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ASSOCIATED CONTINUOUS HAHN POLYNOMIALS

Dedicated to our friend P. G. (Tim) Rooney on his 65th birthday

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ABSTRACT. Explicit solutions to the recurrence relation for associated continuous Hahn polynomials are derived using ${}_3F_2$ contiguous relations. These solutions are used to obtain a new continued fraction and the associated absolutely continuous measure. An exceptional case is shown to yield entry 33 in Chapter 12 of Ramanujan's second notebook.

1. **Introduction.** Contiguous relations for hypergeometric functions are a fundamental source for obtaining explicit results for orthogonal polynomials and their corresponding continued fractions (eg. see [9], [10], [11], [14], [17], [18], [20]). In this paper we demonstrate this viewpoint for the case of associated continuous Hahn polynomials.

Continuous Hahn polynomials $\{P_n(x))\}_0^\infty$ are expressed in terms of hypergeometric functions of the type ${}_3F_2$ of unit argument (see Askey [1]) with

$$P_n(x) = i^n \frac{(a+c,a+d)_n}{n!} {}_3F_2\left(\begin{matrix} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{matrix} \middle| 1 \right), \tag{1.1}$$

where $(a)_n := \Gamma(n+a)/\Gamma(a)$, and $(a_1,a_2,\ldots,a_p)_n := \prod_{j=1}^p (a)_j$.

The orthogonality relation for the case when a,b,c,d have positive real part is

$$\begin{aligned} &\int_{-\infty}^{\infty} P_n(x)P_m(x)w(x)\,dx \\ &\quad = \frac{\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)}{(2n+a+b+c+d-1)\Gamma(n+a+b+c+d-1)n!}\delta_{m,n}, \end{aligned} \tag{1.2}$$

where

$$w(x) = \Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix)/2\pi,$$

and $(-\infty,\infty)$ is replaced by a complex contour for more general parameter values.

These polynomials must necessarily also satisfy a three term recurrence relation. This can be derived as a limiting case of the three term recurrence relation for Wilson polynomials $P_n(a,b,c,d,t^2)$, [17]. We replace the Wilson parameters (a,b,c,d) by

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$(a+u, b+u, c-u, d-u)$, the variable t by $t+iu$ and let $u \longrightarrow \infty$. We then obtain the three term recurrence relation

$$\begin{aligned}(1.3)\qquad &\frac{(n+s-1)(n+1)}{(2n+s)(2n+s-1)}P_{n+1}(x)\\ &+\frac{(n+a+c-1)(n+a+d-1)(n+b+d-1)(n+b+c-1)}{(2n+s-2)(2n+s-1)}P_{n-1}(x)\\ &=\left[x+i\frac{(c+d-b-a)}{4}+i\frac{(s-2)(s-2a-2d)(s-2a-2c)}{4(2n+s)(2n+s-2)}\right]P_n(x)\end{aligned}$$

where $s=a+b+c+d$.

At first glance it appears that continuous Hahn polynomials depend on four parameters a, b, c, d. However, after a translation of $x\big(x \longrightarrow x-i(c+d-a-b)/4\big)$ it can be seen that there are only three essential parameters. We take these parameters to be $a+c, a+d, s$ and denote them by A, B, D respectively.

With this parameter notation and the above translated x we can rewrite (1.1)–(1.3) in terms of monic continuous Hahn polynomials $\{p_n(x)\}$, that is

$$(1.4)\qquad p_n(x)=i^n\frac{(D-1,A,B)_n}{(D-1)_{2n}}\,{}_3F_2\left(\begin{matrix}-n, n+D-1, ix+(2A+2B-D)/4\\ A, B\end{matrix}\,\Big|\,1\right).$$

The $p_n(x)$'s satisfy the three term recurrence relation

$$\begin{aligned}(1.5)\qquad &p_{n+1}(x)-\left(x+\frac{i(D-2)(D-2A)(D-2B)}{4(2n+D)(2n+D-2)}\right)p_n(x)\\ &=-\frac{n(n+A-1)(n+B-1)(n+D-2)(n+D-A-1)(n+D-B-1)}{(2n+D-1)(2n+D-2)^2(2n+D-3)}p_{n-1}(x)\end{aligned}$$

and they are orthogonal on a complex contour Γ. See Askey [1] for details. Their orthogonality relation is, [1]

