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GENERALIZED BLOCH MAPPINGS IN COMPLEX HILBERT SPACE

FLETCHER D. WICKER

1. Introduction. Anderson, Clunie and Pommerenke defined and studied the family of Bloch functions on the unit disc (see [**1**]). This family strictly contains the space $\mathscr{H}^\infty$ of bounded analytic functions. However, all Bloch functions are normal and therefore enjoy the "nice" properties of normal functions. The importance of the Bloch function concept is the combination of their richness as a family and their "nice" behavior.

This paper gives a generalization of the Bloch function concept and some basic properties of the generalized Bloch mappings. These generalized Bloch mappings are Fréchet holomorphic, defined on the unit ball in a complex Hilbert space and take values in another complex Hilbert space. The Caratheodory-Reiffen metric on the unit ball of a complex Hilbert space is utilized in the definition of Bloch mappings. Properties of the Caratheodory-Reiffen metric are discussed in Section 2.

As in the one dimensional case, the set of Bloch mappings forms a complex Banach space with respect to the Bloch norm. This is proven in Section 3. In Section 4, the main result is that closure of the subspace of homogeneous polynomials is characterized by the boundary behavior of the Fréchet derivative's norm. Together the results in Sections 3 and 4 extend Theorem 4 of [**4** page 457].

2. Preliminaries. Let W be a complex Hilbert space with inner product denoted by $(\, , \,)$ and norm $|| \; ||_W$. The unit ball in W is $B \equiv B(\theta, 1)$, where θ is the zero vector in W. Let V be another complex Hilbert space with norm $|| \; ||_V$.

For each nonnegative integer m, $\mathscr{L}(^mW; V)$ will denote the Banach space of all continuous m-linear mappings from W to V, where $\mathscr{L}(^0W; V) = V$. The norm of $A \in \mathscr{L}(^mW; V)$ is

$$(1) \qquad |||A||| \equiv \sup \frac{||A(w, \ldots, w)||_V}{||w||_W^m},$$

where the supremum is taken over all $w \neq \theta$.

A continuous m-homogeneous polynomial is a continuous mapping, $P : W \to V$, for which there is some symmetric $A \in \mathscr{L}(^mW; V)$ such that $P(w) = A \cdot (w, \ldots, w)$, for all $w \in W$. A finite sum of m-homogeneous poly-

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nomials is a continuous polynomial. The spaces of m-homogeneous polynomials and continuous polynomials are denoted by $\mathscr{P}({}^mW;\ V)$ and $\mathscr{P}(W;\ V)$, respectively.

A power series from W to V about $w_0 \in W$ is a series in $x \in W$ of the form $\sum_{m=0}^{\infty} P_m(x - w_0)$, where $P_m \in \mathscr{P}({}^mW;\ V)$ for each nonnegative integer m. The radius of convergence of the power series is the largest r, $0 \leqq r \leqq \infty$, such that the power series is uniformly convergent on every $\overline{B(w_0, \rho)}$, for $0 \leqq \rho < r$.

A standard proof (see [**5**, page 111, Theorem 3.16.2]) shows that a power series represents a continuous mapping within its ball of convergence. Thus a power series about θ with radius of convergence $r \geqq 1$ is a continuous mapping of B into V. Any continuous mapping of B into V, which has a power series representation at the origin with radius of convergence $r \geqq 1$, is called a holomorphic mapping. The set of all holomorphic mappings is $\mathscr{H}(B;\ V)$.

If $f \in \mathscr{H}(B;\ V)$, then for each $w \in B$, $f(w) = \sum_{m=0}^{\infty} P_m(w)$. Of course $P_m \in \mathscr{P}({}^mW;\ V)$ and the sequence $\{P_m\}_{m=0}^{\infty}$ depends on f. It is clear that for each $w_0 \in B$ there is a radius of convergence $r \geqq 1 - ||w_0||_W$ such that $f(w) = \sum_{m=0}^{\infty} P_m'(w - w_0)$ for $||w - w_0||_W < r$. Here $\{P_m'\}_{m=0}^{\infty}$ is a new sequence of m-homogeneous polynomials. The m-derivative of f at $w_0 \in B$ is defined to be $D_f^m(w_0) \equiv m!P_m'$. For each nonnegative integer m, $D_f^m(w) \in \mathscr{P}({}^mW;\ V)$. An important property of m-derivatives is the following (see [**5**, page 111, Theorem 3.16.3]).

