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ALMOST CONVERGENCE, SUMMABILITY AND ERGODICITY

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1. Introduction. 1.1. The notion of almost convergence introduced by Lorentz [**15**] has been generalized in several directions (see, for example [**1**; **8**; **11**; **14**; **17**]). It is the purpose of this paper to give a generalization based on the original definition in terms of invariant means. This is effected by replacing the shift transformation by an "ergodic" semigroup $\mathscr{A}$ of positive regular matrices in the definition of invariant mean. The resulting "$\mathscr{A}$-invariant means" give rise to a summability method which we dub $\mathscr{A}$-almost convergence.

In section 3 we study the set $L(\mathscr{A})$ of $\mathscr{A}$-invariant means. We show that there is a sublinear functional P_+ on the space of bounded sequences such that each member of $L(\mathscr{A})$ may be regarded as a Hahn-Banach extension with respect to P_+ of the limit functional on the space of convergent sequences. This generalizes results in [**8**; **13**; **17**]. In section 4 we characterize the space of $\mathscr{A}$-almost convergent sequences, unifying and generalizing results in [**1**; **8**; **15**; **17**].

Matrix methods stronger than $\mathscr{A}$-almost convergence are characterized in section 5. We prove also that when $\mathscr{A}$ is suitably restricted such methods always exist. Section 6 is devoted to examples. Here we use a theorem of Eberlein [**9**] to show that the logarithmic method contains the collective Hausdorff method for bounded sequences.

In section 7 we generalize results in [**3**] and [**4**] concerning the multiplicative behaviour of the $\mathscr{A}$-almost convergent sequences. Section 8 is concerned with matrix transformations of spaces of almost convergent sequences (cf. [**19**]).

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2. Preliminaries. 2.1. We denote by $\mathscr{M}$ the Banach spaces of all bounded sequences of real number $x = (x_0, x_1, \ldots)$ with norm

$$||x|| = \sup_n |x_n|.$$

Each member of $\mathscr{M}$ has a continuous extension to a function on $\beta\mathbf{N}$, the Stone-Čech compactification of the nonnegative integers. The space $C(\beta\mathbf{N})$ of all continuous real-valued functions on $\beta\mathbf{N}$ is thus naturally isomorphic to $\mathscr{M}$. We shall therefore not distinguish between these spaces in the sequel.

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For each $x \in \mathscr{M}$ we write lim sup x (respectively lim inf x) for $\limsup_n x_n$ (respectively, $\liminf_n x_n$). If $x \in c$, the space of convergent sequences, we write lim x for $\lim_n x_n$. We denote by c_0 the space of all sequences which converge to 0. If $x \in \mathscr{M}$, $|x|$ denotes the sequence defined by $|x|_n = |x_n|$.

2.2 For $x \in \mathscr{M}$, let $\tilde{x}$ denote the restriction of x to $\beta\mathbf{N}\backslash\mathbf{N}$. The map $x \mapsto \tilde{x}$ is a continuous surjection of $\mathscr{M}$ onto $C(\beta\mathbf{N}\backslash\mathbf{N})$. If we denote the natural norm in $C(\beta\mathbf{N}\backslash\mathbf{N})$ by $|| \quad ||_0$ then we have $||\tilde{x}||_0 = \limsup |x|$.

Now let A be a positive regular matrix. A may be thought of as a bounded linear operator on $\mathscr{M}$. Since A is regular $A(c_0) \subseteq c_0$. Hence A induces an operator $\tilde{A}$ on $C(\beta\mathbf{N}\backslash\mathbf{N})$ defined by $\tilde{A}\tilde{x} = \widetilde{Ax}$. It has the following easily verifiable properties.

(2.2.1) $\tilde{A} \geqq 0$ (i.e. $\tilde{A}\tilde{x} \geqq 0$ whenever $\tilde{x} \geqq 0$)

(2.2.2) $\tilde{A}\tilde{u} = \tilde{u}$ where $u = (1, 1, \ldots)$

(2.2.3) $||\tilde{A}||_0 = 1$

2.3. A functional $\phi \in \mathscr{M}^*$ is called a mean if $\phi(u) = 1$, $\phi \geqq 0$ (i.e., $\phi(x) \geqq 0$ whenever $x \geqq 0$). By the Riesz Representation Theorem we may think of a mean as a probability measure on $\beta\mathbf{N}$. We shall denote by L the set of all means supported in $\beta\mathbf{N}\backslash\mathbf{N}$. Again by the Riesz Representation Theorem we may think of L as contained in $C(\beta\mathbf{N}\backslash\mathbf{N})^*$. The viewpoint we adopt will be dictated by convenience.

3. $\mathscr{A}$-invariant means. 3.1. Let $\mathscr{A}$ be a semigroup of positive regular matrices. Multiplication can be thought of either as matrix multiplication or as composition of operators on $\mathscr{M}$.

Definition. An $\mathscr{A}$-invariant mean is an element ϕ of L satisfying $\phi(x) = \phi(Ax)$ for all $A \in \mathscr{A}$ and $x \in \mathscr{M}$. We denote by $L(\mathscr{A})$ the (possibly empty) set of all $\mathscr{A}$-invariant means. $\mathscr{A}$ is said to be admissible if $L(\mathscr{A}) \neq \emptyset$.

3.2. In order to develop a theory with content we shall place certain regularity restrictions on the semigroup $\mathscr{A}$. Specifically, we shall usually demand the $\mathscr{A}$ be ergodic in the following sense:

Definition. $\mathscr{A}$ is ergodic if there is a net $\{A_\alpha\}$ of matrix operators on $\mathscr{M}$, called a system of averages for $\mathscr{A}$, which satisfies:

(3.2.1) for each α and each $x \in \mathscr{M}$, $\tilde{A}_\alpha\tilde{x}$ is in the closed convex hull (in $C(\beta\mathbf{N}\backslash\mathbf{N})$) of the set $\{\tilde{A}\tilde{x} | A \in \mathscr{A}\}$, and

(3.2.2) $\lim_\alpha ||\tilde{A}_\alpha\tilde{x} - \tilde{A}_\alpha\tilde{A}\tilde{x}||_0 = 0$ for each $A \in \mathscr{A}$ and $x \in \mathscr{M}$.

