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RINGS WITH DUAL CONTINUOUS RIGHT IDEALS

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Abstract

In this paper the structure of rings with dual continuous right ideals is discussed. The main result is the following: If R is a ring with (Jacobson) radical nil, and all of its finitely generated right ideals are dual continuous, then $R \simeq \left(\begin{smallmatrix} S & M \\ 0 & T \end{smallmatrix}\right)$ where S is a finite direct sum of local rings each of which has its radical square zero, or is a right valuation ring, T is semiprimary right semihereditary ring, and M is an (S, T)-bimodule such that all of its finitely generated T-submodules are projective. A partial converse of this result is obtained: any matrix ring of the above type with $M = 0$ has all of its finitely generated right ideals dual continuous.

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1. Introduction

Mohamed and Singh (1977) introduced the concept of dual continuous modules (for short d-continuous) modules as follows: A module M is called d-continuous if it satisfies the following conditions: (I) for every submodule A of M there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subset A$ and $M_2 \cap A$ is small in M and (II) every epimorphism from M onto a summand of M splits. Any quasi-projective module over a perfect ring is d-continuous but not conversely. Over arbitrary rings the relation between dual continuity and quasi-projectivity is less close. However d-continuous modules still possess many properties which are analogous to that of quasi-projective modules. The study of d-continuous modules was motivated to generalize a decomposition theorem for quasi-projective modules over perfect rings given by Koehler (1971). A decomposition theorem for d-continuous modules over arbitrary rings was obtained by Mohamed and Singh (1977) and was later improved by Mohamed and Müller (1979) as follows:

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THEOREM. *A d-continuous module M has a decomposition, unique up to isomorphism,* $M = \sum_{i \in I} \oplus A_i \oplus N$ *where each* A_i *is a local module and* $N = \text{Rad } N$.

It follows by the above theorem that a d-continuous module with small radical is a direct sum of local modules. In particular a finitely generated d-continuous module is a finite direct sum of local modules.

Jain and Singh (1975) generalized the concept of hereditary rings; they called a ring R right qp-ring if every right ideal of R is quasi-projective. Making an effective use of Koehler's decomposition theorem, they studied perfect qp-rings. Then Goel and Jain (1976) studied semiperfect qp-rings with nil radical. Having obtained the above decomposition theorem for d-continuous modules, here we discuss rings with d-continuous right ideals.

DEFINITION. A ring R in which every right ideal (resp. finitely generated right ideal) is d-continuous is called a right dc-ring (resp. right dcf-ring).

In the present work we study the structure of dc-rings and dcf-rings with nil radical. The structure of arbitrary dc-rings is still open.

All rings considered have unities and all modules are unital right modules. Rad M and Soc M will denote the Jacobson radical and socle of a module M respectively. For any ring R, Rad R_R will be denoted by $J(R)$ or simply J. A module M is local if Rad M is a maximal submodule. A ring R is local if R_R is a local module, that is R/J is a division ring. For the definitions and basic properties of semiperfect and semiprimary rings, we refer to Faith (1976). If X is a subset of a ring R, then $r(X)$ (resp. $l(X)$) will denote the right (resp. left) annihilator of X in R. For any ring R, Soc $R_R \subset l(J)$ and if R is local, then Soc $R_R = l(J)$. For definition and basic properties of quasi-projective modules we refer to Miyashita (1966) or Wu and Jans (1967).

2. Some general results

The following results about d-continuous modules are given in Mohamed and Singh (1977) and are listed here for easy reference.

THEOREM 2.1. *A ring R is (semi) perfect if and only if every (finitely generated) quasi-projective R-module is d-continuous.*

COROLLARY 2.2. *A ring R is semiperfect if and only if* R_R *is d-continuous.*

LEMMA 2.3. *Let A and B be submodules of a d-continuous module M such that $M = A + B$. Then there exist submodules A_0 and B_0 such that $A_0 \subset A$, $B_0 \subset B$ and $M = A_0 \oplus B_0$.*

LEMMA 2.4. *Let A and B be summands of a d-continuous module M. Then any exact sequence $A \xrightarrow{f} B \to 0$ splits. If in addition A is indecomposable and $B \neq 0$, then f is an isomorphism.*

LEMMA 2.5. *If $M \times M$ is d-continuous, then M is quasi-projective.*

PROPOSITION 2.6. *Let M be any module and A, B be two small submodules of M such that $M/A \oplus M/B$ is d-continuous, then $M/A \simeq M/B$.*

Next we prove

LEMMA 2.7. *Let $M = A + B$ be a d-continuous module. If A and B are indecomposable and noncomparable, then $A \cap B = 0$.*

PROOF. By Lemma 2.3, $M = A_0 \oplus B_0$ where $A_0 \subset A$ and $B_0 \subset B$. Since A is indecomposable $A_0 = 0$ or $A_0 = A$. However $A_0 = 0$ implies

$$A \subset M = B_0 \subset B,$$

a contradiction. Hence $A_0 = A$. Similarly $B_0 = B$. Hence $A \cap B = 0$.