LEMMA 1. (Cauchy Inequality) *Let $f \in \mathscr{H}(B;\ V)$, $r > 0$ and $w \in B$ be such that $\overline{B(w, r)}$ is contained in B. Then for each positive integer n,*

$$|||D_f^n(w)||| \leqq \frac{n!}{r^n} \sup ||f(x)||_V,$$

where the supremum is taken over all x such that $||x - w||_W = r$.

Let G be a bounded domain in W. D will denote the unit disc in the complex plane C. Let $\mathscr{H}(G;\ D)$ be the set of holomorphic mappings from G to D. The Caratheodory-Reiffen metric on G is defined by

$$\alpha_G(w, \zeta) \equiv \sup \{|D_f(w)\zeta|\,;\, f \in \mathscr{H}(G;\ D)\},$$

where $w \in G$, $\zeta \in W$ and $||$ denotes the usual norm in C. The distance between points w and w' in G is defined by

$$\rho_G(w, w') \equiv \inf \int_a^b \alpha_G(\gamma(t), D\gamma(t))dt,$$

where the infimum runs over all piecewise continuously differentiable curves, $\gamma : [\alpha, b] \rightarrow G$, with $\gamma(a) = w$ and $\gamma(b) = w'$.

G' is a bounded domain of another complex Hilbert space V. The following extensions of results by Earle and Hamilton ([**2**, page 62, Theorem 2]) are due to Hahn [**3**].

LEMMA 2. (Schwarz-Pick Lemma) *Let $f : G \to G'$ be a holomorphic mapping. Then*

$$\alpha_{G'}(f(w), D_f(w)\zeta) \leqq \alpha_G(w, \zeta)$$

and

$$\rho_{G'}(f(w), f(w')) \leqq \rho_G(w, w')$$

for all $w, w' \in G$ and $\zeta \in W$.

LEMMA 3. *Let B be the open unit ball in the complex Hilbert space W. Then for all $w \in B$, $\zeta \in W$,*

$$(1) \quad \alpha_B(w, \zeta) = \frac{\{(1 - ||w||_W{}^2)||\zeta||_W{}^2 + |(w, \zeta)|^2\}^{1/2}}{(1 - ||w||_W{}^2)}$$

and

$$\rho_B(w, w') = \tanh^{-1} \delta_B(w, w'),$$

where

$$\delta_B(w, w') = \frac{\{|(w, w')|^2 - ||w||_W{}^2||w'||_W{}^2 + ||w - w'||_W{}^2\}^{1/2}}{|1 - (w, w')|}.$$

This metric on B satisfies the following inequality:

$$(2) \quad \frac{||\zeta||_W}{(1 - ||w||_W{}^2)^{1/2}} \leqq \alpha_B(w, \zeta) \leqq \frac{||\zeta||_W}{(1 - ||w||_W{}^2)}.$$

3. The space $\mathscr{B}$(B; V). For each $w \in B$ and $f \in \mathscr{H}(B; V)$, let

$$Q_f(w) \equiv \sup_{\zeta \neq \theta} \frac{||D_f(w)\zeta||_V}{\alpha_B(w, \zeta)},$$

where $\zeta \in W$ and

$$N_f \equiv \sup_{w \in B} Q_f(w).$$

Definition 1. A mapping $f \in \mathscr{H}(B; V)$ is called a *Bloch mapping* if $N_f < \infty$.

Because $|| \ ||_V$ is a norm, $Q_f(w)$ has the following properties:

$$(3) \quad Q_{f_1+f_2}(w) \leqq Q_{f_1}(w) + Q_{f_2}(w)$$

and

$$(4) \quad Q_{\lambda f}(w) = |\lambda| Q_f(w)$$

for a given $\lambda \in C$ and holomorphic mappings f_1, f_2 and f in $\mathscr{H}(B; V)$. Taking the supremum over all vectors $w \in B$ in (3) and (4) yields

$$(5) \quad N_{f_1+f_2} \leqq N_{f_1} + N_{f_2}$$

and

$$(6) \quad N_{\lambda f} = |\lambda| N_f$$

respectively. It follows from (5) and (6) that the set of Bloch mappings is a linear subspace of $\mathscr{H}(B; V)$. This subspace is denoted by $\mathscr{B}(B; V)$ and the linear structure of $\mathscr{B}(B; V)$ will be investigated.

