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UNIVERSAL VARIETIES OF (0, 1)-LATTICES

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This article fully characterizes categorically universal varieties of (0, 1)-lattices (that is, lattices with a least element 0 and a greatest element 1 regarded as nullary operations), thereby concluding a series of partial results [**3**, **5**, **8**, **10**, also **14**] which originated with the proof of categorical universality for the variety of all (0, 1)-lattices by Grätzer and Sichler [**6**].

A category **C** of algebras of a given type is *universal* if every category of algebras (and equivalently, according to Hedrlín and Pultr [**7** or **14**], also the category of all graphs) is isomorphic to a full subcategory of **C**. The universality of **C** is thus equivalent to the existence of a full embedding $\Phi : \mathbf{G} \longrightarrow \mathbf{C}$ of the category **G** of all graphs and their compatible mappings into **C**. When Φ assigns a finite algebra to every finite graph, we say that **C** is *finite-to-finite universal.*

Since every [finite] monoid occurs as the endomorphism monoid of a [finite] graph [**14**], any [finite-to-finite] universal category **C** contains a [finite] algebra whose endomorphisms form a given [finite] monoid; thus every [finite-to-finite] universal category **C** is also *[finite-to-finite] monoid universal.* G. Birkhoff's result that every group is isomorphic to the group of all automorphisms of a distributive lattice, one of the origins of the present studies, shows that the variety that of distributive lattices is *group universal.*

Since all singletons are subalgebras in any ordinary lattice, the universality of any category **C** of lattices is prevented by the associated constant homomorphisms. Even though the class $N(\mathbf{C})$ of all non-constant lattice homomorphisms in **C** need not be a category, it is still possible to ask whether or not $N(\mathbf{C})$ includes a universal full subcategory; if it does then **C** is said to be *almost universal.* The problem of almost universality for varieties of lattices appears to be linked, in a way not yet fully understood, to the problem of universality for varieties of (0, 1)-lattices; also, all examples of almost universal varieties of lattices [**10**, **15**] originate from universal varieties of (0, 1)-lattices.

Every [finite-to-finite] universal category of algebras contains a proper class of pairwise non-isomorphic algebras representing a given monoid [and also a countably infinite set of non-isomorphic finite algebras representing any finite monoid], see [**14**]. This is why only non-universal varieties of (0, 1)-lattices offer any hope for results similar to that by McKenzie and Tsinakis [**11**] which shows that any distributive (0, 1)-lattices L is, up to an anti-isomorphism, determined by its endomorphism monoid.

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THEOREM . *The following are equivalent for any variety* $\mathbf{V}$ *of* (0, 1)*-lattices:*
(1) $\mathbf{V}$ *is [finite-to-finite] universal,*
(2) $\mathbf{V}$ *is [finite-to-finite] monoid universal,*
(3) $\mathbf{V}$ *contains a [finite]* (0, 1)*-lattice* L *with no prime ideal,*
(4) $\mathbf{V}$ *contains a finitely generated [finite] non-distributive simple* (0, 1)*-lattice* K.

Proof segment. Any [finite-to-finite] monoid universal variety $\mathbf{V}$ of (0, 1)-lattices contains a [finite] (0, 1)-lattice L with no prime ideals, for instance any lattice whose endomorphism monoid is isomorphic to a finite non-trivial group. Any such L has a finitely generated [finite] (0, 1)-sublattice M with no prime ideal, by the equational compactness of the two-element lattice. Since the total congruence of M is principal, maximal congruences of M exist, so that a simple finitely generated [finite] non-distributive (0, 1)-lattice K can be found amongst the homomorphic images of M. This shows that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) in either case. □

We remark that an equivalent form of this result holds for any $\mathbf{V} = B(\mathbf{V}')$ formed by all (0, 1)-preserving homomorphisms of lattices possessing bounds 0, 1 and contained in any given variety $\mathbf{V}'$ of ordinary lattices.

The proof of (4) $\Rightarrow$ (1) is divided into five sections. The finite-to-finite universal category $\mathbf{U}$ to be fully embedded into $\mathbf{V}$ is presented in Section 1. Our full embedding $F : \mathbf{U} \longrightarrow \mathbf{V}$ varies according to whether or not the length of K is greater than 3; following a common background Section 2, we produce the two variants of F in parallel Sections 3 and 4. The concluding Section 5 adapts the functorial construction from [**5**] in order to provide for a full embedding F which is finite-to-finite.

Throughout the paper, the symbol K is reserved for a finitely generated non-distributive simple (0, 1)-lattice.

1. Finite-to-finite universal category U. Let $2\mathbf{D}$ denote the category of all triples (D, p_0, p_1) consisting of a distributive (0, 1)-lattice D augmented by two selected lattice (0, 1)-homomorphisms p_0 and p_1 of D onto the two-element chain $\{0, 1\}$; a lattice (0, 1)-homomorphism $f : D \longrightarrow D'$ is a morphism in $2\mathbf{D}$ of (D, p_0, p_1) into $D', p'_0, p'_1)$ if and only if $p'_i \cdot f = p_i$ for $i = 0, 1$.

The main result of Koubek [**9**] implies the universality of $2\mathbf{D}$, but his construction assigns infinite members of $2\mathbf{D}$ to finite graphs.

We refer to [**1**] for the fact that the category $5\mathbf{D}$ consisting of distributive (0, 1)-lattices which are similarly augmented by five (0, 1)-homomorphisms p_i onto the two-element chain is finite-to-finite universal. In view of this, we need to construct a finiteness preserving full embedding of $5\mathbf{D}$ into $2\mathbf{D}$. We do this next, within the framework of Priestley's duality, for a somewhat more general category $D\mathbf{D}$ consisting of all morphisms $f : (D, p_0, p_1) \longrightarrow (D', p'_0, p'_1)$ such that f is a lattice (0, 1)-homomorphism of D into D' satisfying $\{p'_0 \cdot f, p'_1 \cdot f\} =$

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$\{p_0, p_1\}$. The full embedding of 5**D** into D**D** presented below will, in fact, be also a full embedding of 5**D** into 2**D**.

Recall that a *Priestley space* is any compact totally order disconnected ordered topological space $(X, \tau, \leqq)$, that is, a partially ordered compact space whose topology is the weak topology induced by all order preserving maps of $(X, \tau, \leqq)$ into the two-element discrete chain $\{0, 1\}$. All continuous order preserving maps between such spaces, called *Priestley maps*, form a category dually isomorphic to the category of all (0, 1)-homomorphisms of distributive (0, 1)-lattices, see Priestley [**12**, **13**].

Since the poset $(X, \leqq)$ of the Priestley space dual to a (0, 1)-lattice D consists of all prime ideals of D ordered by their inclusion and because the inverse image map f^{-1} is the Priestley map dual to a lattice (0, 1)-homomorphism f, the Priestley dual $(X, \tau, \leqq)$ of an object (D, p_0, p_1) of 2**D** is a Priestley space augmented by two distinguished elements c_0 and c_1. Any morphism f of 2**D** corresponds to a Priestley map g preserving c_0 and c_1.

Let 2**P** and 5**P** denote thus augmented Priestley duals of the respective categories 2**D** and 5**D**.

The order ideal (or filter) generated by a subset Y of $(X, \leqq)$ will be respectively denoted as $(Y]$ (or $[Y)$).

The following was shown in [**1**].

LEMMA 1.1 [**1**]. *The category* 5**P** *contains a full subcategory* **Q** *dually isomorphic to a universal category. The category* **Q** *is determined by Priestley spaces* $(X, \tau, \leqq, A)$ *with* $A = \{a_0, a_1, \ldots, a_4\}$ *consisting of minimal clopen points such that* $([A)] = X$, *and* $|(\{x\}] \cap A| \neq 1$ *for any* $x \in X \setminus A$. *Every morphism* g *of* **Q** *satisfies* $g^{-1}\{g(a_i)\} = \{a_i\}$ *for all* $i \in 5$. □

For any object $Q = (X, \tau, \leqq, A)$ of **Q** define $\Phi(Q) = (Y, \sigma, \leqq, \{c_0, c_1\})$ as follows:

$$F = \bigcup\{E_i | i \in 5\} \cup \{c_0, c_1\} \cup D,$$

$$Y = (X \setminus A) \cup F,$$

where all unions are disjoint, $D = \{d_i | i < 52\}$ and, for $i \in 5$,

$$E_i = \{e_{i,k} | 1 \leqq k \leqq 14\}.$$

The partial order on $(Y, \leqq)$ is the least order for which

(i) $d_{2i} \leqq d_{2i+1}$ and $d_{2i+2} \leqq d_{2i+1}$ for $i \in 26$ with the addition modulo 52;

(ii) $d_0 \leqq c_0 \leqq d_{51}$ and $d_{26} \leqq c_1 \leqq d_{25}$;

(iii) for every $i \in 5$, $e_{i,2j} \leqq e_{i,2j-1}, e_{i,2j+1}$ when $1 \leqq j \leqq 6$, and $e_{i,14} \leqq e_{i,13}$;

(iv) for every $i \in 5$, $d_{8+2i} \leqq e_{i,1}$ and $e_{i,14} \leqq d_{43-2i}$;

(v) for every $i \in 5$, $e_{i,8} \leqq x \in X \setminus A$ if and only if $a_i \leqq x$ in $(X, \leqq)$;

(vi) $x \leqq y$ for $x, y \in X \setminus A$ whenever $x \leqq y$ in $(X, \leqq)$.

The topology σ of $\Phi(Q)$ is the union topology given by the discrete topology on the finite set F and by the clopen subspace $X \setminus A$ of Q. It is easily seen that $\Phi(Q)$ is an object of $2\mathbf{P}$ (cf. also [**1**]).