$$(1.6)\qquad \int_\Gamma p_n(x)p_m(x)d\alpha(x)=\frac{n!\,(A,B,D-1,D-A,D-B)_n}{(D-1,D)_{2n}}\delta_{m,n}$$

and the absolutely continuous component of $d\alpha$ is

$$(1.7)\qquad 2\pi\frac{d\alpha(x)}{dx}=\mathbf{\Gamma}\left(\begin{matrix}D, ix+\frac{A+B}{2}-\frac{D}{4}, ix+\frac{3}{4}D-\frac{A+B}{2}, -ix+\frac{D}{4}+\frac{A-B}{2}, -ix+\frac{D}{4}+\frac{B-A}{2}\\ A, B, D-A, D-B\end{matrix}\right)$$

where the Γ function on the right side of (1.7) is defined via

$$\mathbf{\Gamma}\left(\begin{matrix}a_1, a_2, \ldots, a_n\\ b_1, b_2, \ldots, b_m\end{matrix}\right)=\frac{\prod_{j=1}^n\Gamma(a_j)}{\prod_{k=1}^m\Gamma(b_k)}.$$

Properties of the above three parameter Hahn polynomials were first obtained by Atakishiyiev and Suslov in [3]. They used the real parameters α, β, γ where $A=\beta+1$, $B=(\alpha+\beta+2-2i\gamma)/2, D=\alpha+\beta+2$ and our x is their $-x/2$. Note that symmetric

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cases occur when $D = 2A$ or $D = 2B$. An exceptional symmetric case, not given by (1.7), occurs when $D = 2$. This exceptional case corresponds to Ramanujan's entry 33 of Chapter 12 in his second notebook; see B. Berndt et al [6]. Another exceptional case occurs when $D = 1$. We shall discuss these cases in detail in Section 3.

The recurrence relation for monic associated continous Hahn polynomials is obtained by a translation of n in the coefficients of (1.5). Equivalently we may introduce a fourth parameter C and consider monic polynomials $p_n(x; C)$ satisfying the initial conditions $p_{-1}(x; C) = 0, p_0(x; C) = 1$ and the recurrence relation

(1.8)
$$p_{n+1}(x;C) - \Big(x + \frac{i(D-2A)(D-2B)(D-2C)}{4(2n+D)(2n+D-2)}\Big)p_n(x;C) + \frac{(n+A-1)(n+B-1)(n+C-1)(n+D-A-1)(n+D-B-1)(n+D-C-1)}{(2n+D-3)(2n+D-2)^2(2n+D-1)} \cdot p_{n-1}(x;C) = 0.$$

Additional properties may be deduced from appropriate limits of results for the associated Wilson polynomials [9],[14]. Here we choose to consider this problem from the viewpoint of contiguous relations for ${}_3F_2$ functions derived by Bailey [5] and Wilson [17].

Section 2 contains a detailed construction of solutions of the three term recurrence relation (1.8). We construct a total of six solutions. In Section 3 we use these solutions to construct the minimal solution [8] and evaluate the corresponding J-fraction. We then use the Stieltjes inversion formula to find the absolutely continuous component of the measure with respect to which the polynomials are orthogonal. Different solutions are minimal in different parts of the complex plane. Our analysis is valid for complex parameters A, B, C, D and complex variable x as long as the coefficent of p_{n-1} in (1.8) does not vanish for $n = 1, 2, \ldots$. In general the polynomials are orthogonal with respect to a complex valued measure but its absolutely continuous component is always supported on $(-\infty, \infty)$. If the parameters B, C, D, are real but $2A - D$ is purely imaginary then the polynomials are orthogonal with respect to a real measure. Cases of orthogonality with respect to a real measure occur if and only if one of $2A - D$, $2B - D$, $2C - D$, or all three are purely imaginary.

Note that the case considered in this paper is connected with the case of associated Jacobi polynomials [19] by the following procedure. In (1.8) replace x by Ax then divide by A, renormalize the resulting recursion to monic form and let $A \longrightarrow \infty$. A final renormalization to monic form yields the three term recurrence relation for monic associated Jacobi polynomials with the Wimp [19] parameters α, β, c given by $D = 2c + \alpha + \beta + 2$, $C = c + 1$, $B = c + \beta + 1$.

A basic (or q) analog of this work is in preparation. The q polynomials are orthogonal on a bounded set in contrast with the polynomials in the present work, which are orthogonal on the whole real line. The fact that the polynomials are orthogonal on $(-\infty, \infty)$ raises interesting technical difficulties that we managed to overcome. The minimal so-

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lutions of the three term recurrence relation also exhibit interesting behavior that do not seem to have been encountered in specific examples so far.