Definition 2. Let $f \in \mathscr{B}(B; V)$. Define the *Bloch norm* of f to be

$$||f||_{\mathscr{B}} \equiv ||f(\theta)||_V + N_f.$$

LEMMA 4. *$\mathscr{B}(B; V)$ is a normed linear space with respect to the Bloch norm.*

Proof. It is enough to prove that $||f||_{\mathscr{B}} = 0$ implies $f(w) = \theta$ for all $w \in B$. Clearly, $||f||_{\mathscr{B}} = 0$ implies $f(\theta) = \theta$ and $D_f(w) = 0$ for all $w \in B$. Therefore, $f(w)$ is a constant map and $f(w) = f(\theta) = \theta$.

LEMMA 5. *If $f \in \mathscr{H}^\infty$, then $f \in \mathscr{B}(B; V)$. Moreover,*

$$(7) \qquad ||f||_{\mathscr{B}} \leqq 2||f||_\infty.$$

Proof. If $f \in \mathscr{H}^\infty$, then $f : B \to B(\theta, M) = G$, where $M = ||f||_\infty$. It follows from Lemma 2 that for $w \in B$ and $\zeta \in W$

$$(8) \qquad \alpha_G(f(w), D_f(w)\zeta) \leqq \alpha_B(w, \zeta).$$

The left hand side of (8) has the form

$$\alpha_G(f(w), D_f(w)\zeta) = \frac{\{(M^2 - ||f(w)||_V^2)||D_f(w)\zeta||_V^2 + |(D_f(w)\zeta, f(w))^2|\}^{1/2}}{(M^2 - ||f(w)||_V^2)}.$$

Hence one has

$$\alpha_G(f(w), D_f(w)\zeta) \geqq \frac{||D_f(w)\zeta||_V}{(M^2 - ||f(w)||_V^2)^{1/2}}$$

which together with (8) yields

$$\frac{||D_f(w)\zeta||_V}{\alpha_B(w, \zeta)} \leqq M$$

for all $\zeta \in W$, $\zeta \neq \theta$ and $w \in B$. By Definition 1, (7) follows.

Under Bloch mappings, the size of the image of a set is restricted by the Bloch norm of the map.

THEOREM 1. *Let $f \in \mathscr{B}(B; V)$ and $0 \leqq r < 1$; then*

$$(9) \qquad \sup ||f(w)||_V \leqq ||f||_{\mathscr{B}}(1 + \tanh^{-1} r),$$

where the supremum is over all $||w||_W \leqq r$.

Proof. Let $\gamma : I \to B$ be any curve from θ to w, where $||w||_W \leqq r$. Then $f \circ \gamma : I \to V$ is a curve from $f(\theta)$ to $f(w)$. This implies

$$(10) \qquad ||f(w)||_V - ||f(\theta)||_V \leqq ||f(w) - f(\theta)||_V \leqq \int_0^1 ||d(f \circ \gamma)/dt||_V dt.$$

But $d(f \circ \gamma)/dt = D_f(\gamma(t))\gamma_* t$, where $\gamma_* t$ is the tangent vector to $\gamma(t)$. Using the definition of $Q_f(w)$ and N_f yields $||D_f(\gamma(t))\gamma_* t|| \leqq N_f \alpha_B(\gamma(t), \gamma_* t)$. Putting this into (10) yields

$$(11) \qquad ||f(w)||_V - ||f(\theta)||_V \leqq N_f L(\gamma),$$

where $L(\gamma)$ is the length of γ relative to the metric α_B. Taking the infimum of (11) over all piecewise continuously differentiable curves from θ to w, and applying Lemma 3 one has

$$||f(w)||_V - ||f(\theta)||_V \leqq N_f \tanh^{-1} r,$$

from which (9) is easily derived.