Since every morphism $\varphi : Q \longrightarrow Q'$ of $\mathbf{Q}$ satisfies $\varphi(X \setminus A) \subseteq X' \setminus A$, the extension of $\varphi \upharpoonright (X \setminus A)$ to $\Phi(Q)$ by the identity mapping id_F of F is a continuous order preserving mapping $\Phi(\varphi)$ satisfying $\Phi(\varphi)(c_i) = c_i$ for $i \in \{0, 1\}$. The functor Φ is obviously one-to-one.

Let $\psi : \Phi(Q) \longrightarrow \Phi(Q')$ be an order preserving continuous mapping such that Q and Q' are objects of $\mathbf{Q}$ and $\{\psi(c_0), \psi(c_1)\} = \{c_0, c_1\}$. We aim to show that $\psi = \Phi(\varphi)$ for some $\varphi : Q \longrightarrow Q'$.

First we note that there are exactly two comparability paths of minimal length connecting c_0 to c_1, namely, the paths $D_0 = \{d_0, \ldots, d_{25}\}$ and $D_1 = \{d_{26}, \ldots, d_{51}\}$.

If $\psi(c_i) = c_{1-i}$ for $i \in \{0, 1\}$ then (ii) implies that $\psi(d_0) = d_{26}$ and $\psi(d_{51}) = d_{25}$. Hence ψ must interchange the two shortest paths in such a way that, for every $i \in 52$, $\psi(d_i) = d_{i+26}$ with the addition modulo 52. In particular, the elements d_{12} and d_{39} – connected by the comparability path E_2 of length 15 – are mapped to d_{38} and d_{13}, respectively. The two shortest comparability paths connecting the latter two elements must, respectively, include E_2 or E_3 and thus be of length 17, which is impossible.

Thus $\psi(c_i) = c_i$ for $i \in \{0, 1\}$. Similarly to the previous argument, (ii) implies that $\psi(d_k) = d_k$ for all $k \in 52$. Since, for each $i \in 5$, the shortest comparability path connecting d_{8+2i} to d_{43-2i} is that consisting entirely of elements of E_i, the restriction of ψ to each E_i must be the identity mapping. Altogether, ψ is the identity on the poset F.

If $x \in X \setminus A \subseteq Y$ satisfies $a_i \leqq x$ for some $i \in 5$, then $a_j \leqq x$ for some $j \in 5$ distinct from i. By (v), $e_{i,8} \leqq x$ and $e_{j,8} \leqq x$ in $\Phi(Q)$; since ψ fixes all elements of F and because no elements of F occur above distinct $e_{i,8}$ and $e_{j,8}$, it follows that $\psi(x) \in X' \setminus A$. By the definition, $(X' \setminus A] \subseteq X' \cup \{e_{i,8} | i \in 5\}$, and from $([A)] = X$ we finally conclude that

$$\psi((X \setminus A) \cup \{e_{i,8} | i \in 5\}) \subseteq (X' \setminus A) \cup \{e_{i,8} | i \in 5\}.$$

Since the latter space is homeomorphic and order isomorphic to Q', the mapping $\psi \upharpoonright Q$ is a morphism in $5\mathbf{P}$, and $\psi = \Phi(\psi \upharpoonright Q)$ as was to be shown.

This completes the proof of the result below which, in conjunction with the finite-to-finite universality of the category of graphs [**14**] yields the finite-to-finite universality of $2\mathbf{D}$.

PROPOSITION 1.2. *Let $D\mathbf{D}$ be the category of all triples (D, p_0, p_1) in which D is a distributive $(0, 1)$-lattice and $p_0, p_1 : D \longrightarrow \{0, 1\}$ are distinct $(0, 1)$-homomorphisms, whose morphisms are those lattice $(0, 1)$-homomorphisms $f : D \longrightarrow D'$ for which $\{p_0' \cdot f, p_1' \cdot f\} = \{p_0, p_1\}$. Then there is a full embedding Ψ of the category of all graphs into $D\mathbf{D}$ such that $\Psi(G)$ is a finite lattice whenever the graph G is finite.* □

Since all objects (D, p_0, p_1) occurring in the image of the functor Ψ from 1.2 possess incomparable prime filters $p_i^{-1}\{1\}$, the following useful consequence follows.

COROLLARY 1.3. *The full subcategory* **U** *of* 2**D** *consisting of all* 2**D**-*objects* (D, p_0, p_1) *with* p_0 *incomparable to* p_1 *is finite-to-finite universal.* □

2. Solid sublattices of K^n. For any simple finitely generated (0, 1)-lattice K, this section aims to exhibit certain subdirect powers of K whose endomorphisms are easily controlled.

Let K be a simple finitely generated (0, 1)-lattice.

Identifying an integer $n \geqq 0$ with the set $\{0, 1, \ldots, n-1\}$, we define the n-th power K^n of K as the set of all functions $\varphi : n \longrightarrow K$ with all (0, 1)-lattice operations in K^n carried out componentwise.

For any subset $S \subseteq n$ and a (0, 1)-sublattice L of K^n, the *S-restriction* p_s : $L \longrightarrow K^S$ is the (0, 1)-homomorphism defined by $p_S(\varphi) = \varphi \upharpoonright S$ for all $\varphi \in L$. For singletons $S = \{i\} \subseteq n$, the lattice $K^{\{i\}}$ will be canonically identified with K and $p_{\{i\}}$ with the restriction $p_i : L \longrightarrow K$ of the orindary i-th projection to L.

Any (0, 1)-sublattice L of K^n such that $p_i(L) = K$ for all $i \in n$ is said to be *subdirect in* K^n. Since K is simple, the kernel $\pi_i = \operatorname{Ker} p_i$ is a maximal congruence on any such sublattice L. If L is subdirect in K^n and if, moreover, $\pi_i = \pi_j$ only when $i = j$, then $L \subseteq K^n$ is an irredundant subdirect representation of L; whenever this is the case, we call L a *solid* (0, 1)-sublattice of K^n.

For any $S \subseteq n$, the kernel π_S of the S-restriction p_S satisfies $\pi_S = \bigcap\{\pi_i | i \in S\}$.

LEMMA 2.1. *Let L be a solid* (0, 1)-*sublattice of K^n. Then for every congruence Θ on L there exists a unique subset $S \subseteq n$ such that $\Theta = \pi_S$.*

Proof. The congruence lattice Con(L) of L is distributive, $\{\pi_i | i \in n\}$ is the set of all coatoms of Con(L), and the intersection $\cap\{\pi_i | i \in n\}$ is the identity congruence; thus Con(L) is isomorphic to the Boolean lattice 2^n. □

LEMMA 2.2. *For every* (0, 1)-*homomorphism $f : L \longrightarrow M$ of solid* (0, 1)-*sublattices L and M of K^n and K^m, respectively, there exists a unique family $\{S(j) \subseteq n | j \in m\}$ of subsets of n accompanied by a uniquely determined family $\{h_j | j \in m\}$ of* (0, 1)-*embeddings $h_j : p_{S(j)}(L) \longrightarrow K$ such that $f(\varphi)(j) = h_j(\varphi \upharpoonright S(j))$ for all $\varphi \in L$ and $j \in m$.*

Proof. Assume first that $m = 1$, that is, $M = K$. By Lemma 2.1, we have $\operatorname{Ker} f = \pi_S$ for unique $S \subseteq n$; hence $f = h \cdot p_S$ with the uniquely determined surjective $p_S : L \longrightarrow p_S(L)$ and a (0, 1)-embedding $h : p_S(L) \longrightarrow K$. Hence $f(\varphi) = h(\varphi \upharpoonright S)$ for all $\varphi \in L$.

The case of an arbitrary m is obtained by applying the above to the (0, 1)-homomorphism $p_j \cdot f : L \longrightarrow K$ for $j \in m$. □

A function $\varphi \in K^n$ is *skeletal* if $\varphi(i) \in \{0,1\} \subseteq K$ for all $i \in n$, and *antiskeletal* if $\varphi(i) \in K \setminus \{0,1\}$ for all $i \in n$.

LEMMA 2.3. *If $f : L \longrightarrow M$ is as in Lemma 2.2, then $f(\varphi)$ is antiskeletal for any antiskeletal $\varphi \in L$.*

Proof. Immediate from the fact that all h_j are (0, 1)-embeddings. □

Lemma 2.2 can be strengthened for those solid sublattices of K^n which consist of non-decreasing functions alone.

Definition of the lattice I. Let $I = I(K^n)$ denote the (0, 1)-sublattice of K^n formed by all its non-decreasing functions, that is, by all $\varphi \in K^n$ satisfying $\varphi(i) \leqq \varphi(j)$ whenever $i \leqq j \in n$.

In what follows, $n - m$ stands for the arithmetic difference of the integers n and m. □

Definition of the lattice I_m. For each $m \in n$ set

$$I_m = \{\varphi \in I | \forall i \in n - m \quad (\varphi(i) \neq 0 \Rightarrow \varphi(i+m) = 1)\}.$$

Any I_m with $m > 0$ is a solid (0, 1)-sublattice of K_n, while $I_0 = \{\chi_j : j \in n{+}1\}$ is the chain of all skeletal functions in I, indexed by $j \in n+1$ in such a way that $\chi_j(i) = 0$ if and only if $i < j$.

LEMMA 2.4. *Let L, M be solid (0, 1)-sublattices of K^n such that $L, M \subseteq I$ and $I_0 \subseteq L$. Then for any (0, 1)-homomorphism $f : L \longrightarrow M$, the two functions*

$$g(j) = \min S(j) \quad \text{and} \quad \bar{g}(j) = \max S(j)$$

determined by the set family $\{S(j) \subseteq n | j \in n\}$ of f, are non-decreasing.