2. **Recurrence relation solutions.** We obtain solutions to the recurrence relation

$$\begin{aligned} &X_{n+1}(x)-(x-a_n)X_n(x)+b_n^2X_{n-1}(x)=0\\ &a_n=-i\frac{(D-2A)(D-2B)(D-2C)}{4(2n+D)(2n+D-2)} \qquad (2.1)\\ &b_{n+1}^2=\frac{(n+A)(n+B)(n+C)(n+D-A)(n+D-B)(n+D-C)}{(2n+D-1)(2n+D)^2(2n+D+1)}\end{aligned}$$

by first considering the contiguous relations [17, (13),(14),(18)]. These are

$$a(d+e-a-b-c-1)[F(a+)-F]-(d-a)(e-a)[F(a-)-F]-bcF=0 \qquad (2.2)$$

$$a[F(a+)-F]-b[F(b+)-F]=0 \qquad (2.3)$$

$$(d-a)(e-a)[F(a-)-F]-(d-b)(e-b)[F(b-)-F]+(b-a)cF=0 \qquad (2.4)$$

where

$$F={}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix}\,\middle|\,1\right) \text{ and } F(a\pm)={}_3F_2\left(\begin{matrix}a\pm 1,b,c\\ d,e\end{matrix}\,\middle|\,1\right).$$

Note that the series defining the ${}_3F_2$ in F converges when Re $(d+e-a-c)>0$. From (2.3) (with $b\longrightarrow b-1$), (2.4) (with $b\longrightarrow b+1$), and (2.2) (with $a\leftrightarrow b$) we eliminate $F(b+)$, $F(b-)$ to obtain the relation

$$\begin{aligned}&\frac{a(d-b)(e-b)}{(a-b+1)(a-b)}F(a+,b-)+\frac{b(d-a)(e-a)}{(a-b-1)(a-b)}F(a-,b+)\\ &\qquad=\left[\frac{a(d-b)(e-b)}{(a-b+1)(a-b)}+\frac{b(d-a)(e-a)}{(a-b-1)(a-b)}-c\right]F.\end{aligned} \qquad (2.5)$$

This relation has also been derived by Bailey [5] by using an independent method.

If we now make the parameter identification

$$\begin{aligned}&a=-n-C+1,\quad b=n+D-C,\\ &c=ix+(2A+2B-2C-D+2)/4,\quad d=A-C+1,\quad e=B-C+1\end{aligned} \qquad (2.6)$$

and renormalize, we obtain a solution to (2.1) given by

$$X_{n,1}(x)=i^n\frac{(A,B,D-C)_n}{(D-1)_{2n}}\,{}_3F_2\left(\begin{matrix}-n-C+1,n+D-C,ix+(2A+2B-2C-D+2)/4\\ A-C+1,B-C+1\end{matrix}\,\middle|\,1\right). \qquad (2.7)$$

From (2.7) we obtain two similar solutions with the replacements $A\leftrightarrow C$ or $B\leftrightarrow C$. Note that (2.7) reduces to (1.4) in the case $C=1$.

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A different solution is obtained from (2.5) by using the parameter identification

(2.8)
$$a = n + C, \quad b = -n + 1 + C - D,$$
$$c = -ix - (2A + 2B - 2C - D - 2)/4, \quad d = C - A + 1, \quad e = C - B + 1.$$

After renormalizing we obtain

(2.9)
$$X_{n,2}(x) = (-i)^n \frac{(C, D-A, D-B)_n}{(D-1)_{2n}} \cdot {}_3F_2\left(\begin{matrix} n+C, -n+1+C-D, -ix-(2A+2B-2C-D-2)/4 \\ C-A+1, C-B+1 \end{matrix} \,\middle|\, 1 \right)$$

and two similar solutions from the replacements $A \leftrightarrow C$ or $B \leftrightarrow C$.

Two further solutions of the above type are obtained by the replacements $(A, B) \longrightarrow (D-A, D-B)$ in (2.7) and (2.9). Thus a total of eight different solutions to (2.1) results from (2.5). This can also be seen by exploiting the symmetry properties of (2.1). In particular (2.1) is obviously invariant under the interchange $A \leftrightarrow C$ but also invariant under the replacement $A \longrightarrow D - A$ together with $x \longrightarrow -x$ and $X_n \longrightarrow (-1)^n X_n$.