COROLLARY 1. *If the sequence of Bloch mappings, $\{f_n\}_{n=1}^\infty$ converges to the Bloch mapping f in the Bloch norm, then*

a) *$\{f_n\}_{n=1}^\infty$ converges to f uniformly on any closed ball $\overline{B(\theta, r)}$, $0 \leqq r < 1$, and*

b) *for each positive integer m and $0 \leqq r < 1$, $\lim_{n\to\infty} |||D_{f-f_n}{}^m(w)||| = 0$, uniformly for $||w||_W \leqq r$.*

Proof. a) It suffices to show that $\{f_n\}_{n=1}^\infty$ converges uniformly to f on $\overline{B(\theta, r)}$, for fixed $0 \leqq r < 1$. By Theorem 1, for all w in the ball $\overline{B(\theta, r)}$,

$$||f(w) - f_n(w)||_V \leqq ||f - f_n||_{\mathscr{B}}(1 + \tanh^{-1} r).$$

b) For $0 \leqq r < 1$, let $r < r' < 1$ and $\rho = (r' - r)/2$, then $\overline{B(w, \rho)} \subset B$ for all $||w||_W \leqq r$. By Cauchy's Inequality (Lemma 1),

$$|||D_{f-f_n}{}^m(w)||| \leqq \frac{m!}{\rho^m} \sup ||f(x) - f_n(x)||_V,$$

where the supremum is over the ball $||x||_W \leqq (r + r')/2$. The result follows from Part a).

THEOREM 2. *$\mathscr{B}(B; V)$ is a Banach space with respect to the Bloch norm.*

Proof. Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence of Bloch mappings. There is a constant J such that $||f_n||_{\mathscr{B}} \leqq J$ for all positive integers n. Hence by Theorem 1, for each $w_0 \in B$, if $||w_0||_W \leqq r < 1$,

$$||f_n(w_0)||_V \leqq J(1 + \tanh^{-1} r)$$

for all positive integers. Thus $\{f_n\}_{n=1}^\infty$ is locally uniformly bounded (see [**5**, page 113]). For fixed $w_0 \in B$ with $||w_0||_W \leqq r < 1$, Theorem 1 again implies

$$||f_n(w_0) - f_m(w_0)||_V \leqq ||f_n - f_m||_{\mathscr{B}}(1 + \tanh^{-1} r).$$

Hence $\{f_n(w_0)\}_{n=1}^\infty$ is a Cauchy sequence in V. Because V is complete the pointwise limit, $\lim_{n\to\infty} f_n(w_0) = f(w_0)$, exists for all $w_0 \in B$ and $f \in \mathscr{H}(B; V)$ by Theorem 3.18.1 of [**5**, page 113]. Furthermore with (9) of Theorem 1 it is easy to show that $f_n(w)$ converges to $f(w)$ uniformly on any closed ball $\overline{B(\theta; r)}$, $0 \leqq r < 1$.

Let $\epsilon > 0$. There exists a positive integer $N(\epsilon)$ such that $n, m \geqq N(\epsilon)$ implies $N_{f_n - f_m} < \frac{1}{2}\epsilon$. This means for each $n, m \geqq N(\epsilon)$, $||w||_W < 1$ and $\zeta \in W$, $\zeta \neq \theta$ one has

$$||D_{f_n - f_m}(w)\zeta||_V < \tfrac{1}{2}\epsilon\alpha_B(w, \zeta).$$

Also, by the uniform convergence on closed balls and the Cauchy inequality given in the proof of Theorem 3.17.1 of [**5** page 112], for arbitrary $r < 1$ there exists a positive integer $K(\epsilon, r)$ such that when $k \geqq K(\epsilon, r)$, $||w||_W \leqq r$ and $\zeta \in W$, $\zeta \neq \theta$, one has

$$||D_{f - f_k}(w)\zeta||_V < \tfrac{1}{2}\epsilon||\zeta||_W.$$

Using equation (2), the above inequality becomes

$$||D_{f - f_k}(w)\zeta||_V < \tfrac{1}{2}\epsilon(1 - ||w||_W{}^2)^{1/2}\alpha_B(w, \zeta) < \tfrac{1}{2}\epsilon\alpha_B(w, \zeta).$$

Hence for $n \geqq N(\epsilon)$, $||w||_W < 1$ and $\zeta \in W$, $\zeta \neq \theta$, there exists a positive integer $k > \max\,(N(\epsilon), K(\epsilon, ||w||_W))$ such that

$$||D_{f - f_n}(w)\zeta||_V \leqq ||D_{f - f_k}(w)\zeta||_V + ||D_{f_k - f_n}(w)\zeta||_V < \epsilon\alpha_B(w, \zeta).$$

Therefore for $n \geqq N(\epsilon)$, $N_{f - f_n} < \epsilon$.