Our definition is derived from the definition of an ergodic semigroup given in [**7**]. It follows that if $\mathscr{A}$ is abelian or, more generally amenable, then $\mathscr{A}$ is ergodic [**5**].

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We assume for the rest of this section that $\mathscr{A}$ is ergodic.

3.3. We wish to characterize the set $L(\mathscr{A})$. As a first step in this direction let l denote the functional defined on c by $l(x) = \lim x$. It will turn out that there is a sublinear functional defined on $\mathscr{M}$ such that $L(\mathscr{A})$ consists precisely of all the Hahn-Banach extensions of l with respect to this functional.

Definition. For $x \in \mathscr{M}$ we define

$$P_+(x) = \lim_\alpha \sup \lim \sup A_\alpha x$$

$$P_-(x) = \lim_\alpha \inf \lim \inf A_\alpha x.$$

We summarize the elementary properties of P_+ and P_- in the following

LEMMA.

(3.3.1) $P_\pm(x) \geqq 0$ *if* $x \geqq 0$,

(3.3.2) $P_\pm(ax) = aP_\pm(x)$ *if* $a \geqq 0$, $a \in \mathbf{R}$,

(3.3.3) $P_+(x + y) \leqq P_+(x) + P_+(y)$,

(3.3.4) $P_\pm(x - Ax) = 0$ *for all* $A \in \mathscr{A}$, $x \in \mathscr{M}$, *and*

(3.3.5) $P_+(x) = -P_-(-x)$.

Proof. We prove only 3.3.4.

$$|P_+(x - Ax)| \leqslant \lim_\alpha \sup \lim \sup |A_\alpha x - A_\alpha Ax|$$

$$= \lim_\alpha \sup ||\tilde{A}_\alpha \tilde{x} - \tilde{A}_\alpha \tilde{A} \tilde{x}||_0$$

$$= 0$$

by 3.2.2. Similarly, $P_-(x - Ax) = 0$.

3.4. THEOREM. *Let* $\phi \in \mathscr{M}^*$. *Then* $\phi \in L(\mathscr{A})$ *if and only if* $P_-(x) \leqq \phi(x) \leqq P_+(x)$ *for each* $x \in \mathscr{M}$.

Proof. Suppose that $\phi \in L(\mathscr{A})$. If $x \in \mathscr{M}$, then $\phi(x) = \phi(A_\alpha x)$ because by 3.2.1 $\tilde{A}_\alpha \tilde{x}$ is a uniform limit of convex combinations of functions of the form $\tilde{A}\tilde{x}$, $A \in \mathscr{A}$ and ϕ is supported in $\beta\mathbf{N}\backslash\mathbf{N}$. Moreover since ϕ is a mean

$$\phi(A_\alpha x) \leqq \phi(\sup \{\tilde{A}\tilde{x}(t) | t \in \beta\mathbf{N}\backslash\mathbf{N}\} \cdot u) = \lim \sup A_\alpha x.$$

Since α is arbitrary we conclude that $\phi(x) \leqq P_+(x)$. 3.3.5 now shows that $P_-(x) \leqq \phi(x)$.

Suppose conversely that $\phi \in \mathscr{M}^*$ and $P_-(x) \leqq \phi(x) \leqq P_+(x)$ for all $x \in \mathscr{M}$. If $x \geqq 0$ we have $0 \leqq P_-(x) \leqq \phi(x)$. Hence $\phi \geqq 0$. Since $P_\pm(u) = 1$, ϕ is a mean. The $\mathscr{A}$-invariance of ϕ follows from 3.3.4. That ϕ is supported in

$\beta\mathbf{N}\backslash\mathbf{N}$ is a consequence of the easily verified fact that $P_{\pm}(x) = 0$ if $x \in c_0$. Hence $\phi \in L(\mathscr{A})$.

Several convenient corollaries follow from the preceding theorem.

3.5. COROLLARY. *$L(\mathscr{A})$ consists exactly of all the Hahn-Banach extensions of l to $\mathscr{M}$ with respect to P_+. In particular, $L(\mathscr{A})$ is not empty.*

3.6. COROLLARY. (a) $P_+(x) = \lim_\alpha \limsup A_\alpha x = \inf_\alpha \limsup A_\alpha x$.
(b) $P_-(x) = \lim_\alpha \liminf A_\alpha x = \sup_\alpha \liminf A_\alpha x$.

Proof. We prove only (a). It suffice to show that $P_+(x) = \inf_\alpha \limsup A_\alpha x$. The inequality $\geqq$ being obvious, we need only show that $P_+(x) \leqq \limsup A_\alpha x$ for each α. This follows from the proof of Theorem 3.4 and Corollary 3.7.

3.7. COROLLARY. *For each $x \in \mathscr{M}$, $P_+(x) = \sup \{\phi(x) | \phi \in L(\mathscr{A})\}$ and $P_-(x) = \inf \{\phi(x) | \phi \in L(\mathscr{A})\}$.*

Proof. If $\phi \in L(\mathscr{A})$ then by Theorem 3.4 $\phi(x) \leqq P_+(x)$ so $P_+(x) \geqq \sup \{\phi(x) | \phi \in L(\mathscr{A})\}$. However, the Hahn-Banach Theorem and Corollary 3.5 imply that for given x we may choose $\phi \in L(\mathscr{A})$ such that $\phi(x) = P_+(x)$. This proves the first half of the corollary. The second half is proved similarly.

The idea of the above proof is found in [**8**].

4. $\mathscr{A}$-almost convergence.

