i = 0$ in Corollary 2.

THEOREM 4. *Suppose that each $g_i(r)$ is bounded in $[R_0, \infty)$ and that $|\gamma_i| < (N+2)/(N-2)$, $i = 1, \dots, M$. Then (3) is a necessary condition for (1) to have a positive solution with finite energy in a neighbourhood of infinity.*

PROOF. The function v defined by $v(x) = \sum_{i=1}^M u_i(x)$ solves a linear elliptic equation $-\Delta v = Hv$ in an exterior domain Ω, where by (2)

$$H(x) \leq C \sum_{i=1}^{M} [v(x)]^{|\gamma_i|-1} \tag{6}$$

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for some positive constant C. Since $|\gamma_i| - 1 < 4/(N-2)$, Hölder's inequality with exponents

$$p_i = \frac{4}{(N-2)(|\gamma_i|-1)} \qquad q_i = \frac{4}{4-(N-2)(|\gamma_i|-1)}$$

applied in a ball $B_r(x)$ of centre x and small radius r shows that there exists a constant $C_1 > 0$, independent of r and x, such that

$$\|H\|_{N/2,B_r(x)} \leq C_1 \sum_{i=1}^{M} r^{2/q_i} \|v\|^{|\gamma_i|-1}_{2N/(N-2),B_r(x)}.$$

Since $v \in L^{2N/(N-2)}(\mathbb{R}^N)$ from the Sobolev embedding $D_0^{1,2}(\Omega) \hookrightarrow L^{2N/(N-2)}(\Omega)$, it follows that $\|H\|_{N/2,B_r(x)} \longrightarrow 0$ as $r \longrightarrow 0$ uniformly in Ω. From results of Brezis and Kato [1, Remark 2.1 and Theorem 2.3], this implies that $v \in L^q(\Omega)$ for all $q \geq 2N/(N-2)$, from which the norms

$$\|v\|_{q,B_2(x)} \text{ and } \|Hv\|_{s,B_2(x)}, \tag{7}$$

for sufficiently large q and s, are bounded functions of x and have limits zero as $|x| \longrightarrow \infty$. Then $v(x)$ is bounded in Ω as a consequence of standard *a priori* estimates for the equation $-\Delta v = Hv$ [2, Theorem 8.17]. It follows that $\|v\|_{2,B_2(x)}$, as well as the norms (7), has limit zero as $|x| \longrightarrow \infty$. Interior Hölder estimates [2, Theorem 8.24] imply that $v(x) \longrightarrow 0$, and so also each $u_i(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$. Consequently the maximum principle for $-\Delta u_i \geq 0$ yields $u_i(x) \geq C|x|^{2-N}$ for $|x| \geq R$, where C and R denote positive constants. The proof of Theorem 1 can then be repeated to obtain (3).

THEOREM 5. *Suppose that g_i in (2) is specialized to $g_i(r) = 0(r^{-b_i})$ as $r \longrightarrow \infty$ and that*

$$\begin{cases} 1 < |\gamma_i| < \frac{N+2}{N-2} & \text{if } b_i \geq 2 \\ \frac{N+2-2b_i}{N-2} < |\gamma_i| < \frac{N+2}{N-2} & \text{if } 0 < b_i < 2, \end{cases} \tag{8}$$

$i = 1, \ldots, M$. Then a positive finite energy solution of (1) in a neighbourhood of infinity is necessarily minimal.

PROOF. Kelvin's transformation

$$y = \frac{x}{|x|^2}, \quad v_i(y) = |x|^{N-2} u_i(x), \quad i = 1, \ldots, M$$

maps (1) into

$$-\Delta v_i = H_i(y) v_i, \quad y \in \Omega' \tag{9}$$

where Ω' is a deleted neighbourhood of the origin and

$$H_i(y) = |y|^{-N-2} [v_i(y)]^{-1} f_i\Big(\frac{y}{|y|^2}, |y|^{N-2} v(y)\Big).$$

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Let $V(y) = \sum_{i=1}^{M} v_i(y)$, and use (2) to obtain

$$H_i(y) \leq C|y|^{\rho_i}[V(y)]^{|\gamma_i|-1}, \quad y \in \Omega', \tag{10}$$

for some constant $C > 0$, where

$$\rho_i = |\gamma_i|(N-2) - N - 2 + b_i.$$

A proof that $H_i \in L^s(\Omega')$ for some $s > N/2$, $i = 1, \ldots, M$, will be sketched below. Then a theorem of Serrin [8, p. 220] applied to (9) near $y = 0$ shows that either $v_i(y)$ or $|y|^{N-2}v_i(y)$ is bounded above and below by positive constants in a deleted neighbourhood of $y = 0$. However, $u_i(x)$ cannot be bounded below by a positive constant in an exterior domain by the finite energy hypothesis, and hence it must be that $|x|^{N-2}u_i(x)$ is bounded above and below by positive constants for sufficiently large $|x|$.

To show that $H_i \in L^s(\Omega')$ for $s > N/2$, we fix s satisfying

$$8 - 2b_i - \frac{2N}{s} < (N-2)(|\gamma_i| - 1) < \frac{2N}{s}, \tag{11}$$

which is possible by assumption (8). Define

$$p_i = \frac{2N}{s(N-2)(|\gamma_i|-1)}, \qquad q_i = \frac{2N}{2N - s(N-2)(|\gamma_i|-1)}$$

and apply Hölder's inequality to (10), giving

$$\|H_i\|_s^s \leq \|\,|y|^{s\rho_i}\|_{q_i} \|V\|_{2N/(N-2)'}^{s(|\gamma_i|-1)} \tag{12}$$

where $\|\ \|_s$ denotes the norm in $L^s(\Omega')$. The assumption $u_i \in D_0^{1,2}(\Omega)$ implies that $V \in D_0^{1,2}(\Omega')$, whence $V \in L^{2N/(N-2)}(\Omega')$ by Sobolev embedding. The left inequality (11) is equivalent to $s\rho_i q_i > -N$, and therefore (12) yields $H_i \in L^s(\Omega')$.

We remark, if $|\gamma_i| < (N - b_i)/(N-2)$ for some i, then a positive solution of (1) (of any type whatsoever) in any external domain cannot be minimal. Theorem 1 shows this since condition (3) fails in this case.

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