Proof. $\chi_{g(j)}, \chi_{\bar{g}(k)+1} \in L$ and $f(\chi_{g(j)}), f(\chi_{\bar{g}(k)+1}) \in M \subseteq I$ for $j \leqq k \in n$. Thus $1 = h_j(\chi_{g(j)} \restriction S(j)) = f(\chi_{g(j)})(j) \leqq f(\chi_{g(j)})(k) = h_k(\chi_{g(j)} \restriction S(k))$ and, since h_k is a (0, 1)-embedding, $g(j) \leqq g(k)$.
Similarly we have

$$0 = h_k(\chi_{\bar{g}(k)+1} \restriction S(k)) = f(\chi_{\bar{g}(k)+1})(k) \geqq f(\chi_{\bar{g}(k)+1})(j)$$
$$= h_j(\chi_{\bar{g}(k)+1} \restriction S(j)), \text{ whence } \bar{g}(k) + 1 \geqq \bar{g}(j) + 1.$$ □

Definition of the lattice I_mA. For any (0, 1)-sublattice A of I and any integer $m \in n$, let I_mA be the (0, 1)-lattice generated by $I_m \cup A$.

The lattice I_mA enjoys the following separation property.

LEMMA 2.5. *For every $\varphi \in I_mA$, there exists a non-decreasing sequence $\alpha_0 \leqq \alpha_1 \leqq \ldots \leqq \alpha_{n-1}$ in A such that*

(*) $\qquad \varphi(j) \leqq \alpha_j(j) \leqq \alpha_j(j+m) \leqq \varphi(j+m)$ *for all $j \in n - m$.*

Proof. If $\varphi \in A$ then set $\alpha_i = \varphi$ for all $i \in n$. If $\varphi \in I_m$ and k is the first integer with $\varphi(k) \neq 0$ then set $\alpha_i = 0 \in A$ for all $i < k$ and $\alpha_i = 1 \in A$ for $k \leqq i \in n$. This proves the claim for the generators of I_mA. Now, for arbitrary $\varphi, \psi \in I_mA$, if $\{\alpha_i | i \in n\}$ and $\{\beta_i | i \in n\}$ are the respective sequences in A satisfying $(*)$, then $\{\alpha_i \vee \beta_i | i \in n\}$ and $\{\alpha_i \wedge \beta_i | i \in n\}$ are the required sequences in A for $\varphi \vee \psi$ and $\varphi \wedge \psi$, respectively.

Definition of the lattice $K^*_{j,m}$. For any $x \in K$, let $x^* \in K^n$ denote the constant function defined by $x^*(i) = x$ for all $i \in n$; for $m \in n$ and $j \leqq n - m$, let $x^*_{j,m} = (\chi_j \wedge x^*) \vee \chi_{j+m}$. Set also

$$K^* = \{x^* | x \in K\} \quad \text{and } K^*_{j,m} = \{x^*_{j,m} | x \in K\}.$$

LEMMA 2.6. *$K^*_{j,k} \cap I_mA \subseteq (\chi_j \wedge A) \vee \chi_{j+k} \subseteq I_0A$ for all $k > m$ and $j \in n - k$.*

Proof. If $x^*_{j,k} = (\chi_j \wedge x^*) \vee \chi_{j+k} \in I_mA$, then by Lemma 2.5 there exists a non-decreasing sequence $\alpha_0 \leqq \alpha_1 \leqq \ldots \leqq \alpha_{n-1}$ in A such that

$$\begin{aligned} x = x^*_{j,k}(j) &\leqq \alpha_j(j) \leqq \alpha_j(j+k-1) \leqq \alpha_{j+k-1-m}(j+k-1) \\ &\leqq x^*_{j,k}(j+k-1) = x. \end{aligned}$$

Hence $\alpha_j(i) = x$ for all $i = j, \ldots, j+k-1$ and we have $x^*_{j,k} = (\chi_j \wedge \alpha_j) \vee \chi_{j+k} \in I_0A$. □

Finally, we note that comparable endomorphisms of K are easy to control: the claim below shows why this is the case.

LEMMA 2.7. *Let f, g be $(0,1)$-endomorphisms of K satisfying $f(x) \leqq g(x)$ for all $x \in K$. If for every e in some set E of generators of K there exists $x \in K$ with $f(x) \leqq e \leqq g(x)$ then $f = g$.*

Proof. Let $R = \{(a,b) \in K \times K; f(x) \leqq a, b \leqq g(x) \text{ for some } x \in K\}$. Obviously, R is symmetric. To see that R is also reflexive, for any $c \in K$ select a $(0,1)$-lattice term t such that $c = t(e_0, e_1, \ldots, e_{n-1})$ and $\{e_0, e_1, \ldots, e_{n-1}\} \subseteq E$. Since $f(t) = t = g(t)$ for $t \in \{0,1\}$, we may assume that $n > 0$ and, by the hypothesis, there exist $x_0, \ldots, x_{n-1} \in K$ such that $f(x_i) \leqq e_i \leqq g(x_i)$, from which $f(t(x_0, \ldots, x_{n-1})) \leqq c \leqq g(t(x_0, \ldots, x_{n-1}))$ follows by the monotonicity of $(0,1)$-lattice terms.

If $(a,b) \in R$ and $c \in K$ then $f(x) \leqq a, b \leqq g(x)$ and $f(y) \leqq c \leqq g(y)$ for some $x, y \in K$ and, consequently, $f(x \vee y) \leqq a \vee c, b \vee c \leqq g(x \vee y)$; thus $(a \vee c, b \vee c) \in R$ and, dually, $(a \wedge c, b \wedge c) \in R$.

The smallest congruence $\Theta(R) \in \mathrm{Con}(K)$ containing R is, therefore, just the transitive closure of R. If $f(x) \leqq a, 1 \leqq g(x)$ for some $x \in K$ then $x = 1$ since g is a $(0,1)$-embedding of K into itself, and $a = 1$ follows from $f(1) = 1$. Therefore $\Theta(R)$ is not the total congruence; because K is simple, $\Theta(R) \supseteq R$ must be trivial and $f = g$ follows. □

3. Constructions for K of length greater than 3. Let K be a finitely generated simple (0, 1)-lattice containing a chain $0 < a < b < c < 1$.

Let $D_1 \subseteq K \setminus \{0, 1\}$ be a finite generating set of K. Denote $D = D_1 \setminus \{b\}$ and $E = D \cup \{b\}$. Thus $E \subseteq K \setminus \{0, 1\}$ is a finite generating set of K and $D \cap \{b\} = \phi$.

Set $n = 4m + 7$ with $m = |D| + 2$.

Let Δ denote the set of all binary relations $\delta \subseteq D \times \{1, 2, \ldots, m\}$ such that $\delta(d) = \{k|(d, k) \in \delta\} \neq \phi$ for all $d \in D$, and, $\delta(d) \cap \delta(d') = \phi$ for $d \neq d'$. For any $\delta \in \Delta$, let $\delta(D) = \bigcup\{\delta(d)|d \in D\}$ denote the range of δ in $\{1, 2, \ldots, m\}$.

Definition of the lattice $A(\delta)$. Let $A(\delta)$ be the (0, 1)-sublattice of K^n generated by the set

$$\{d^*_{4k,3}|(d, k) \in \delta\} \cup \{b^*, \beta\},$$

where $\beta(0) = a, \beta(n-1) = c$, and $\beta(i) = b$ for $i = 1, 2, \ldots, n-2$.

It is easily seen that for each $\alpha \in A(\delta)$ we have $\alpha(0) \in \{0, a, b, 1\}$ and $\alpha(n-1) \in \{0, b, c, 1\}$, $\alpha(i) \in \{0, b, 1\}$ for $i \in \{1, 2, 3, n-4, n-3, n-2\}$, and $\alpha(i) \in \{0, b, d, b \vee d, b \wedge d, 1\}$ for $4 \leqq i \leqq n-5$, with the appropriate $d \in D \cup \{b\}$.

LEMMA 3.1. *For every antiskeletal $\varphi \in I_2A(\delta)$ we have $\varphi(2) = \varphi(n-3) = b$. If $\mu \wedge \tau$ or $\mu \vee \tau$ with $\mu, \tau \in I_2A(\delta)$ is antiskeletal, then either μ or τ is antiskeletal.*

Proof. To any antiskeletal $\varphi \in I_2A(\delta)$, Lemma 2.5 assigns a non-decreasing sequence $\alpha_0 \leqq \alpha_1 \leqq \ldots \leqq \alpha_{n-1}$ of members of $A(\delta)$ such that

$$0 < \varphi(0) \leqq \alpha_0(0) \leqq \alpha_0(2) \leqq \varphi(2) \leqq \ldots \leqq \varphi(n-3) \leqq \alpha_{n-3}(n-3)$$
$$\leqq \alpha_{n-3}(n-1) \leqq \varphi(n-1) < 1.$$

Since $\alpha_0(2)$, $\alpha_{n-3}(n-3) \in \{0, b, 1\}$ by the definition of $A(\delta)$, these inequaliteis imply that $\alpha_0(2) = \alpha_{n-3}(n-3) = b$, whence the first assertion.

As for the second one, assume that, say, $\mu \wedge \tau$ is antiskeletal. Then $\mu(0) \neq 0 \neq \tau(0)$ and, simultaneously, $\mu(n-1) \neq 1$ or $\tau(n-1) \neq 1$. □

STATEMENT 3.2. *Let $\delta, \epsilon \in \Delta$, and let $f : I_2A(\delta) \longrightarrow I_2A(\epsilon)$ be a (0, 1)-homomorphism. Then $\delta(D) \subseteq \epsilon(D)$, and the functions $g(j) = \min S(j)$ and $\bar{g}(j) = \max S(j)$ associated with f satisfy*

(1) $g(i+2) - g(i) \geqq 2$ *and* $\bar{g}(i+2) - \bar{g}(i) \geqq 2$ *for all* $i \in n-2$;

(2) $S(2k) = \{2k\}$ *for all* $2k \in n$.

Proof. First we observe that Lemma 2.3 and Lemma 3.1 imply that $f(b^*)(i) = b$ for all $i \in \{2, \ldots, n-3\}$.