The proof is completed by showing that $f \in \mathscr{B}(B; V)$. Because the sequence $\{||f - f_n||_{\mathscr{B}}\}_{n=1}^{\infty}$ converges to zero, the sequence $\{||f||_{\mathscr{B}} - ||f_n||_{\mathscr{B}}\}_{n=1}^{\infty}$ converges to zero. Therefore, $||f||_{\mathscr{B}} \leqq ||f_n||_{\mathscr{B}} + 1$ for a suitable large positive integer n.

4. The space $\mathscr{B}_0(B; V)$. All homogeneous polynomials are Bloch mappings and have an interesting property.

LEMMA 6. *$\mathscr{P}(B; V) \subset \mathscr{B}(B; V)$ and if $P \in \mathscr{P}({}^mB; V)$, then for all $w \in B$*

$$(12) \quad Q_P(w) \leqq \frac{m^{m+1}}{m!}\,(1 - ||w||_W{}^2)^{1/2}|||P|||.$$

Proof. There exists an $A \in \mathscr{L}({}^mW; V)$ such that $P(w) = A(w, \ldots, w)$ and $D_P(w) = mA(w, \ldots, w, .)$. So for $\zeta \neq \theta$,

$$\frac{||D_P(w)\zeta||_V}{\alpha_B(w, \zeta)} = \frac{m||A(w, \ldots, w, \zeta)||_V}{||\zeta||_W} \cdot \frac{||\zeta||_W}{\alpha_B(w, \zeta)}.$$

Taking the supremum over all $\zeta \neq \theta$ yields

$$Q_P(w) \leqq m \sup \frac{||A(w, \ldots, w, \zeta)||_V}{||\zeta||_W} \cdot \sup \frac{||\zeta||_W}{\alpha_B(w, \zeta)}.$$

Because $||w||_W < 1$ and using (2),

$$Q_P(w) \leqq (1 - ||w||_W{}^2)^{1/2} m \sup_{\zeta \neq \theta} \frac{||A(w, \ldots, w, \zeta)||_V}{||w||_W{}^{m-1}||\zeta||_W},$$

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The result is obtained by expanding the supremum as follows:

$$Q_P(w) \leqq (1 - ||w||_W{}^2)^{1/2} m \sup_{w_i \neq \theta} \frac{||A(w_1, \ldots, w_m)||_V}{||w_1||_W \ldots ||w_m||_W}$$

$$\leqq (1 - ||w||_W{}^2)^{1/2} \frac{m^{m+1}}{m!} |||P|||.$$

The last inequality is obtained from [**6**, page 7].

Property (12) of the homogeneous polynomials leads to an interesting idea.

Definition 3. The set of Bloch mappings f such that $\lim_{||w||_W \to 1} Q_f(w) = 0$ uniformly as $||w||_W \to 1$, $w \in B$, is denoted by $\mathscr{B}_0(B; V)$.

Lemma 6 implies that all homogeneous polynomials belong to $\mathscr{B}_0(B; V)$.

LEMMA 7. *The set $\mathscr{B}_0(B; V)$ is a closed subspace of $\mathscr{B}(B; V)$.*

Proof. That $\mathscr{B}_0(B; V)$ is a subspace is immediate from Definition 3. Let $\{f_n\}_{n=1}^\infty$ be a sequence in $\mathscr{B}_0(B; V)$ which converges to $f \in \mathscr{B}(B; V)$. For all $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ such that $n \geqq N(\epsilon)$ implies $N_{f-f_n} < \frac{1}{2}\epsilon$. Thus for each $||w||_W < 1$, $Q_f(w) < Q_{f_n}(w) + \frac{1}{2}\epsilon$. Fix $n_0 \geqq N(\epsilon)$, since $f_{n_0} \in \mathscr{B}_0(B; V)$, there exists an $r < 1$ such that $Q_{f_{n_0}}(w) < \frac{1}{2}\epsilon$ for $r < ||w||_W < 1$. Hence $Q_f(w) < \epsilon$.