4.1. Let $\mathscr{A}$ be an admissible semigroup of positive regular matrices.

Definition. $x \in \mathscr{M}$ is said to be $\mathscr{A}$-almost convergent to a real number a if $\phi(x) = a$ for all $\phi \in L(\mathscr{A})$. This is written $F(\mathscr{A}) - \lim x = a$ or simply $F - \lim x = a$ when the semigroup is clear from the context. The space of all $\mathscr{A}$-almost convergent sequences is denoted by $F(\mathscr{A})$ or simply F.

It is clear that $F = F_0 \oplus \mathbf{R}u$ where F_0 consists of all sequences which are $\mathscr{A}$-almost convergent to 0 and $\mathbf{R}$ is the set of real numbers. F_0 is a closed subspace of $\mathscr{M}$ since it is an intersection of kernels of continuous linear functionals. Hence F is closed as well. Note also that both F and F_0 are invariant under $\mathscr{A}$.

If $x \in \mathscr{M}$ is summable by some matrix $A \in \mathscr{A}$, say $\lim Ax = a$, then $F(\mathscr{A}) - \lim x = a$ as is easily seen. Hence the matrices in $\mathscr{A}$ are consistent for bounded sequences.

4.2. The archetypal example of $\mathscr{A}$-almost convergence is, of course, almost convergence itself. Here $\mathscr{A}$ consists of the iterates of the shift matrix S defined by $(Sx)_m = x_{m+1}$. Another example is Banach-Hausdorff summability [**8**]. These and other examples will be discussed more fully below.

4.3. Now suppose that $\mathscr{A}$ is ergodic. We are in a position to characterize the $\mathscr{A}$-almost convergent sequences. If $E \subseteq \mathscr{M}$, let $\overline{\mathrm{sp}}\, E$ denote the closed span of E. If $E \subseteq C(\beta\mathbf{N}\backslash\mathbf{N})$ let $\overline{\mathrm{sp}}^0 E$ denote its closed span in $C(\beta\mathbf{N}\backslash\mathbf{N})$.

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THEOREM. *Let $a \in \mathbf{R}$. The following conditions on a sequence $x \in \mathscr{M}$ are equivalent.*

(4.3.1) $F - \lim x = a$.

(4.3.2) $\lim_\alpha \|\tilde{A}_\alpha \tilde{x} - a\tilde{u}\|_0 = 0$.

(4.3.3) $P_+(x) = P_-(x) = a$.

(4.3.4) $\tilde{x} \in \overline{\text{sp}}^0\{\tilde{y} - \tilde{A}\tilde{y} | y \in \mathscr{M}, A \in \mathscr{A}\} + a\tilde{u}$.

(4.3.5) $x \in \overline{\text{sp}}(c_0 + \{y - Ay | y \in \mathscr{M}, A \in \mathscr{A}\}) + au$.

Proof. The equivalence of the first three conditions follows immediately from the definitions of $F - \lim$ and $P_\pm$ and from Corollary 3.7. For the remainder of the proof we assume without loss of generality that $a = 0$.

Suppose now that 4.3.2 is satisfied, i.e. suppose that $\tilde{A}_\alpha \tilde{x} \to 0$ in the norm of $C(\beta\mathbf{N} \backslash \mathbf{N})$. Then $\tilde{x} = \lim_\alpha (\tilde{x} - \tilde{A}_\alpha \tilde{x})$ so that 4.3.4 will follow if we can show that for each α, $\tilde{x} - \tilde{A}_\alpha \tilde{x} \in \overline{\text{sp}}^0\{\tilde{y} - \tilde{A}\tilde{y} | y \in \mathscr{M}, A \in \mathscr{A}\}$. By 3.2.1, $\tilde{A}_\alpha \tilde{x}$ is a limit of convex combinations of images of $\tilde{x}$ under members of $\mathscr{A}$. If $\sum \alpha_i \tilde{A}_i \tilde{x}$ is such a combination then

$$\tilde{x} - \sum \alpha_i \tilde{A}_i \tilde{x} = \sum \alpha_i (\tilde{x} - \tilde{A}_i \tilde{x}) \in \overline{\text{sp}}^0\{\tilde{y} - \tilde{A}\tilde{y} | y \in \mathscr{M}, A \in \mathscr{A}\}.$$

This proves 4.3.4.

To prove that 4.3.4 implies 4.3.5 suppose that

$$\tilde{x} \in \overline{\text{sp}}^0\{\tilde{y} - \tilde{A}\tilde{y} | y \in \mathscr{M}, A \in \mathscr{A}\}.$$

It will suffice to produce a sequence in $c_0 + \text{sp}\{y - Ay | y \in \mathscr{M}, A \in \mathscr{A}\}$ which converges weakly to x. (Here sp denotes linear span.)

Now clearly there is a sequence $\{w^n\} \subseteq \text{sp}\,\{y - Ay | y \in \mathscr{M}, A \in \mathscr{A}\}$ such that $\|\tilde{x} - \tilde{w}^n\|_0 \to 0$ and we can assume that $\|\tilde{w}^n\|_0 \leqq \|x\| + 1/2$.

For each n we can choose an integer K_n such that $|w_k^n| \leqq \|x\| + 1$ whenever $k \geqq K_n$. Further, $\{K_n\}$ can be chosen to increase to infinity.

Define z^n by

$$z_k^n = \begin{cases} x_k - w_k^n, & \text{if } k < K_n \\ 0, & \text{if } k \geqq K_n. \end{cases}$$

Clearly $z^n \in c_0$ for each n.

Our object is to show that $z^n + w^n$ converges to x weakly in $\mathscr{M}$. For this we must prove that $\{z^n + w^n\}$ is uniformly bounded and converges to x pointwise on $\beta\mathbf{N}$.

Now

$$|z_k^n + w_k^n| = \begin{cases} |x_k|, & \text{if } k < K_n \\ |w_k^n|, & \text{if } k \geqq K_n. \end{cases}$$

Hence $\|z^n + w^n\| \leqq \|x\| + 1$ for all n.