Suppose that $g(i+2) - g(i) \leqq 1$ for some $i \in n-2$. For any $x \in K$, denote $\psi_x = f(x^*_{g(i),2})$. Then $\psi_x(k) = h_k(x^*_{g(i),2} \restriction S(k))$; since $g(i) \leqq g(k) \leqq g(i) + 1$ for $k \in \{i, i+1, i+2\}$, and because each h_k is one-to-one,

$$0 < \psi_x(i) \leqq \psi_x(i+2) < 1 \quad \text{for all } x \in K \setminus \{0, 1\}.$$

We note that the mapping $F : K \longrightarrow K$ defined by $F(x) = \psi_x(i)$ for all $x \in K$ is an ordinary endomorphism of K for which $F(1) = 1$. From $g(i) = \min S(i)$ it follows that $b^* \upharpoonright S(i) \leqq b^*_{g(i),2} \upharpoonright S(i)$; hence $b = f(b^*)(i) \leqq \psi_b(i) = F(b)$.

Furthermore, Lemma 2.5 yields the existence of an $\alpha_x \in A(\epsilon)$ such that, for each $x \in K \setminus \{0, 1\}$,

$$0 < \psi_x(i) \leqq \alpha_x(i) \leqq \alpha_x(i+2) \leqq \psi_x(i+2) < 1.$$

If $i \leqq 3$, then $\alpha_x(i) \in \{a, b\}$, whence $F(x) = \psi_x(i) \leqq b$ for all $x \in K \setminus \{1\}$, which is clearly impossible, since $K \setminus \{1\}$ would then become a prime ideal in K.

If $4 \leqq i \leqq n-5$, then $\alpha_x(i) \in \{b, d, b \wedge d, b \vee d\}$ for some $d \in D \cup \{b\}$, and hence $\psi_x(i) \leqq b \vee d$ for all $x \in K \setminus \{1\}$. For $b \vee d < 1$ we get a contradiction identical to that above. Hence $b \vee d = 1$, $\alpha_x(i) \neq b \vee d$, and $\psi_x(i) \leqq b$ or $\psi_x(i) \leqq d$ for all $x \in K \setminus \{1\}$. Recalling the existence of the chain $b < c < 1$ in K and also the fact that F is injective, we conclude that $b \leqq F(b) < F(c) < 1$; thus ψ_c violates the latter requirement.

If $i \in \{n-3, n-4\}$ then $F(x) = g_x(i) \leqq \alpha_x(i+2) \leqq c$ for all $x \in K \setminus \{1\}$, so that $K \setminus \{1\}$ would be a prime ideal of K, a contradiction again.

This demonstrates (1) for g. The proof for $\bar{g}$ is similar, while (2) follows from (1) since n is odd.

Finally, suppose that $k \in \delta(d) \setminus \epsilon(D)$. Using (2), we obtain

$$f(d^*_{4k,3})(4k) = h_{4k}(d^*_{4k,3} \upharpoonright S(4k)) = h_{4k}(d), \text{ and}$$

$$f(d^*_{4k,3})(4k+2) = h_{4k+2}(d^*_{4k,3} \upharpoonright S(4k+2)) = h_{4k+2}(d).$$

By Lemma 2.5, there exists an $\alpha \in A(\epsilon)$ for which

$$0 < h_{4k}(d) \leqq \alpha(4k) \leqq \alpha(4k+2) \leqq h_{4k+2}(d) < 1.$$

The definition of $I_2(\epsilon)$ and $k \notin \epsilon(D)$ yield $\alpha(4k) = \alpha(4k+2) = b$. On the other hand, we already know that

$$b = f(b^*)(4k) = h_{4k}(b^* \upharpoonright S(4k)) = h_{4k}(b), \text{ and}$$

$$b = f(b^*)(4k+2) = h_{4k+2}(b^* \upharpoonright S(4k+2)) = h_{4k+2}(b);$$

hence $h_{4k}(d) \leqq h_{4k}(b)$ and $h_{4k+2}(d) \geqq h_{4k+2}(b)$. Since all homomorphisms h_j are injective, $d = b$ follows. This, however, contradicts $b \notin D$. □

STATEMENT 3.3. *Let $\delta, \epsilon \in \Delta$. If $\delta \subseteq \epsilon$ then $I_2A(\delta)$ is a $(0,1)$-sublattice of $I_2A(\epsilon)$ and the canonical inclusion map is the only $(0,1)$-homomorphism $f : I_2A(\delta) \longrightarrow I_2A(\epsilon)$ satisfying $f(\beta) = \beta$*

Proof. From $\delta \subseteq \epsilon$ it follows that a generating set of $A(\delta)$ is contained in that of $A(\epsilon)$; hence also $I_2A(\delta) \subseteq I_2A(\epsilon)$.

Assume that f is as required, with $\{S(j) | j \in n\}$, g and $\bar{g}$ as in Statement 3.2. We have only to prove that $g(1) = 1$ and $\bar{g}(n-2) = n-2$, for 3.2(1) will then ensure that, in addition to $S(2k) = \{2k\}$ for all $2k \in n$, also $S(2k+1) = \{2k+1\}$ for all $2k + 1 \in n$.

By Lemma 2.4, $0 \leqq g(1) \leqq \bar{g}(1) \leqq \bar{g}(2) \leqq 2$; thus for $g(1) = 2$ we have $\bar{g}(1) = 2$, and $\bar{g}(n-2) = n-1$ by 3.2(1). Therefore only the cases of $g(1) = 0$ and of $\bar{g}(n-2) = n-1$ need be considered.

In the first case we have

$$f(b^*)(1) = h_1(b^* \upharpoonright S(1)) > h_1(\beta \upharpoonright S(1)) = f(\beta)(1) = b = f(b^*)(2),$$

while in the second

$$\begin{aligned} f(b^*)(n-3) = b = \beta(n-2) &= f(\beta)(n-2) \\ &= h_{n-2}(\beta \upharpoonright S(n-2)) > h_{n-2}(b^* \upharpoonright S(n-2)) \\ &= f(b^*)(n-2), \end{aligned}$$

contrary to $f(b^*)$ being a non-decreasing function.

We have thus shown that f takes on a simpler form, namely

$$f(\varphi)(i) = h_i(\varphi(i)) \quad \text{for all } \varphi \in I_2A(\delta) \quad \text{and } i \in n,$$

where $\{h_i | i \in n\}$ is now a sequence of (0, 1)-endomorphisms of K. The latter sequence is non-decreasing because

$$h_i(x) = f(x^*_{i,2})(i) \leqq f(x^*_{i,2})(i+1) = h_{i+1}(x)$$

for all $x \in K$ and $i \in n-1$.

In particular, $h_i(b) = b$ for all $i \in \{2, \ldots, n-3\}$ by Lemma 3.1, and hence also $h_0(b) \leqq b \leqq h_{n-1}(b)$.

Let $d \in D$ and $k \in \delta(d)$; therefore $d^*_{4k,3} \in I_2A(\delta)$ and $4 \leqq 4k \leqq 4k+2 \leqq n-5$. By Lemma 2.5, for the element $f(d^*_{4k,3}) \in I_2A(\epsilon)$ there exists an $\alpha \in A(\epsilon)$ such that

$$\begin{aligned} h_{4k}(d) &= f(d^*_{4k,3})(4k) \leqq \alpha(4k) \leqq \alpha(4k+2) \leqq f(d^*_{4k,3})(4k+2) \\ &= h_{4k+2}(d), \end{aligned}$$

where $\alpha(4k)$, $\alpha(4k+2) \in \{b, d, d \vee b, d \wedge b\}$.

Suppose that $\alpha(4k) \nleqq d$. Then $\alpha(4k+2) \geqq \alpha(4k) \geqq b$ and $h_{4k+2}(d) \geqq \alpha(4k+2)$ imply $h_{4k+2}(d) \geqq b = h_{4k+2}(b)$; thus $d \geqq b$ because h_{4k+2} is injective. Hence $\alpha(4k) \leqq b \vee d = d$, a contradiction. For $d \ngeqq \alpha(4k+2)$ we similarly obtain $h_{4k}(d) \leqq \alpha(4k) \leqq b = h_{4k}(b)$, and the contradictory $d \leqq b$ follows by the injectivity of h_{4k}.

Therefore $h_{4k}(d) \leqq d \leqq h_{4k+2}(d)$; since $\{h_i | i \in n\}$ is an increasing sequence, $h_0(d) \leqq d \leqq h_{n-1}(d)$ is obtained for all $d \in D$ from the defintion of $I_2A(\epsilon)$. Since $h_0(b) \leqq b \leqq h_{n-1}(b)$, and because $E = D \cup \{b\}$ generates K, Lemma 2.7 applies to show that every h_i coincides with the identity endomorphism of K. Hence $f(\varphi)(i) = \varphi(i)$ for all $i \in n-1$ as claimed. □

Definition of the lattice $L_{\delta,\epsilon}$. For δ, $\epsilon \in \Delta$, let $L_{\delta,\epsilon}$ be the set of all pairs $(\varphi, \psi) \in I_2A(\delta) \times I_2A(\epsilon)$ such that

$$(\varphi, \psi) \leqq (\beta, \beta) \quad \text{or } (\varphi, \psi) \geqq (\beta, \beta) \quad \text{or } \varphi = \beta \quad \text{or } \psi = \beta.$$

The concatenation of the two components will be used to interpret elements of $L_{\delta,\epsilon}$ with K^{2n} as functions given by

$$(\varphi_0, \varphi_1)(kn + i) = \varphi_k(i) \quad \text{for } k \in 2 \quad \text{and } i \in n.$$

It is easily verified that $L_{\delta,\epsilon}$ thus becomes a solid (0, 1)-sublattice of K^{2n}.