The following is well known.

LEMMA 2.8. *If R is a right valuation ring with J nil, then any right ideal of R is two-sided.*

3. Main results

We first note that any dc-ring (or dcf-ring) is semiperfect by Corollary 2.2. This fact will be used without any further reference.

THEOREM 3.1. *The following are equivalent for a ring R with J nil:*

(i) *R is a right dcf-ring such that eRe is a division ring for every indecomposable idempotent e.*

(ii) *R is a semiprimary right semihereditary ring.*

PROOF. Assume (i). Let A be a right ideal of R such that A_R is local. (Such a right ideal will be called local right ideal). As R is semiperfect, there exists an indecomposable idempotent e of R with an R-epimorphism $f\colon eR \to A$. By Lemma 2.7 either $eR \subset A$ or $A \subset eR$ or $eR \cap A = 0$. If $eR \subset A$, then $eR = A$ since A is indecomposable. Let $A \subset eR$. Then $f(e) = exe \in eRe$. Since exe is a unit in eRe, we get $A = eR$. It remains to discuss the case when $eR \cap A = 0$. Since $eR \oplus A$ is d-continuous, $A \simeq eR$ by Lemma 2.4. This all shows that A is projective. Now let B be a finitely generated right ideal of R. By the decomposition theorem of d-continuous modules, B is a finite direct sum of local right ideals. Hence B is projective, and R is right semihereditary.

Let $R = e_1R \oplus \cdots \oplus e_nR$, for some orthogonal indecomposable idempotents e_i. We have shown that any local right ideal of R is isomorphic to some e_iR, $i = 1, \ldots, n$. Now, let C and D be distinct local right ideals of R such that $C \simeq D$. We claim that C and D are not comparable. On the contrary, assume that $C \subset D$. Then C is small in D. Let $D \simeq e_iR$. This yields a nonzero R-endomorphism ϕ of e_iR with $\phi(e_iR) \subset e_iJ$. Consequently $e_iJe_i \neq 0$, a contradiction. This proves our claim. Thus if k is the number of nonisomorphic indecomposable summands of R_R, then every ascending (or descending) chain of local right ideals of R contains at most k terms.

Assume that $J^{k+1} \neq 0$. Choose $x_1 \in J^{k+1}$ such that x_1R is a local right ideal. Now, $x_1 \in J^kJ$ implies $x_1 = b_1\alpha_1 + \cdots + b_t\alpha_t$, $b_i \in J^k$ and $\alpha_i \in J$. Since R is a dcf-ring

$$b_1R + \cdots + b_tR = A_1 \oplus \cdots \oplus A_m$$

where each A_i is a local right ideal contained in J^k. Then

$$x_1 = a_1\beta_1 + \cdots + a_m\beta_m$$

where $a_i \in A_i$ and $\beta_i \in J$. Let $a_j\beta_j \neq 0$. Then the mapping $x_1r \to a_j\beta_jr$ is an epimorphism from x_1R onto $a_j\beta_jR$. Since R is semihereditary, the epimorphism splits and as x_1R is indecomposable we get $x_1R \simeq a_j\beta_jR$. Hence x_1R is embedded properly in A_j. Let $A_j = x_2R$. Repeating the process we can find a local right ideal $x_3R \subset J^{k-1}$ such that x_2R is embedded properly in x_3R. Continuing, we get a strictly ascending chain of local right ideals with $k + 1$ terms, a contradiction. Hence $J^{k+1} = 0$ and R is semiprimary. Thus (i) implies (ii).

Conversely, let R be semiprimary right semihereditary. Obviously eRe is a division ring for any indecomposable idempotent e of R. Since every finitely generated projective module over a semiperfect ring is d-continuous by Theorem 2.1, we get R is a right dcf-ring.