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Let $t \in \beta\mathbf{N}\backslash\mathbf{N}$ be fixed. Then $z^n(t) + w^n(t) = w^n(t) \to x(t)$. If $k \in \mathbf{N}$ is fixed, choose N so that $K_n \geqq k$ whenever $n \geqq N$. Then $z_k^n + w_k^n = x_k$ for $n \geqq N$. It follows that $z^n + w^n \to x$ pointwise on $\beta\mathbf{N}$.

To see that 4.3.5 implies 4.3.1, let $\phi \in L(\mathscr{A})$. $\phi(c_0) = \{0\}$ because ϕ is supported in $\beta\mathbf{N}\backslash\mathbf{N}$. Also $\phi(y - Ay) = 0$ for all $y \in \mathscr{M}$ and $A \in \mathscr{A}$. Linearity and continuity of ϕ now imply that $\phi(x) = 0$. Since ϕ is arbitrary, $F - \lim x = 0$.

4.4. *Remark.* It is clear that in 4.3.4 and 4.3.5 we may replace $\mathscr{A}$ by any subset of $\mathscr{A}$ which generates it as a semi-group.

5. Matrices containing $\mathscr{A}$-almost convergence. 5.1. *Definition.* Suppose that $\mathscr{A}$ is admissible and B is a regular matrix. B is said to contain $\mathscr{A}$-almost convergence if $c_B \supseteq F$ where $c_B = \{x \in \mathscr{M} | Bx \in c\}$ is the bounded convergence field of B.

5.2. THEOREM. *Let $\mathscr{A}$ be ergodic. The regular matrix B contains $\mathscr{A}$-almost convergence if and only if each matrix of the form $B(I - A)$, $A \in \mathscr{A}$, maps $\mathscr{M}$ into c_0. Moreover, in this case* $\lim Bx = F(\mathscr{A}) - \lim x$ *for each $x \in F(\mathscr{A})$.* (I is the identity matrix.)

Remark. It will be clear from the proof and Remark 4.4 that it suffices to consider only those matrices $B(I - A)$ where A lies in some set of generators for $\mathscr{A}$.

5.3. Recall that a matrix is a Schur matrix if it maps $\mathscr{M}$ into c. We shall need the following elementary facts about Schur matrices (see [**16**]).

(5.3.1) If $D = (d_{m,n})$ is a Schur matrix, then $\lim_m d_{m,n} = \delta_n$ exists for each n and $\lim Dx = \sum_{n=0}^{\infty} \delta_n x_n$ for each $x \in \mathscr{M}$.

(5.3.2) A matrix D maps $\mathscr{M}$ into c_0 if and only if

$$\lim_m \sum_{n=0}^{\infty} |d_{m,n}| = 0.$$

5.4. We recall also the fact that a matrix D maps c_0 into itself if and only if

(5.4.1) $||D|| = \sup_m \sum_{n=0}^{\infty} |d_{m,n}| < +\infty$, and

(5.4.2) $\lim_m d_{m,n} = 0$ for each n.

5.5. *Proof of Theorem* 5.2. Suppose that $c_B \supseteq F$. Since $x - Ax \in F_0$ for each x by 4.3.5, $B(I - A)(\mathscr{M}) \subseteq c$, i.e. $B(I - A)$ is a Schur matrix. Now $B(I - A)(c_0) \subseteq c_0$ because B, A and I are regular. Hence $\lim_m \{B(I - A)\}_{m,n} = 0$ for each n by 5.4.2. It follows from 5.3.1 that $B(I - A)(\mathscr{M}) \subseteq c_0$.

Suppose conversely that $B(I - A)(\mathscr{M}) \subseteq c_0$ for each $A \in \mathscr{A}$. Since B is regular it maps $c_0 + \{x - Ax | x \in \mathscr{M}, A \in \mathscr{A}\}$ into c_0. It follows from 4.3.5 and the fact that B is a bounded operator on $\mathscr{M}$ that $B(F_0) \subseteq c_0$. Hence

$B(F) \subseteq c$ and $F - \lim x = \lim Bx$ for all $x \in F$ since $F = F_0 \oplus \mathbf{R}u$ and B is regular.

Remark. It is an immediate consequence of Theorem 5.2 and 5.3.2 that B sums every almost convergent sequence if and only if B is strongly regular. This is a theorem of Lorentz [**15**]. (Cf. also [**18**; **1**].)

5.6. Theorem 5.2 gives a necessary and sufficient condition for a regular matrix to contain $\mathscr{A}$-almost convergence. However, it leaves unanswered the question of whether there are any such matrices. We have been unable to answer this question in general. We give a partial answer in the following

THEOREM. *Let A be a positive regular matrix and assume that A is triangular, i.e., $a_{m,n} = 0$ when $m < n$. Let $\mathscr{A} = \{A^n | n = 0, 1, \ldots\}$. Then there is a regular matrix B which contains $\mathscr{A}$-almost convergence. Moreover B may be chosen to be positive.*

Remark. We must produce a matrix B such that $B(I - A)(\mathscr{M}) \subseteq c_0$. We shall assume without loss of generality that $\sum_{n=0}^{\infty} a_{m,n} \leqq 1$, $a_{m,m} < 1$ for each m because A differs from such a matrix only by a matrix which maps $\mathscr{M}$ into c_0.

5.7. The proof of the theorem will be preceded by a definition and two lemmas.

Definition. A matrix D satisfies condition (K) if

(5.7.1) $d_{m,n} = 0$ for $m < n$,

(5.7.2) $d_{m,m} > 0$ for all m, and

(5.7.3) $d_{m,n} \leqq 0$ for $0 \leqq n < m$.

LEMMA. *If D satisfies condition* (K) *then D^{-1} is positive.*

Proof. It suffices to prove the lemma for finite (square) matrices. We do this by induction on the size of the matrix. The lemma is obvious for 1×1 matrices.

Now suppose the lemma is true for $k \times k$ matrices. Let D be a $(k + 1) \times (k + 1)$ matrix satisfying (K). Let $C = D^{-1}$. Then C is triangular. Since the matrix obtained from C by deleting the last row and last column is the inverse of the matrix obtained from D by deleting the last row and last column and this latter matrix is a $k \times k$ matrix satisfying (K), it follows that $c_{i,j} \geqq 0$ for $i = 0, 1, \ldots, k - 1$ and all j. Thus, it remains to show that $c_{k,j} \geqq 0$ for $j = 0, 1, \ldots, k$.