STATEMENT 3.4. *Let δ, ϵ, δ', $\epsilon' \in \Delta$ be such that $\delta \subseteq \delta'$, $\epsilon \subseteq \epsilon'$, $\delta(D) \nsubseteq \epsilon'(D)$, and $\epsilon(D) \nsubseteq \delta'(D)$. Then $L_{\delta,\epsilon}$ is a $(0, 1)$-sublattice of $L_{\delta',\epsilon'}$; moreover, the only $(0, 1)$-homomorphism $f : L_{\delta,\epsilon} \longrightarrow L_{\delta',\epsilon'}$ is the canonical inclusion map.*

Proof. The existence of the (0, 1)-inclusion $L_{\delta,\epsilon} \subseteq L_{\delta',\epsilon'}$ is a straightforward consequence of Statement 3.3.

Let $f : L_{\delta,\epsilon} \longrightarrow L_{\delta',\epsilon'}$ be a (0, 1)-homomorphism whose associated families, given by Lemma 2.2, are $\{S(j) \subseteq 2n | j \in 2n\}$ and $\{h_j | j \in 2n\}$.

For the distinguished sequence β in either component lattice, denote

$$f(\beta, \beta) = (\varphi, \psi), \quad f(\beta, 1) = (\mu, \nu) \quad \text{and} \quad f(1, \beta) = (\tau, \sigma).$$

By Lemma 2.3, (φ, ψ) is antiskeletal. Since $(\varphi, \psi) = (\mu, \nu) \wedge (\tau, \sigma)$, both $\varphi = \mu \wedge \tau$ and $\psi = \nu \wedge \sigma$ are antiskeletal, hence by Lemma 3.1, μ or τ is antiskeletal, and also ν or σ is antiskeletal.

Next we show that if μ is antiskeletal then $S(i) \subseteq n$ for all $i \in n$.

Indeed, Lemma 3.1 gives

$$b = \varphi(2) = (\varphi, \psi)(2) = f(\beta, \beta)(2) = h_2((\beta, \beta) \restriction S(2)),$$
$$b = \mu(2) = (\mu, \nu)(2) = f(\beta, 1)(2) = h_2((\beta, 1) \restriction S(2)),$$

whence $S(2) \subseteq n$ by the injectivity of h_2. Consequently, for $i \leqq 2$, we have

$$h_i((0, \beta) \restriction S(i)) = f(0, \beta)(i) \leqq f(0, \beta)(2) = h_2((0, \beta) \restriction S(2)) = 0,$$

whence $S(i) \subseteq n$ for $i \leqq 2$; for $2 \leqq i \in n$ we have

$$h_i((1, \beta) \restriction S(i)) = f(1, \beta)(i) \geqq f(1, \beta)(2) = h_2((1, \beta) \restriction S(2)) = 1,$$

whence again $S(i) \subseteq n$.

Analogously we find that

- if ν is antiskeletal then $S(n+i) \subseteq n$ for all $i \in n$,
- if τ is antiskeletal then $S(i) \subseteq n$ for all $i \in n$,
- if σ is antiskeletal then $S(n+i) \subseteq 2n \setminus n$ for all $i \in n$.

If $S(n+i) \subseteq n$ for all $i \in n$, then the maping $f_1 : I_2A(\sigma) \longrightarrow I_2A(\epsilon')$ defined by $f_1(\xi) = f(\xi, \beta) \upharpoonright (2n \setminus n)$ is a (0, 1)-homomorphism of these lattices; hence by Statement 3.2, $\delta(D) \subseteq \epsilon'(D)$, in contradiction to the hypothesis.

Similarly, if $S(i) \subseteq 2n \setminus n$ for all $i \in n$, then $f_2(\eta) = f(\beta, \eta) \upharpoonright n$ defines a (0, 1)-homomorphism of $I_2A(\epsilon)$ to $I_2A(\delta')$, contrary to the hypothesis of $\epsilon(D) \not\subseteq \delta'(D)$.

Therefore $S(i) \subseteq n$ and $S(n+i) \subseteq 2n \setminus n$ for all $i \in n$.

The disjunction defining $L_{\delta',\epsilon'}$ easily implies that

$$f(0, \beta) = (0, \rho) \text{ with } \rho \leqq \beta,$$

$$f(1, \beta) = (1, \rho) \text{ with } \rho \geqq \beta,$$

$$f(\beta, 0) = (\lambda, 0) \text{ with } \lambda \leqq \beta, \text{ and}$$

$$f(\beta, 1) = (\lambda, 1) \text{ with } \lambda \geqq \beta;$$

thus also $f(\beta, \beta) = (\beta, \beta)$.

Furthermore, for every $(\varphi, \psi) \in L_{\delta,\epsilon}$ we have

$$\begin{aligned} f(\varphi, \psi) &= f((\varphi, 1) \wedge (1, \psi)) = f(\varphi, 1) \wedge f(1, \psi) \\ &= (f(\varphi, 1) \upharpoonright n, 1) \wedge (1, f(1, \psi) \upharpoonright (2n \setminus n)) \\ &= (f(\varphi, 1) \upharpoonright n, f(1, \psi) \upharpoonright (2n \setminus n)), \end{aligned}$$

which means that the component (0, 1)-homomorphisms $f_\delta : I_2A(\delta) \longrightarrow I_2A(\delta')$ and $f_\epsilon : I_2A(\epsilon) \longrightarrow I_2A(\epsilon')$ defined by $f_\delta(\varphi) = f(\varphi, 1) \upharpoonright n$ and $f_\epsilon(\psi) = f(1, \psi) \upharpoonright (2n \setminus n)$ satisfy $f_\delta(\beta) = f_\epsilon(\beta) = \beta$. By Statement 3.3, both f_δ and f_ϵ are the canonical inclusions, and the claim follows. □

Definition of the lattice $L_{\delta,\epsilon}/L_{\delta,\epsilon}$. Next we identify each quadruple $(\varphi_0, \varphi_1, \varphi_2, \varphi_3) \in (K^n)^4$ with its concatenation in K^{4n} by writing

$$(\varphi_0, \varphi_1, \varphi_2, \varphi_3)(kn+i) = \varphi_k(i) \quad \text{for all } k \in 4 \text{ and } i \in n,$$

and denote as $L_{\delta,\epsilon}/L_{\delta,\epsilon}$ the (0, 1)-sublattice of K^{4n} consisting of those functions $(\varphi, \psi, \sigma, \tau) \in K^{4n}$ for which $(\varphi, \psi), (\sigma, \tau) \in L_{\delta,\epsilon}$ and either $(\varphi, \psi) = (0, 0)$ or $(\sigma, \tau) = (1, 1)$.

Clearly, $L_{\delta,\epsilon}/L_{\delta,\epsilon}$ is a solid sublattice of K^{4n}.

STATEMENT 3.5. *Let δ, ϵ and δ', ϵ' be as in Statement* 3.4. *Under any* (0, 1)-*homomorphism $f : L_{\delta,\epsilon}/L_{\delta,\epsilon} \longrightarrow L_{\delta',\epsilon'}$, the element $f(0, 0, 1, 1)$ is comparable with (β, β), and*

(1) *if* $f(0,0,1,1) \leqq f(0,0,\beta,\beta)$ *then* $f(0,0,1,1) = (0,0)$,
(2) *if* $f(0,0,1,1) \geqq f(\beta,\beta,1,1)$ *then* $f(0,0,1,1) = (1,1)$.

Proof. Let $\{S(j) \subseteq 4n | j \in 2n\}$ and $\{h_j | j \in 2n\}$ be the sets and (0, 1)-embeddings associated with f by Lemma 2.2; hence $f(\varphi,\psi,\sigma,\tau)(j) = h_j((\varphi,\psi,\sigma,\tau) \restriction S(j))$ for every $j \in 2n$.

We prove that in $(\lambda,\mu) = f(0,0,1,1)$, neither λ nor μ can coincide with β; this will ensure that (λ,μ) and (β,β) are comparable.

Assume that $\lambda = \beta$. Then, for each $j \in n$,

$$h_j((0,0,1,1) \restriction S(j)) = f(0,0,1,1)(j) = \beta(j) \notin \{0,1\};$$

therefore $S(j) \cap 2n \neq \emptyset \neq S(j) \cap (4n \setminus 2n)$.

Denote $(\xi,\eta) = f(0,0,\beta,\beta)$. For every $j \in n$ we thus have

$$\begin{aligned} 0 = h_j((0,0,0,0) \restriction S(j)) &< h_j((0,0,\beta,\beta) \restriction S(j)) \\ &< h_j((0,0,1,1) \restriction S(j)) = \beta(j). \end{aligned}$$

Hence $0 < \xi(j) < 1$ for all $j \in n$, and thus $\xi \in I_2A(\delta')$ must be antiskeletal. By Lemma 3.1, $h_2((0,0,\beta,\beta) \restriction S(2)) = f(0,0,\beta,\beta)(2) = \xi(2) = b = \beta(2) = h_2((0,0,1,1) \restriction S(2))$. Since $(0,0,\beta,\beta) \restriction S(2)$ and $(0,0,1,1) \restriction S(2)$ are distinct, this contradicts the fact that h_2 is an embedding. Similarly we find that $\mu \neq \beta$.

Since all h_j are embeddings, the hypothesis of (1) implies that $(0,0,1,1) \restriction S(j) \leqq (0,0,\beta,\beta) \restriction S(j)$ and hence also $S(j) \subseteq 2n$ for all $j \in 2n$; as a result, $f(0,0,1,1)(j) = 0$ for all $j \in 2n$. The proof of (2) is dual. □

4. Constructions for short K. The term *short* (0, 1)-*lattice* will be used to refer to any of the four finite lattices M_3, K_4, Q, and the dual Q^d of Q, shown in Figure 1.