LEMMA 8. *Let $\{f_n\}_{n=1}^\infty$ be a sequence in $\mathscr{B}_0(B; V)$ and $f \in \mathscr{B}(B; V)$. The sequence converges to f in the Bloch norm if and only if*

a) *$\{f_n\}_{n=1}^\infty$ converges to f uniformly on any closed ball in B, and*

b) *for each $\epsilon > 0$, there exists a positive integer $N(\epsilon)$ and an $0 \leqq r < 1$ such that for $n \geqq N(\epsilon)$ and $r \leqq ||w||_W < 1$, $Q_{f_n}(w) < \epsilon$.*

Proof. Suppose $\{f_n\}_{n=1}^\infty$ converges to f in the Bloch norm, then Corollary 1 implies a). Also for $\epsilon > 0$, there exists an $N(\epsilon)$ such that $n, m \geqq N(\epsilon)$ implies $N_{f_n - f_m} < \frac{1}{2}\epsilon$. Thus for all $||w||_W < 1$, $Q_{f_n}(w) < Q_{f_m}(w) + \frac{1}{2}\epsilon$. Fix $m' \geqq N(\epsilon)$, then since $f_{m'} \in \mathscr{B}_0(B; V)$, there exists an $r < 1$ such that for $r \leqq ||w||_W < 1$, $Q_{f_{m'}}(w) < \frac{1}{2}\epsilon$. The conclusion is $Q_{f_n}(w) < \epsilon$ for $n \geqq N(\epsilon)$ and $r \leqq ||w||_W < 1$.

Conversely, suppose a) and b) hold. An argument similar to the one in Lemma 7 shows $f \in \mathscr{B}_0(B; V)$. Let $\epsilon > 0$, there exists $N(\epsilon)$ and $r < 1$ such that $Q_{f_n}(w) < \frac{1}{2}\epsilon$ and $Q_f(w) < \frac{1}{2}\epsilon$ for $r \leqq ||w||_W < 1$ and $n \geqq N(\epsilon)$. By a) there exists $K(\epsilon, r)$ such that for $k \geqq K(\epsilon, r)$ and $||w||_W < r$, $Q_{f-f_k}(w) < \epsilon$. Thus, for $n \geqq \max(N(\epsilon), K(\epsilon, r))$ and $||w||_W < 1$, $Q_{f-f_n}(w) < \epsilon$. Hence, $N_{f-f_n} < \epsilon$ and $\{f_n\}_{n=1}^\infty$ converges to f in the Bloch norm.

THEOREM 3. *$\mathscr{B}_0(B; V)$ is a closed subspace of $\mathscr{B}(B; V)$ which is the closure of $\mathscr{P}(B; V)$.*

Proof. $\mathscr{B}_0(B; V)$ is closed by Lemma 7. Furthermore, since $\mathscr{P}(B; V) \subset \mathscr{B}_0(B; V)$, it suffices to show each $f \in \mathscr{B}_0(B; V)$ is the limit of a sequence of polynomials. To this end let $\{r_n = 1 - 1/n\}_{n=1}^\infty$ and define $T_n : B \to B$ to be

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$T_n(w) = r_n w$ for each positive integer n and $w \in B$. The sequence $\{f_n = f \circ T_n\}_{n=1}^{\infty}$ has elements which are holomorphic on $\bar{B}$. Thus, there is a polynomial $P_n \in \mathscr{P}(B; V)$ such that

$$\sup_{w \in \bar{B}} ||f_n(w) - P_n(w)||_V < 1/n$$

for each positive integer n. Now $\{P_n\}_{n=1}^{\infty}$ converges to f in the Bloch norm. Indeed, let $\epsilon > 0$, then there exists a positive integer $N(\epsilon)$ such that $n \geqq N(\epsilon)$ implies

$$\sup_{w \in \bar{B}} ||f_n(w) - P_n(w)||_V < \tfrac{1}{4}\epsilon$$

By Lemma 5, (7), applied to the mapping $f_n - P_n$, $||f_n - P_n||_{\mathscr{B}} < \frac{1}{2}\epsilon$, for all $n \geqq N(\epsilon)$. On the other hand clearly $\{f_n\}_{n=1}^{\infty}$ and f satisfy a) and b) of Lemma 8, so there exists a $K(\epsilon)$ such that $k \geqq K(\epsilon)$ implies $||f - f_k||_{\mathscr{B}} < \frac{1}{2}\epsilon$. Taking $k \geqq \max(N(\epsilon), K(\epsilon))$, one has

$$||f - P_k||_{\mathscr{B}} \leqq ||f - f_k||_{\mathscr{B}} + ||f_k - P_k||_{\mathscr{B}} < \epsilon,$$

which completes the proof.

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2306 Brant Street,
San Diego, California