If $j = k$, $c_{k,j} = d_{k,k}^{-1} > 0$.

Suppose that $j < k$. Then

$$0 = (DC)_{k,j} = \sum_{i=0}^{k} d_{k,i} c_{i,j} = \sum_{i=j}^{k} d_{k,i} c_{i,j}.$$

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Hence

$$d_{k,k}c_{k,j} = -\sum_{i=j}^{k-1} d_{k,i}c_{i,j}.$$

But for $j \leqq i \leqq k - 1$, $c_{i,j} \geqq 0$ by induction while $d_{k,i} \leqq 0$ since D satisfies condition (K). Hence $d_{k,k}c_{k,j} \geqq 0$ and since $d_{k,k} > 0$, $c_{k,j} \geqq 0$.

5.8. Now let $C = (I - A)^{-1}$. $I - A$ satisfies condition (K) so $C \geqq 0$ by Lemma 5.7.

LEMMA. $||C|| = \infty$.

Proof. Suppose $||C||$ were finite. Then $I - A$ would be an invertible transformation on $\mathscr{M}$. By 4.3.5 we would have $u \in F_0$. This is absurd.

5.9. *Proof of Theorem* 5.6. We claim first that for each $k = 1, 2, \ldots c_{n+k,n} \leqq c_{n,n}$. For $k = 1$ we have

$$c_{n+1,n} = \frac{a_{n+1,n}}{1 - a_{n+1,n+1}} c_{n,n} \leqslant c_{n,n}$$

because $a_{n+1,n} \leqq 1 - a_{n+1,n+1}$. Assuming the claim is true for $c_{n+1,n}, \ldots, c_{n+k-1,n}$, we have

$$c_{n+k,n} = \frac{1}{1 - a_{n+k,n+k}} \{a_{n+k,n}c_{n,n} + \ldots + a_{n+k,n+k-1} \cdot c_{n+k-1,n}\}$$
$$\leqslant \frac{1}{1 - a_{n+k,n+k}} \{a_{n+k,n} + \ldots + a_{n+k,n+k-1}\} c_{n,n} \leqslant c_{n,n},$$

proving the claim.

Set $\gamma_m = \sum_{n=0}^{\infty} c_{m,n}$. By Lemma 5.8 there is an increasing sequence $\{m_k\}$ of positive integers such that $\gamma_{m_k} \to \infty$. Define a matrix B by $b_{k,n} = c_{m_k,n}/\gamma_{m_k}$. Then clearly $B \geqq 0$, $\sum_{n=0}^{\infty} b_{k,n} = 1$ for every k and $\lim_k b_{k,n} = 0$ for every n by the claim and the choice of m_k. Thus B is a positive regular matrix. Moreover

$$\sum_{n=0}^{\infty} |\{B(I - A)\}_{k,n}| = 1/\gamma_{m_k} \to 0.$$

By the remark following Theorem 5.2 and 5.3.2, B contains $\mathscr{A}$-almost convergence.

5.10. Before giving some examples we draw a corollary from Theorem 5.6.

COROLLARY. *If $\mathscr{A}$ is generated by a single triangular matrix, then $L(\mathscr{A})$ is infinite dimensional.*

More generally, if $\mathscr{A}$ is an admissible semigroup such that there is a regular matrix containing $\mathscr{A}$-almost convergence, then $L(\mathscr{A})$ is infinite dimensional.

Proof. By Theorem 5.6, we need only prove the second statement. If $L(\mathscr{A})$ were finite dimensional, then F_0 and hence F would have finite codimension in $\mathscr{M}$. If B is a regular matrix such that $c_B \supseteq F$, then c_B would have finite codimension in $\mathscr{M}$ also. However, it is well-known that there is no such regular matrix.

6. Examples. 6.1. Let $\mathscr{A}$ consist of the positive regular Hausdorff matrices. Then $\mathscr{A}$-almost convergence coincides with the notion of Banach-Hausdorff summation introduced in [**8**]. (That any $\mathscr{A}$-invariant mean is a Banach limit follows from the fact that $\mathscr{A}$ contains a strongly regular matrix.) The well-known decomposition of a regular Hausdorff matrix into its positive and negative parts shows that any bounded sequence which is summed by some regular Hausdorff matrix is Banach-Hausdorff summable. Thus Banach-Hausdorff summation is a generalization of the collective Hausdorff method for bounded sequences.

Let H denote the Hölder-Cesaro matrix. Eberlein has announced the following surprising result [**9**]:

6.1.1. If A is any regular Hausdorff method, then

$$\lim_n ||\tilde{H}^n - \tilde{H}^n\tilde{A}||_0 = 0.$$

The foregoing theorem states that the sequence $\{H^n\}$ of iterates of H is a system of averages for $\mathscr{A}$. Since H is itself a Hausdorff matrix, an immediate consequence of 6.1.1 is

6.1.2. The notions of Banach-Hausdorff summation and $\{H^n : n = 0, 1, \ldots\}$-almost convergence coincide.

Garten and Knopp [**11**] term a sequence x H_∞-summable if $\lim_n \liminf H^n x = \lim_n \limsup H^n x$. If follows that from 6.1.1 and 4.3.3 that a bounded sequence x is H_∞-summable if and only if it is Banach-Hausdorff summable. Thus H_∞-summation is a good deal stronger than ordinary Hölder summation for it sums any bounded sequence summed by *any* regular Hausdorff method.

6.2. The results of the preceding paragraph allow us to prove the following

THEOREM. *The regular matrix A contains Banach-Hausdorff summation if and only if*

$$\lim_m \sum_{n=0}^{\infty} \left| \frac{n}{n+1} a_{m,n} - \sum_{l=n+1}^{\infty} \frac{a_{m,l}}{l+1} \right| = 0. \tag{6.2.1}$$

For example, the logarithmic matrix $L^{(1)}$ defined by

$$L_{m,n}^{(1)} = \begin{cases} \dfrac{1}{n \log m}, & \text{if } 1 \leqslant n \leqslant m \text{ and } m > 1 \\ 0, & \text{otherwise,} \end{cases}$$

is such a matrix.