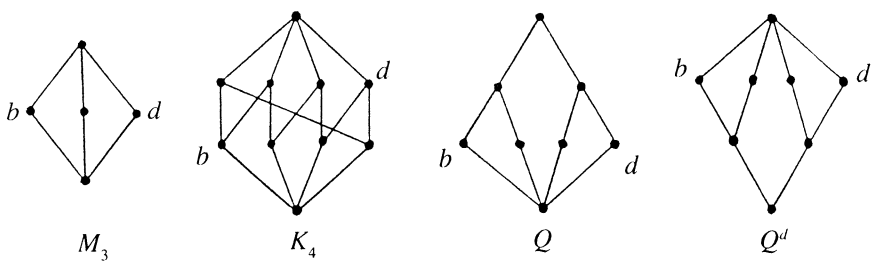

Figure 1

PROPOSITION 4.1. *Each short lattice K is simple and contains a complemented pair $\{b,d\}$ such that the identity* id_K *is the only* (0, 1)-*endomorphism f of K satisfying $f(b) = b$ and $f(d) = d$.*

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Proof. By a direct inspection of Figure 1. □

PROPOSITION 4.2. *Let L be a* $(0, 1)$*-lattice of length* $\leqq 3$ *with no prime ideals. Then the variety* $\mathrm{Var}(L)$ *generated by L contains a short* $(0, 1)$*-lattice K.*

Proof. Assume that L does not contain a copy of M_3. Then L is of length 3 and all elements in $L \setminus \{0, 1\}$ are atoms or coatoms (or both). Clearly,

(a) for every atom $a \in L$ and every coatom $c \in L$, either a and c are complementary in L or $a \leqq c$.

(b) For any coatom $c \in L$, there are at most two atoms of L in the ideal $(c]$, for otherwise $(c] \subseteq L$ would contain a copy of M_3, and

(c) for any coatom $c \in L$ there exist at most two atoms a_0 and a_1 incomparable to c, and $a_0 \vee a_1 < 1$.

Indeed, for $i \in 3$, let a_i be three atoms incomparable to c. If $a_i \vee a_j = 1$ then a_i, a_j, c would be pairwise complementary, and thus generate a sublattice of L isomorphic to M_3. For $a_0 \vee a_1 = d < 1$, the elements a_2, c, d would be pairwise complementary.

From (b), (c) and their duals we conclude that

(d) L has at most four atoms and at most four coatoms.

If L had only one coatom c then $(c]$ would be a prime ideal; therefore

(e) L has at least two atoms and at least two coatoms.

(f) If L has exactly two coatoms c and d, then L is isomorphic to Q.

Indeed, if one of the two coatoms were join-irreducible, then the principal ideal generated by the other one would be prime. Thus both c and d are joins of pairs of atoms, and these pairs must be disjoint, for otherwise both $(c]$ and $(d]$ would be prime ideals.

(g) If some coatom $c \in L$ is join-irreducible, then L has only two atoms (and hence, by the dual of (f), L is isomorphic to Q^d).

Indeed, assume that a_i, for $i \in 3$, are three distinct atoms and a_2 is the only atom in $(c]$. Then $d = a_0 \vee a_1$ is a coatom by (c). Also, $\{c, d\}$ is a complemented pair, for otherwise $c \wedge d = a_2$ and $\{a_0, a_1, a_2\} \subseteq (d]$ would contradict (b). If $L = [a_2) \cup (d]$ then $(d]$ is a prime ideal of L. Any $b \in L \setminus [a_2) \cup (d]$ satisfies $b \nleqq c$, and it follows that $\{b, c\}$ is a complemented pair. Since L does not contain a copy of M_3 while $\{b, c\}$ and $\{c, d\}$ are its complemented pairs, from $b \nleqq d$ it follows that $b \wedge d > 0$; thus $b \wedge d = a_i$ for some $i \in 2$. But then b is a coatom, and (a) implies that $\{b, c, a_{1-i}\}$ generates a sublattice of L isomorphic to M_3 after all.

Finally,

(h) if all atoms and coatoms are reducible, then L is isomorphic to K_4.

Indeed, by (f) and its dual, L has at least three atoms and at least three coatoms. If both numbers were three, then L would be Boolean. Since three atoms produce at most three join-reducible coatoms, a fourth coatom would have to be join irreducible, in contradiction to (g). The only remaining case is that L has four reducible atoms and four reducible coatoms; in view of (b) and (c), the lattice L must be isomorphic to K_4. □

In the remainder of this section, K will denote a short (0, 1)-lattice; all other notation is that of Sections 2 and 3.

LEMMA 4.3. *To every* $(0,1)$*-homomorphism* $f : L \longrightarrow M$ *of solid* $(0,1)$*-sublattices* L *and* M *of* K^n *and* K^m*, respectively, there exists a function* $g : m \longrightarrow n$ *and a family* $\{h_j | j \in m\}$ *of automorphisms of* K *such that*

$$f(\varphi)(j) = h_j(\varphi \cdot g(j)) \quad \textit{for all } \varphi \in L \textit{ and } j \in m.$$

Conseqently, $f(\varphi)$ *is skeletal for any skeletal* $\varphi \in L$.

Proof. For any $S \subseteq n$, the lattice $p_S(L)$ is a solid (0, 1)-sublattice of K^S. Since $|p_S(L)| > |K|$ whenever $|S| > 1$, the lattice $p_S(L)$ can be embedded into K only when $|S| = 1$. The set family $\{S(j) : j \in m\}$ associated with f by Lemma 2.2 thus consists of singletons; that is, $S(j) = \{g(j)\}$ for all $j \in m$, where g is the function from Lemma 3.2. The formula for f then follows immediately from that of Lemma 2.2, and implies that f preserves skeletality. □

LEMMA 4.4. *Let* $f : I_1B \longrightarrow I_1A$ *be a* $(0,1)$*-homomorphism for* $(0,1)$*-sublattices* $A, B \subseteq I = I(K^n)$*. If* A *is distributive, then the function* $g : n \longrightarrow n$ *associated with* f *is the identity of* n.

Proof. By Lemma 2.4, the function g is non-decreasing. Assume that $g(i) = g(i+1)$ for some $i \in n$. Then, for every $x \in K$,

$$\begin{aligned} h_i(x) &= h_i(x^*_{g(i),1}(g(i))) = f(x^*_{g(i),1})(i) \leqq f(x^*_{g(i),1})(i+1) \\ &= h_{i+1}(x^*_{g(i),1}(g(i+1))) = h_{i+1}(x), \end{aligned}$$

and $h_i(x) = h_{i+1}(x)$ is thus obtained for every $x \in K$.

From Lemma 2.6 it follows that

$$\varphi_x = (\chi_i \wedge f(x^*_{g(i),1})) \vee \chi_{i+2} \in K^*_{i,2} \cap I_1A \subseteq I_0A,$$

thus the mapping $H : K \longrightarrow I_0A$ defined by $H(x) = \varphi_x$ is an ordinary (not necessarily (0, 1)-preserving) non-constant homomorphism of K into I_0A. This, however, is impossible since $p_i(I_0A) = p_i(A)$ for all $i \in n$ implies that the lattice I_0A is distributive. □

Definition of the lattice $B(\delta)$. Let Δ be the set of all non-void subsets of $\{0, 1, 2, 3\} = 4$. For b, $d \in K$ of Proposition 4.1 and any $\delta \in \Delta$, let $B(\delta)$ be the (0, 1)-sublattice of K^6 generated by the set

$$\{d^*_{j,2} | j \in 5\} \cup \{d^*_{k,3} | k \in \delta\} \cup \{b^*\}.$$

It is easily seen that $\delta \subseteq \epsilon$ implies that $B(\delta)$ is a (0, 1)-sublattice of $B(\epsilon)$ and, consequently, that $I_1B(\delta)$ is a (0, 1)-sublattice of $I_1B(\epsilon)$.

STATEMENT 4.5. *If* $f : I_1B(\delta) \longrightarrow I_1B(\epsilon)$ *is a* $(0, 1)$-*homomorphism then* $\delta \subseteq \epsilon$ *and* f *is the canonical inclusion mapping.*

Proof. Since $p_i(B(\epsilon)) = \{0, b, d, 1\}$ is distributive for every $i \in 6$, Lemma 4.4 applies and yields

$$f(\varphi)(i) = h_i(\varphi(i)) \text{ for } i \in 6,$$

where each h_i is an automorphism of the lattice K.

Since $h_i(b) = f(b^*)(i) \leqq f(b^*)(i+1) = h_{i+1}(b)$ for all $i \in 5$, and because each h_i is invertible, $h_i(b) = c$ for some $c \in K \setminus \{0, 1\}$ and all $i \in 6$; that is, $f(b^*) = c^* \in I_1B(\epsilon)$. Since all generators of $B(\epsilon)$ other than b^* belong to I_3, the lattice $I_1B(\epsilon)$ is included in the (0, 1)-lattice $I_3\{0, b^*, 1\}$, and $c = b$ follows by Lemma 2.5.

Similarly, from $h_j(d) = f(d^*_{j,2})(j) \leqq f(d^*_{j,2})(j+1) = h_{j+1}(d)$ for all $j \in 5$ we conclude that $h_j(d) = e$ for some $e \in K \setminus \{0, 1\}$ and all $j \in 6$. Thus, by Lemma 2.6,

$$f(d^*_{j,2}) = e^*_{j,2} \in K^*_{j,2} \cap I_1B(\epsilon) \subseteq I_0B(\epsilon),$$

hence $e \in \{b, d\}$; from $h_j(d) \neq h_j(b) = b$ it follows that $e = d$.

We have proved that $h_i(b) = b$ and $h_i(d) = d$ for all $i \in 6$. By Proposition 4.1, h_i is the identity on K for all $i \in 6$ and, consequently, f is the canonical inclusion.