THEOREM 3.2. *The following are equivalent for a ring R with J nil*:

(i) *R is a right dc-ring such that eRe is a division ring for every indecomposable idempotent e of R.*

(ii) *R is a semiprimary right hereditary ring.*

PROOF. Assume (i). By the above theorem, R is semiprimary and every local right ideal is projective. Let A be a right ideal of R. Since R is semiprimary, Rad A is small in A. Hence $A = \Sigma_{i\in I} \oplus A_i$ where each A_i is a local right ideal. Thus A is projective. Hence R is right hereditary and (ii) follows.

The converse is on similar lines as in Theorem 3.1.

LEMMA 3.3. *Let R be a right dcf-ring with J nil. If e is an indecomposable idempotent of R, then either $(eJe)^2 = 0$ or eRe is a right valuation ring.*

PROOF. The result is obvious if eRe is a division ring. Let $eJe \neq 0$. Assume that eRe is not a right valuation ring. Then there exist $a, b \in eRe$ such that $aeRe$ and $beRe$ are not comparable. Consequently aeR and beR are not comparable. Then $aeR \cap beR = 0$ by Lemma 2.7. Let $A = r(a) \cap eR$ and $B = r(b) \cap eR$. Since A and B are small submodules of eR and $eR/A \oplus eR/B$ is d-continuous, $eR/A \simeq eR/B$ by Lemma 2.6. Hence eR/A is quasi-projective. It follows by Wu and Jans (1967) that $eReA = A$. Similarly $eReB = B$. Thus $eR/A \simeq eR/B$ implies that $A = B$. Let exe be a nonzero element in eJe. There exist a nonnegative integer k such that $a(exe)^k \neq 0$ and $a(exe)^{k+1} = 0$. Now

$$a(exe)^k eR \cap beR \subset aeR \cap beR = 0.$$

Then, as proved above,

$$r\left(a(exe)^k\right) \cap eR = B = A.$$

Therefore $a(exe) = 0$. Hence $eJe \subset r(a)$. So that $aeRe$ is a minimal right ideal in the ring eRe.

Let S be the right socle of eRe. We have proved that S contains more than one minimal right ideal and eRe/S is a right valuation ring. We claim that $S = eJe$. On the contrary, let $c \in eJe - S$ and let $C = r(c)$ in eRe. If possible, assume that $S \subset C$. As $ceRe \simeq eRe/C$, the family of all right subideals of $ceRe$ is lineraly ordered by inclusion. However, this is a contradiction since $S \subset ceRe$ and S is not a minimal right ideal. Therefore $S \not\subset C$, and hence $C \subset S$. For any $b \in eRe - S$, $bR \underset{\neq}{\supset} S$. So that $beRe/C$ is not simple. Hence $\operatorname{Soc}(eRe \overset{\neq}{/} C) = S/C$. Now

$$S = \operatorname{Soc}(ceRe) \simeq \operatorname{Soc}(eRe/C) = S/C.$$

Thus $cS = S$. As c is nilpotent, we get $S = 0$, a contradiction. Hence $S = eJe$ and therefore $(eJe)^2 = 0$. This completes the proof.

THEOREM 3.4. *Let R be a local ring with J nil. Then R is a right dcf-ring if and only if*

(i) $J^2 = 0$, *or*

(ii) *R is a right valuation ring.*

PROOF. Necessity follows by the above lemma. Conversely, it is obvious that any local ring with $J^2 = 0$ is a right dcf-ring—in fact it has every proper right ideal semisimple. Assume that R is of type (ii). Let A be a finitely generated right ideal of R. Since R is a right valuation ring, $A = aR$ for some element $a \in R$. By Lemma 2.8, $r(a)$ is a two-sided ideal of R. Hence aR is quasi-projective by Wu and Jans (1967). Since R is semiperfect, A is d-continuous by Theorem 2.1. This completes the proof.

COROLLARY 3.5. *Any local right dcf-ring with J nil is a right dc-ring whenever* $J \neq \operatorname{Rad} J$.

PROOF. If $J^2 = 0$, the result is obvious. Let R be a right valuation ring with $J \neq \operatorname{Rad} J$. Let $x \in J - \operatorname{Rad} J$. As $\operatorname{Rad} J$ is a maximal submodule of J, we get $J = xR$. Hence R is a principal right ideal ring with descending chain condition. Hence R is a right dc-ring.