Proof. Since Banach-Hausdorff summation coincides with $\{H^n : n = 0, 1, \ldots\}$-almost convergence, Theorem 5.2 shows that A contains Banach-Hausdorff

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summation if and only if $A(I - H)(\mathscr{M}) \subseteq c_0$. A short calculation combined with 5.3.2 shows that this is the case if and only if 6.2.1 holds. That the logarithmic matrix is such a matrix is verified in a straightforward way.

Remarks. Fuchs has shown that there is no matrix which contains the collective Hausdorff method for *all* sequences [**10**]. Theorem 6.2 shows that for bounded sequences the situation is quite different – such matrices do indeed exist. Also, since H_∞-summation is strictly stronger than ordinary Hölder summation, the second part of the theorem generalizes (for bounded sequences) the classical theorem that logarithmic summation is stronger than Hölder summation (see [**12**]).

Borwein [**2**, Theorem 5] has shown that if x is any sequence summed by $L^{(1)}$ and A is any regular Hausdorff method, then x is summed by $L^{(1)}A$. The second part of the Theorem 6.2 shows that for bounded sequences we can say considerably more, viz. if x is a bounded sequence and A is any regular Hausdorff matrix, then $L^{(1)}x$ and $L^{(1)}Ax$ differ only by a sequence which converges to 0.

6.3. We say that a bounded sequence x is Banach-logarithmic $(B - L^{(1)})$ summable to a if $\phi(x) = a$ whenever $\phi \in L$ has the property that $\phi(y) = \phi(L^{(1)}y)$ for all $y \in \mathscr{M}$. This is simply almost convergence with respect to the semigroup generated by $L^{(1)}$.

The question now arises: "Which matrices contain $B - L^{(1)}$ summation?" Theorem 5.2 gives one kind of answer to this question. In order to give concrete examples, we define the logarithmic matrix of order k by

$$L^{(k)}_{m,n} = \begin{cases} \dfrac{1}{n \log n \ldots \log_{k-1} n \cdot \log_k m}, & \text{if } k \leqslant n \leqslant m \\ 0, & \text{otherwise.} \end{cases}$$

Here $\log_k x$ is defined inductively as $\log_1 x = \log x$, $\log_k x = \log_{k-1}(\log x)$.

We now have

6.4. Theorem. *For each $k = 1, 2, \ldots, L^{(k+1)}$ contains $B - L^{(k)}$ summation. ($B - L^{(k)}$ summation is defined in the obvious way.)*

Proof. Let $k > 1$ be fixed. We must show that $L^{(k+1)}(I - L^{(k)})$ maps $\mathscr{M}$ into c_0. By 5.3.2 this means we must show that

$$\lim_m \sum_{n=0}^{\infty} |L^{(k+1)}(I - L^{(k)})_{m,n}| = 0,$$

i.e.

$$(*) \qquad \lim_m \sum_{n=k+1}^{m} \left| \frac{1}{n \ldots \log_k n \log_{k+1} m} - \sum_{l=n}^{m} \frac{1}{l \ldots \log_k l \cdot \log_{k+1} m} \cdot \frac{1}{n \ldots \log_{k-1} n \log_k l} \right| = 0.$$

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Now the sum involved in (*) is

$$(**)\qquad \frac{1}{\log_{k+1} m}\sum_{n=k+1}^{m}\frac{1}{n\ldots\log_{k-1} n}\left|\frac{1}{\log_k n}-\sum_{l=n}^{m}\frac{1}{l\ldots\log_{k-1} l(\log_k l)^2}\right|.$$

For convenience, let us write

$$f(x)=\frac{1}{x\cdot\log x\ldots\log_{k-1}x(\log_k x)^2}.$$

Then

$$(**)\leqslant\frac{1}{\log_{k+1} m}\sum_{n=k+1}^{m}\frac{1}{n\ldots\log_{k-1} n}$$
$$\left\{\left|\frac{1}{\log_k n}-\int_n^m f(x)dx\right|+\left|\sum_{l=n}^{m} f(l)-\int_n^m f(x)dx\right|\right\}$$
$$\leqslant\frac{1}{\log_{k+1} m}\left\{\frac{1}{\log_k m}\sum_{n=k+1}^{m}\frac{1}{n\ldots\log_{k-1} n}+\sum_{n=k+1}^{\infty}\frac{1}{n^2\ldots(\log_k n)^2}\right\}$$
$$\sim\frac{1}{\log_{k+1} m}\to 0 \text{ as } m\to\infty.$$

6.5. The question of whether there is a matrix which contains $B - L^{(k)}$ summability for all k is answered by the following general

THEOREM. *Let $\{A^{(n)}\}$ be a sequence of positive regular triangular matrices. Let $\mathscr{A}_n$ be the semigroup generated by $A^{(n)}$ and assume that $F(\mathscr{A}_{n+1})\supseteq F(\mathscr{A}_n)$ for $n = 1, 2, \ldots$. Then there is a regular matrix B such that $c_B\supseteq F(\mathscr{A}_n)$ for each n.*

Proof. Let $C^{(s)} = I - A^{(s)}$. By Theorem 5.6 there are positive regular matrices $B^{(k)}$ such that $B^{(k)}C^{(s)}(\mathscr{M})\supseteq c_0$ for $s = 1, \ldots, k$. Let us write $b(m, n; k)$ for $B_{m,n}{}^{(k)}$.