Let $k \in \delta$. By Lemma 2.6, $f(d^*_{k,3}) = d^*_{k,3} \in I_1B(\epsilon)$ implies that $d^*_{k,3} = (\chi_k \wedge \gamma) \vee \chi_{k+3}$ for some $\gamma \in B(\epsilon)$. All generators of $B(\epsilon)$ other than b^* are incomparable to b^* and form a chain; hence γ is a generator of the distributive lattice $B(\epsilon)$, or $\gamma \in \{\varphi \vee b^*, \varphi \wedge b^*, (\varphi \vee b^*) \wedge \psi\}$, where $\varphi < \psi$ are generators of $B(\epsilon)$ other than b^*. From $(\varphi \vee b^*)(i) \neq d \neq (\varphi \wedge b^*)(i)$ for all $i \in 6$ it now follows that $\gamma = (\varphi \vee b^*) \wedge \psi = \varphi \vee (b^* \wedge \psi)$. But then, for each $i \in \{k, k+1, k+2\}$, we have $\gamma(i) = d$ and, since $\{b, d\}$ is a complemented pair, $\varphi(i) = \psi(i) = d$. Therefore $\psi = d^*_{k,3}$ and, from the definition of $B(\epsilon)$, it follows that $k \in \epsilon$. □

Definition of the lattice $L_{\delta,\epsilon}$. Analogously to the definition presented in Section 3, for any δ, $\epsilon \subseteq 4$ we denote as $L_{\delta,\epsilon}$ the set of all $(\varphi, \psi) \in I_1B(\delta) \times I_1B(\epsilon)$ satisfying

$$(\varphi, \psi) \leqq (b^*, b^*) \text{ or } (\varphi, \psi) \geqq (b^*, b^*) \text{ or } \varphi = b^* \text{ or } \psi = b^*;$$

as before, identify $L_{\delta,\epsilon}$ with a solid (0, 1)-sublattice of K^{2n} in such a way that $\varphi \in K^n$ and $\psi \in K^{2n \setminus n}$.

STATEMENT 4.6. *Let* δ, ϵ, δ', $\epsilon' \in \Delta$ *satisfy* $\delta \subseteq \delta'$, $\epsilon \subseteq \epsilon'$, $\delta \not\subseteq \epsilon'$, *and* $\epsilon \not\subseteq \delta'$. *Then* $L_{\delta,\epsilon} \subseteq L_{\delta',\epsilon'}$, *and the only (0, 1)-homomorphism* $f : L_{\delta,\epsilon} \longrightarrow L_{\delta',\epsilon'}$ *is the inclusion map.*

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Proof. The existence of the (0, 1)-inclusion $L_{\delta,\epsilon} \subseteq L_{\delta',\epsilon'}$ follows easily from $\delta \subseteq \delta'$ and $\epsilon \subseteq \epsilon'$.

Let $f : L_{\delta,\epsilon} \longrightarrow L_{\delta',\epsilon'}$ be a (0, 1)-homomorphism with the function $g : 2n \longrightarrow 2n$ and the family $\{h_j | j \in 2n\} \subseteq \mathrm{Aut}(K)$ according to Lemma 4.3.

If $g(0) \in n$, then for every $i \in n$ we have $h_i((1, b^*)(g(i))) = f(1, b^*)(i) \geqq f(1, b^*)(0) = h_0((1, b^*)(g(0))) = 1$, whence $g(i) \in n$.

Similarly,

– if $g(n) \in n$ then $g(n+i) \in n$ for all $i \in n$,

– if $g(0) \in 2n \setminus n$ then $g(i) \in 2n \setminus n$ for all $i \in n$,

– if $g(n) \in 2n \setminus n$ then $g(n+i) \in 2n \setminus n$ for all $i \in n$.

Suppose that $g(n+i) \in n$ for all $i \in n$. Then the mapping $H : I_1B(\delta) \longrightarrow I_1B(\epsilon')$ defined by $H(\varphi) = f(\varphi, b^*) \upharpoonright (2n \setminus n)$ is a (0, 1)-homomorphism, while Statement 4.5 yields $\delta \subseteq \epsilon'$, a contradiction.

If $g(i) \in 2n \setminus n$ for all $i \in n$, then $H(\psi) = f(b^*, \psi) \upharpoonright n$ is a (0, 1)-homomorphism of $I_1B(\epsilon)$ to $I_1B(\delta')$ which, according to Statement 4.5, contradicts $\epsilon \not\subseteq \delta'$.

Therefore $g(i) \in n$ and $g(n+i) \in 2n \setminus n$ for all $i \in n$ and, as in the proof of Statement 3.4, $f(\varphi, \psi) = (f(\varphi, 1) \upharpoonright n, f(1, \psi) \upharpoonright (2n \setminus n)) = (f_\delta(\varphi), f_\epsilon(\psi))$. By Statement 4.5, the component (0, 1)-homomorphisms $f_\delta : I_1B(\delta) \longrightarrow I_1B(\delta')$ and $f_\epsilon : I_1B(\epsilon) \longrightarrow I_1B(\epsilon')$ are the canonical inclusions. □

Using the latter lattice $L_{\delta,\epsilon}$ in a manner similar to that used in Section 3, we form the set $L_{\delta,\epsilon}/L_{\delta,\epsilon}$ of all quadruples $(\varphi, \psi, \sigma, \tau)$ with $(\varphi, \psi), (\sigma, \tau) \in L_{\delta,\epsilon}$ and $(\varphi, \psi) = (0, 0)$ or $(\sigma, \tau) = (1, 1)$, and interpret it again, in an obvious fashion, as a solid (0, 1)-sublattice of K^{4n}.

STATEMENT. 4.7. *If* δ, ϵ, δ', $\epsilon' \subseteq 4$ *are as in Statement* 4.6, *then any* (0, 1)-*homomorphism* $f : L_{\delta,\epsilon}/L_{\delta,\epsilon} \longrightarrow L_{\delta',\epsilon'}$ *satisfies* $f(0, 0, 1, 1) \in \{(0, 0), (1, 1)\}$.

Proof. By Lemma 4.3, $f(0, 0, 1, 1)$ must be skeletal, but (0, 0) and (1, 1) are the only skeletal members of $L_{\delta',\epsilon'}$. □

5. The universality of Var(K). The choice of n made in the definition of $A(\delta)$ at the beginning of Section 3 and the value $n = 6$ chosen in the parallel definition of $B(\delta)$ in Section 4 ensure the existence of δ, ϵ, δ', $\epsilon' \in \Delta$ satisfying

$$\delta \subseteq \delta', \epsilon \subseteq \epsilon', \delta(D) \not\subseteq \epsilon'(D), \epsilon(D) \not\subseteq \delta'(D)$$

as required by Statement 3.4, or, correspondingly,

$$\delta \subseteq \delta', \epsilon \subseteq \epsilon', \delta \not\subseteq \epsilon', \epsilon \not\subseteq \delta',$$

required in Statement 4.6. For a fixed quadruple δ, ϵ, δ', ϵ' satisfying one of these conditions, let

$$M = L_{\delta,\epsilon} \quad L_0 = L_{\delta',\epsilon} \quad L_1 = L_{\delta,\epsilon'} \quad L = L_{\delta',\epsilon'}.$$

Our intention is to define a functor $F : \mathbf{U} \longrightarrow \mathrm{Var}(L)$ from the universal category $\mathbf{U}$ described in Corollary 1, 3 into the variety $\mathrm{Var}(L) = \mathrm{Var}(K)$ generated by K, and prove that F is a full embedding.

We will choose F from amongst the subfunctors of $H = L^{hom(-,2)}$, a functor $H : \mathrm{Var}(2) \longrightarrow \mathrm{Var}(L)$ which is the composite of the set functor $\hom(-,2)$, where $2 = \{0,1\} \subseteq L$, from the variety Var(2) of distributive (0, 1)-lattices, and of the cartesian power functor assigning the (0, 1)-lattice L^X to any set X.

It will be convenient to regard each element d of any distributive (0, 1)- lattice D as the function $d : \hom(D,2) \longrightarrow L$ defined by $d(p) = p(d)$: in this way, $H(D)$ becomes a (0, 1)-extension of D.

Let * denote the diagonal (0, 1)-embedding of L into $H(D)$; thus, for each $z \in L$, the function $z^* : \hom(D,2) \longrightarrow L$ is given by $z^*(p) = z$ for all $p \in \hom(D,2)$.

For any $\mathbf{U}$-object (D, p_0, p_1), define $G(D, p_0, p_1)$ as the set of those $\varphi \in H(D)$ which have the form $\varphi = d \vee z^*$ or $\varphi = d \wedge z^*$ for some $d \in D \setminus \{0,1\}$ and $z \in L$, and which satisfy $\varphi(p_i) \in L_i$ for $i = 0, 1$.

For $i, j \in \{0,1\}$, define

$$D_{i,j} = \{d \in D \setminus \{0,1\} | d(p_0) = i, d(p_1) = j\}.$$

LEMMA 5.1. *For* $G = G(D, p_0, p_1)$ *and* $z \in L$:
(1) *if* $d \in D_{0,0}$ *then* $d \vee z^* \in G \Leftrightarrow z \in M$, *and* $d \wedge z^* \in G \Leftrightarrow z \in L$,
(2) *if* $d \in D_{0,1}$ *then* $d \vee z^* \in G \Leftrightarrow z \in L_0$, *and* $d \wedge z^* \in G \Leftrightarrow z \in L_1$,
(3) *if* $d \in D_{1,0}$ *then* $d \vee z^* \in G \Leftrightarrow z \in L_1$, *and* $d \wedge z^* \in G \Leftrightarrow z \in L_0$,
(4) *if* $d \in D_{1,1}$ *then* $d \vee z^* \in G \Leftrightarrow z \in L$, *and* $d \wedge z^* \in G \Leftrightarrow z \in M$.