By Lemma 3.3 and Theorem 3.4 we have the following:

COROLLARY 3.6. *Let R be a right dcf-ring with J nil. If e is an indecomposable idempotent of R, then eRe is also a right dcf-ring.*

Next we prove

LEMMA 3.7. *Let e be an indecomposable idempotent in a right dcf-ring with J nil. If eR is not an ideal, then eRe is a division ring.*

PROOF. If eR is not an ideal, then there exists $x \in R$ such that $xeR \not\subset eR$. Since xeR is indecomposable, $eR \not\subset xeR$. Then $xeR \cap eR = 0$ by Lemma 2.7. Let $0 \neq eye \in eRe$. Then $xeR \cap eyeR = 0$.

$$xeR \cap eyeR \subset xeR \cap eR = 0.$$

It follows by Proposition 2.6 that

$$eyeR \simeq xeR \simeq eR.$$

This implies that eye is not nilpotent. Hence $eye \notin eJe$. Therefore $eJe = 0$, completing the proof.

The proof of the following lemma is straightforward.

LEMMA 3.8. *Let R be a finite direct sum of rings R_i, then R is a right dc-ring (or dcf-ring) if and only if each R_i is.*

THEOREM 3.9. *Let R be a right dcf-ring with J nil. Then $R \simeq \left(\begin{smallmatrix} S & M \\ 0 & T \end{smallmatrix}\right)$ where*

(i) *S is a finite direct sum of local rings each of which has square of its radical zero or is a right valuation ring.*

(ii) *T is a semiprimary right semihereditary ring.*

(iii) *M is an (S, T)-bimodule such that every finitely generated T-submodule of M is projective.*

PROOF. We can write

$$R = e_1R \oplus \cdots \oplus e_kR \oplus f_1R \oplus \cdots \oplus f_tR$$

where e_i and f_j are orthogonal indecomposable idempotents such that e_iRe_i is not a division ring and f_jRf_j is a division ring. By Lemma 3.7, each e_iR is an ideal. Let $e = e_1 + \cdots + e_k$. Then $1 - e = f_1 + \cdots + f_t$, and $R = eRe \oplus eR(1-e) \oplus (1-e)R(1-e)$.

Let $S = eRe$. Then $S = e_1Re_1 \oplus \cdots \oplus e_kRe_k$, and S is of type (i) by Corollary 3.6 and Theorem 3.4.

Let $T = (1-e)R(1-e) = (1-e)R$. It is obvious that each right ideal of the ring T is a right ideal of R. Hence T is a right dcf-ring. Also gTg is a division ring for every indecomposable idempotent g of T. Hence T is a semiprimary right semihereditary by Theorem 3.1.

Let $M = eR(1-e)$. Consider any finitely generated T-submodule A of M. Then $A = \sum_{i=1}^{m} ex_i(1-e)R$. Since A_R is d-continuous, $A = \sum \oplus A_i$ for some local R-modules A_i. Clearly each A_i is a homomorphic image of $(1-e)R$, and since $A_i \oplus (1-e)R$ is d-continuous, the epimorphism splits. Hence A_R is projective, and therefore A_T is projective.

Clearly $R \simeq \left(\begin{smallmatrix} S & M \\ 0 & T \end{smallmatrix}\right)$. This proves the theorem.

REMARKS. (1) The converse of the above theorem is not true. Let $F[x]$ be the ring of polynomials over a field F, with $x^3 = 0$. Let

$$R = \begin{pmatrix} F[x] & (x^2) \\ 0 & F \end{pmatrix}.$$

Then R satisfies the conditions mentioned in the above theorem.

Let

$$A = \begin{pmatrix} (x) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & (x^2) \\ 0 & 0 \end{pmatrix}.$$

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Then A and B are right ideals of R with $A \cap B = 0$. We have right R-epimorphism f: $A \to B$ defined by $f(x) = x^2$. This f does not split. Hence $A \oplus B$ is not d-continuous by Lemma 2.4. Thus R is not a right dcf-ring.

(2) In general $M \neq 0$. To see this let

$$R = \begin{pmatrix} F[x] & (x) \\ 0 & F \end{pmatrix}$$

where $F[x]$ is the ring of polynomials over a field F, with $x^2 = 0$. This ring is a right dc-ring with $S = F[x]$, $T = F$ and $M = (x) \neq 0$.

(3) $M = 0$ whenever, in addition, R is right continuous or R is a left dcf-ring.

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