Choose m_1 such that

(*a*) $\qquad b(m_1, 1; 1)\leqslant 2^{-1},$

(*b*) $\qquad \sum_{n=0}^{\infty} b(m_1, n; 1)\geqslant 2^{-1}$, and

(*c*) $\qquad \sum_{n=0}^{\infty}|(B^{(1)}C^{(1)})_{m_1,n}|\leqslant 2^{-1}.$

Now suppose that $m_1, \ldots, m_{k-1}$ have been chosen so that for $j = 1, \ldots, k-1$

(*a'*) $\qquad b(m_j, i; j)\leqslant 2^{-j}$ *for* $1\leqslant i\leqslant j$,

(*b'*) $\qquad \sum_{n=0}^{\infty} b(m_j, n; j)\geqslant\frac{1}{j+1}$, and

(*c'*) $\qquad \sum_{n=0}^{\infty}|(B^{(j)}C^{(i)})_{m_j,n}|\leqslant 2^{-j}$ for $1\leqslant i\leqslant j$.

It is easy to see that m_k can then be chosen so that (a'), (b') and (c') are satisfied for $j = 1, \ldots, k$.

Set $b(i, n) = b(m_i, n; i)$. Then clearly B is regular and $BC^{(s)}(\mathscr{M}) \subseteq c_0$ for all s.

7. Multipliers. 7.1. Let $\mathscr{A}$ be an admissible semigroup. We shall suppose throughout this section that there is at least one matrix which contains $\mathscr{A}$-almost convergence.

Definition. $x \in \mathscr{M}$ is called a multiplier if F if $xy \in F$ whenever $y \in F$. (Multiplication is coordinatewise.) We denote the set of multipliers of F by $\mathscr{M}_F$.

Note that $\mathscr{M}_F \subseteq F$ because $u \in F$.

It is the purpose of this section to characterize the space $\mathscr{M}_F$. When F consists of the sequences which are almost convergent in the ordinary sense, this has been done by Chou [**3**] and Chou and the author [**4**]. The results we obtain are direct generalizations of these theorems.

7.2. THEOREM. *If $x \in \mathscr{M}_F$ and $y \in F$, then*

$$F - \lim xy = (F - \lim x)(F - \lim y).$$

Proof. Let A by any member of $\mathscr{A}$. It suffices to show that for all $z \in \mathscr{M}$, $F - \lim x(z - Az) = 0$

Let B be any regular matrix such that $c_B \subseteq F$. Since $x(z - Az) \in F$, $B(x(z - Az)) \in c$ for all $z \in \mathscr{M}$. We must show that in fact $B(x(z - Az)) \in c_0$ for all $z \in \mathscr{M}$. An application of the second part of Theorem 5.2 will then give the desired result.

Let $M[x]$ be the matrix with entries $x_m\delta_{m,n}$ where $\delta_{m,n}$ is the Kronecker delta. Then $B(x \cdot (z - Az)) = B \cdot M[x] \cdot (I - A)(z)$. Thus $B \cdot M[x] \cdot (I - A)$ is a Schur matrix. Hence by 5.3.1 and 5.4.2 we will be done if we can show that $B \cdot M[x] \cdot (I - A)(c_0) \subseteq c_0$. This, however is obvious since B, $M[x]$ and $I - A$ each map c_0 into itself.

7.3. If $\phi \in L$, let supp ϕ denote the support of ϕ.

Definition. $K^F = \bar{U}\{\text{supp } \phi | \phi \in L(\mathscr{A})\}$. K^F is called the support set of the method F.

If $B \subseteq \mathbf{N}$ we denote by B^* the closure of B in $\beta\mathbf{N}$ minus B itself. χ_B is the characteristic sequence (function) of B.

7.4. LEMMA. $K^F = \bigcap \{B^* | B \subseteq \mathbf{N}$ *and* $F - \lim \chi_B = 1\}$

Proof. Suppose that $B \subseteq \mathbf{N}$ and $F - \lim \chi_B = 1$. Then B^* is a closed set in $\beta\mathbf{N}\backslash\mathbf{N}$ and $\phi(B^*) = \phi(\chi_B) = 1$ for each $\phi \in L(\mathscr{A})$. Hence supp $\phi \subseteq B^*$ and $K^F \subseteq B^*$.

Now suppose that $t \in \beta\mathbf{N}\backslash\mathbf{N}$ and $t \notin K^F$. Since K^F is compact and the sets

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$\{B^*|B \subseteq \mathbf{N}\}$ form a basis for the topology of $\beta\mathbf{N}\backslash\mathbf{N}$, there is a set $B \subseteq \mathbf{N}$ such that B^* contains K^F but not t. If $\phi \in L(\mathscr{A})$ then $\phi(\chi_B) = \phi(B^*) = 1$ so $F - \lim \chi_B = 1$. It follows that

$$t \notin \bigcap \{B^*|B \subseteq \mathbf{N},\ F - \lim \chi_B = 1\}.$$

7.5. We are now in a position to prove the following

THEOREM. *The following conditions on a sequence* $x \in \mathscr{M}$ *are equivalent:*

(7.5.1) $x \in \mathscr{M}_F$ *and* $F - \lim x = a$,

(7.5.2) $F - \lim (x - au)^2 = 0$,

(7.5.3) $\tilde{x} \equiv a$ *on* K^F,

(7.5.4) *For each* $\epsilon > 0$ *the characteristic function of the set* $\{n : |x_n - a| \leqq \epsilon\}$ *is* $\mathscr{A}$*-almost convergent to* 1.

Proof. That 7.5.1 implies 7.5.2 follows immediately from Theorem 7.2.

That 7.5.2 implies 7.5.3 is obvious since $(x - au)^2$ is a nonnegative continuous function and each $\phi \in L(\mathscr{A})$ is a regular Borel measure.

Now suppose that $\tilde{x} \equiv a$ on K^F.

Let $y \in F$, say $F - \lim y = b$. Then if $\phi \in L(\mathscr{A})$,

$$\phi(xy) = \int_{\operatorname{supp}\phi} \tilde{x}\tilde{y}d\phi = a \int_{\operatorname{supp}\phi} yd\phi = a \cdot \phi(y) = ab.$$

Thus $xy \in F$ and $x \in \mathscr{M}_F$. Taking $y = u$ in the above computation we see that $F - \lim x = a$.