Proof. Follows easily once it is noted that

$$(d \vee z^*)(p_i) = d(p_i) \vee z = z \quad \text{for } d(p_i) = 0,$$

$$(d \vee z^*)(p_i) = 1 \quad \text{for } d(p_i) = 1,$$

$$(d \wedge z^*)(p_i) = d(p_i) \wedge z = 0 \quad \text{for } d(p_i) = 0,$$

$$(d \wedge z^*)(p_i) = z \quad \text{for } d(p_i) = 1.$$

□

LEMMA 5.2. *Let* $f : (D, p_0, p_1) \longrightarrow (E, q_0, q_1)$ *be a* (0, 1)*-homomorphism. Then for any* $d \vee z^*$, $d \wedge z^* \in G(D, p_0, p_1)$ *we have*
(1) $H(f)(d \vee z^*) = f(d) \vee z^*$, *and*
(2) $H(f)(d \wedge z^*) = f(d) \wedge z^*$.
If f *is a* $\mathbf{U}$*-morphism, then* $H(f)$ *maps* $G(D, p_0, p_1)$ *into* $G(E, q_0, q_1)$.

Proof. For all $\varphi \in H(D)$ and all $q \in \hom(E,2)$ we have $(H(f)(\varphi))(q) = \varphi(q \cdot f)$; therefore

$$(H(f)(d))(q) = d(q \cdot f) = (q \cdot f)(d) = q(f(d)) = f(d)(q),$$

$$(H(f)(z^*)(q) = z^*(q \cdot f) = z = z^*(q),$$

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and the equalities in (1) and (2) follow. Furthermore, $(d \vee z^*)(p_i) = (f(d) \vee z^*)(q_i)$ and $(d \wedge z^*)(p_i) = (f(d) \wedge z^*)(q_i)$ because $f(d)(q_i) = d(p_i)$ for any **U**-morphism f. □

We now define $F(D, p_0, p_1)$ as the (0, 1)-lattice generated by the set $G(D, p_0, p_1)$. For any **U**-morphism $f : (D, p_0, p_1) \longrightarrow (E, q_0, q_1)$ we set $F(f) = H(f) \cap (F(D, p_0, p_1) \times F(E, q_0, q_1))$.

In view of Lemma 5.2, the functor F is well-defined.

LEMMA 5.3. *If $\varphi \in F(D, p_0, p_1)$ and $i = 0, 1$, then $\varphi(p_i) \in L_i$.*

Proof. Since this is true, by definition, for all $\varphi \in G(D, p_0, p_1)$, it suffices to note that L_0 and L_1 are lattices, and that $G(D, p_0, p_1)$ generates $F(D, p_0, p_1)$. □

LEMMA 5.4. $F(D, p_0, p_1) \cap 2^{hom(D,2)} = D$, *that is, the skeletal functions in $F(D, p_0, p_1)$ coincide with the elements of D.*

Proof. For each $\varphi \in F(D, p_0, p_1)$ and $z \in L$ define the z-trim φ_z : $\hom(D, 2) \longrightarrow 2$ by the requirement that

$$\varphi_z(p) = 1 \qquad \text{if and only if} \quad \varphi(p) \geqq z.$$

A simple induction below will show that every $\varphi \in F(D, p_0, p_1)$ has only finitely many distinct z-trims, all of which lie in D. Since $\varphi_z = \varphi$ for any $z \notin \{0, 1\}$ and any skeletal $\varphi \in H(D)$, this will demonstrate the lemma.

The above claim is easily verified for all $\varphi \in G(D, p_0, p_1)$ as follows. If $\varphi = d \vee w^*$ for $d \in D$ and $w \in L$, then $\varphi_z = 1 \in H(D)$ for all $z \leqq w$, while $\varphi_z = d$ for other $z \in L$. For $\varphi = d \wedge w^*$ we have $\varphi_0 = 1 \in H(D)$, $\varphi_z = d$ for $0 < z \leqq w$, and $\varphi_z = 0 \in H(D)$ for all other $z \in L$.

Assume the claim to be valid for φ, $\psi \in F(D, p_0, p_1)$.

Let $\mu = \varphi \wedge \psi$ first. Then, for every $z \in L$ and $p \in \hom(D, 2)$, we have $\mu(p) \geqq z$ if and only if $\varphi(p), \psi(p) \geqq z$; hence $\mu_z = \varphi_z \wedge \psi_z \in D$ for every $z \in L$, and μ has only finitely many distinct z-trims.

Secondly, let $\nu = \varphi \vee \psi$. If $u \vee v \geqq z$ in L and $(\varphi_u \wedge \psi_v)(p) = 1$ then $\varphi(p) \geqq u$ and $\psi(p) \geqq v$, so that $\nu(p) = \varphi(p) \vee \psi(p) \geqq u \vee v \geqq z$; that is, $\nu_z(p) = 1$. Since all values of the functions φ_u, ψ_v and ν_z lie in $\{0, 1\}$, this shows that

$$\nu_z \geqq \bigvee \{\varphi_u \wedge \psi_v | u \vee v \geqq z\} = \sigma,$$

where the join represents an element of D because of the induction hypothesis. Furthermore, if $\nu_z(p) = 1$ then $\varphi(p) \vee \psi(p) \geqq z$; from $\varphi_{\varphi(p)}(p) = 1$ and $\psi_{\psi(p)}(p) = 1$ it follows that $\nu_z \leqq \sigma$. Hence $\nu_z \in D$ and, clearly, ν has finitely many distinct z-trims as required. □

LEMMA 5.5. *Let $\bar{f} : F(D, p_0, p_1) \longrightarrow F(E, q_0, q_1)$ be a (0, 1)-homomorphism. If $\bar{f}(D) \subseteq E$ then the restricted mapping $f : (D, p_0, p_1) \longrightarrow (E, q_0, q_1)$ defined by $f = \bar{f} \cap (D \times E)$ is a **U**-morphism and $\bar{f} = F(f)$.*

Proof. Assume $d \vee z^* \in G(D, p_0, p_1)$ and $q \in \hom(E, 2)$. If $f(d)(q) = 1$ then $\bar{f}(d \vee z^*)(q) = 1 = (f(d) \vee z^*)(q)$. If $f(d)(q) = 0$ then $\bar{f}(d \vee z^*)(q) = z = (f(d) \vee z^*)(q)$, because the mapping I given by $I(x) = \bar{f}(d \vee x^*)(q)$ is, by Lemma 5.1, a (0, 1)-homomorphism of M or L_0 or L_1 or L into L, according to whether d is in $D_{0,0}$ or $D_{0,1}$ or $D_{1,0}$ or $D_{1,1}$; by Statement 3.4 or Statement 4.6, any such I must be an inclusion map. Therefore

$$\bar{f}(d \vee z^*) = f(d) \vee z^* \quad \text{for all } d \vee z^* \in G(D, p_0, p_1),$$

and also dually,

$$\bar{f}(d \wedge z^*) = f(d) \wedge z^* \quad \text{for all } d \wedge z^* \in G(D, p_0, p_1);$$

using Lemma 5.2(1, 2), we have thus shown that $\bar{f}$ and $H(f)$ coincide on $F(D, p_0, p_1)$.

If $f(d)(q_1) = 0$ for some $d \in D_{0,1}$, then by Lemma 5.3 and Lemma 5.1(2), $z = \bar{f}(d \vee z^*)(q_1) \in L_1$ for all $z \in L_0$, a contradiction showing that $f(d)(q_1) = 1$, and thus also, dually, that $f(d)(q_0) = 0$ for all $d \in D_{0,1}$. Therefore $f(D_{0,1}) \subseteq E_{0,1}$ and, by symmetry, also $f(D_{1,0}) \subseteq E_{1,0}$. Hence f is a **U**-morphism, $H(f)$ maps $F(D, p_0, p_1)$ into $F(E, q_0, q_1)$ by Lemma 5.2, and $\bar{f} = F(f)$ follows. □

STATEMENT 5.6. *The functor F is a full embedding.*

Proof. Let $\bar{f} : F(D, p_0, p_1) \longrightarrow F(E, q_0, q_1)$ be a (0, 1)-homomorphism. In view of Lemma 5.5, we only need to prove that $\bar{f}(D) \subseteq E$. By Lemma 5.4, this is equivalent to showing that $\bar{f}(d)$ is skeletal for every $d \in D$.

Let $d \in D$ and $q \in \hom(E, 2)$.

There is a (0, 1)-homomorphism $h : M/M \longrightarrow L$ defined by $h(0, 0, \sigma, \tau) = \bar{f}(d \wedge (\sigma, \tau)^*)(q)$ and $h(\varphi, \psi, 1, 1) = \bar{f}(d \vee (\varphi, \psi)^*)(q)$, to which either Statement 3.5 or Statement 4.7 applies according to the case at hand.

In the first case, $h(0, 0, 1, 1) = \bar{f}(d)(q)$ must be comparable to (β, β) by Statement 3.5. If $h(0, 0, 1, 1) \geqq (\beta, \beta)$ then, by Lemma 5.1 and statement 3.4,

$$h(\beta, \beta, 1, 1) = \bar{f}(d \vee (\beta, \beta)^*)(q) = \bar{f}(d)(q) \vee (\beta, \beta) = h(0, 0, 1, 1),$$

and hence $\bar{f}(d)(q) = h(0, 0, 1, 1) = (1, 1)$ by Statement 3.5(2). Using Statement 3.5(1) when $h(0, 0, 1, 1) \leqq (\beta, \beta)$, we similarly find that $\bar{f}(d)(q) = h(0, 0, 1, 1) = (0, 0)$. Thus $\bar{f}(d)(q)$ is either (0, 0) or (1, 1) for any $d \in D$ and $q \in \hom(D, 2)$.

In the second case, $h(0, 0, 1, 1) = \bar{f}(d)(q)$ must be either (0, 0) or (1, 1) by Statement 4.7.

In either case, $\bar{f}(d)$ is skeletal for every $d \in D$. □

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