The equivalence of 7.5.3 and 7.5.4 is proved exactly as in [**3**].

7.6. As in [**3**], we have

COROLLARY. *Let* χ_B *be the characteristic function of* $B \subseteq \mathbf{N}$. χ_B *is a multiplier if and only if it is almost convergent to* 0 *or* 1.

8. Matrix transformations of spaces of almost convergent sequences.

8.1. We shall need the following lemma of Attala [**1**, Lemma 3.2].

LEMMA. *Let* $\{\tilde{A}_n\}$ *be a sequence of operators on* $C(\beta\mathbf{N}\backslash\mathbf{N})$ *induced by matrices. Then* $||\tilde{A}_n||_0 \to 0$ *if and only if* $||\tilde{A}_n\tilde{x}||_0 \to 0$ *for each* $x \in \mathscr{M}$.

8.2. Let B be an infinite matrix. Let

$$||B|| = \sup_m \sum_{n=0}^{\infty} |b_{m,n}|.$$

We recall the well-known facts that $||B|| < +\infty$ if and only if $B(\mathscr{M}) \subseteq \mathscr{M}$ if and only if $B(c_0) \subseteq \mathscr{M}$. Moreover if $||B|| < +\infty$ then $||B||$ is the norm of B considered as an operator on $\mathscr{M}$.

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8.3. We assume throughout the remainder of the paper that $\mathscr{A}$ is an ergodic semigroup and that $\{A_n\}$ is a system of averages for $\mathscr{A}$ which is a *sequence.*

LEMMA. Let B be an infinite matrix. Then $F(\mathscr{A}) - \lim Bx = 0$ for all $x \in \mathscr{M}$ if and only if $||B|| < +\infty$ and $||\tilde{A}_n\tilde{B}||_0 \to 0$.

Proof. Lemma 8.3 is an immediate consequence of 4.3.2, 8.1 and 8.2.

8.4. Now let $\mathscr{C}$ be an ergodic semigroup. We have the following generalization of a theorem of Schaefer [**19**].

THEOREM. *Let B be an infinite matrix.* $B(F(\mathscr{C})) \subseteq F(\mathscr{A})$ *and* $F(\mathscr{A}) - \lim Bx = F(\mathscr{C}) - \lim x$ *for all* $x \in F(\mathscr{C})$ *if and only if*

(8.4.1) $||B|| < +\infty$,

(8.4.2) $F(\mathscr{A}) - \lim_m \sum_{n=0}^{\infty} b_{m,n} = 1$, *and*

(8.4.3) $F(\mathscr{A}) - \lim_m b_{m,n} = 0$ *for each* n,

(8.4.4) $||\tilde{A}_n\tilde{B}(\tilde{I} - \tilde{C})||_0 \to 0$ *for each* $C \in \mathscr{C}$.

Proof. Suppose that $B(F(\mathscr{C})) \subseteq F(\mathscr{A})$ and $F(\mathscr{A}) - \lim Bx = F(\mathscr{C}) - \lim x$ for each $x \in F(\mathscr{C})$. Then $||B|| < +\infty$ by 8.2 since $c_0 \subseteq F(\mathscr{C})$ and $F(\mathscr{A}) \subseteq \mathscr{M}$. 8.4.2 follows because

$$u \in F(\mathscr{C}),\ F(\mathscr{C}) - \lim u = 1 \text{ and } (Bu)_m = \sum_{n=0}^{\infty} b_{m,n}.$$

8.4.3 follows by similar reasoning applied to the sequence $e^n = (0, \ldots 0, 1, 0, \ldots)$ where 1 appears in the nth position. 8.4.4 is an immediate consequence of 8.3 and the fact that $F(\mathscr{C}) - \lim (x - Cx) = 0$ for all x (4.3.5).

Conversely suppose that 8.4.1 through 8.4.4 are satisfied. By 8.4.3, $F(\mathscr{A}) - \lim Be^n = 0$ for all n. Hence $F - \lim Bx = 0$ whenever $x \in c_0$ since B is linear and $||B|| < +\infty$. By 8.4.4 and Lemma 8.3 $F(\mathscr{A}) - \lim B(x - Cx) = 0$ for all x. We now use the linearity and continuity of B to conclude from 4.3.5 that $F(\mathscr{A}) - \lim Bx = 0$ whenever $F(\mathscr{A}) - \lim x = 0$. This, combined with 8.4.3 proves the theorem.

Remark. Condition 8.4.4 has a concrete reformulation, for if D is any infinite matrix satisfying $D(c_0) \subseteq c_0$ then it is easily seen that

$$||\tilde{D}||_0 = \lim_m \sup \sum_{n=0}^{\infty} |d_{m,n}|.$$

8.5. COROLLARY. $B(c) \subseteq F(\mathscr{A})$ *and* $F(\mathscr{A}) - \lim Bx = \lim x$ *for all* $x \in c$ *if and only if*

(8.5.1) $\|B\| < +\infty$,

(8.5.2) $F(\mathscr{A}) - \lim b_{m,n} = 0$ *for all* n, *and*

(8.5.3) $F(\mathscr{A}) - \lim_m \sum_{n=0}^{\infty} b_{m,n} = 1.$

Proof. We simply take $\mathscr{A} = \{I\}$ in Theorem 8.4.

Remark. This generalizes a theorem of J. P. King [**14**].

8.6. Corollary. *$F(\mathscr{C}) \subseteq F(\mathscr{A})$ if and only if $\|\tilde{A}_n(\tilde{I} - \tilde{C})\|_0 \to 0$ for each $C \in \mathscr{C}$.*

Proof. Take $B = I$ in Theorem 8.4.

Remark. The corollary is Theorem 2.2 of [**1**] when both $\mathscr{C}$ and $\mathscr{A}$ are cyclic.

8.7. *Remark.* We do not include the corollary obtained by the choice $\mathscr{A} = \{I\}$ since a sharper result (viz. Theorem 6.2) has already been given.

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