None of the above are minimal solutions [8] to the recurrence (2.1). In order to obtain such solutions we consider a different set of contiguous relations.

From Wilson [17,(6),(10)] we have

(2.10)
$$F(d+) - F = [-abc/d(d+1)e] F_+(d+)$$

(2.11)
$$eF - (d+e-a-b-c-1)F_+ + \{(d-a)(d-b)(d-c)]/[d(d+1)]\} F_+(d+) = 0$$

where

$$F_\pm = {}_3F_2\left(\begin{matrix} a\pm 1, b\pm 1, c\pm 1 \\ d\pm 1, e\pm 1 \end{matrix} \,\middle|\, 1 \right), \quad F_\pm(d_\pm) = {}_3F_2\left(\begin{matrix} a\pm 1, b\pm 1, c\pm 1 \\ d\pm 2, e\pm 1 \end{matrix} \,\middle|\, 1 \right).$$

From (2.10), (2.11); and (2.11) with $(a, b, c, d, e) \longrightarrow (a-1, b-1, c-1, d-1, e-1)$ we eliminate F_- and $F(d_+)$ to obtain

(2.12)
$$\frac{-(d-a)(d-b)(d-c)}{(d-1)d} \left[\frac{abc}{d(d+1)e} F_+(d+) - F \right] + (e-1)\left[F_-(d-) - \frac{(a-1)(b-1)(c-1)}{(d-2)(d-1)(e-1)} F \right] - (d+e-a-b-c)F = 0.$$

If we now identify the parameters a, b, c, d, e as

(2.13)
$$a = n + A, \quad b = n + B, \quad c = n + C,$$
$$d = 2n + D, \quad e = n + ix + (A + B + C)/2 - D/4 + 1/2$$

and renormalize, we obtain the solution

(2.14)
$$X_{n,3}(x) = i^n \frac{(A, B, C, D-A, D-B, D-C)_n}{(D, D-1)_{2n} \left(ix + \frac{A+B+C}{2} - \frac{D}{4} + \frac{1}{2} \right)_n} \cdot {}_3F_2\left(\begin{matrix} n+A, n+B, n+C \\ 2n+D, n+ix+\frac{1}{2}(A+B+C+1) - \frac{1}{4}D \end{matrix} \,\middle|\, 1 \right).$$

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This will be shown to be a minimal solution if and only if $\operatorname{Im} x < 0$. Another solution using (2.12) results from the choice of parameters

$$
\begin{aligned}
&a = n + D - A, \quad b = n + D - B,\\
&c = n + D - C, \quad d = 2n + D,\\
&e = n - ix - (A + B + C)/2 + 5D/4 + 1/2
\end{aligned} \tag{2.15}
$$

which after renormalization yields the solution

$$
\begin{aligned}
X_{n,4}(x) = (-i)^n &\frac{(A, B, C, D - A, D - B, D - C)_n}{(D, D - 1)_{2n}\left(-ix - \frac{(A+B+C)}{2} + \frac{5D}{4} + \frac{1}{2}\right)_n}\\
&\cdot {}_3F_2\left(\begin{matrix} n + D - A, n + D - B, n + D - C \\ 2n + D, n - ix - \frac{1}{2}(A + B + C - 1) + \frac{5}{4}D \end{matrix} \,\middle|\, 1\right).
\end{aligned} \tag{2.16}
$$

This will be shown to be a minimal solution if and only if $\operatorname{Im} x > 0$.

Two further solutions result from (2.12) with the choice of parameters

$$
\begin{aligned}
&a = -n + 1 - A, \quad b = -n + 1 - B,\\
&c = -n + 1 - C, \quad d = -2n + 2 - D,\\
&e = -n - ix + [6 + D - 2(A + B + C)]/4,
\end{aligned} \tag{2.17}
$$

and the replacement $(A, B, C) \longrightarrow (D - A, D - B, D - C)$ in (2.17). This establishes the following representations, respectively

$$
\begin{aligned}
X_{n,5}(x) = i^n &\left(-n - ix - \frac{(A + B + C)}{2} + \frac{D}{4} + \frac{3}{2}\right)_n\\
&\cdot {}_3F_2\left(\begin{matrix} -n - A + 1, -n - B + 1, -n - C + 1 \\ -2n + 2 - D, -n - ix + \frac{1}{2}(3 - A - B - C) + \frac{1}{4}D \end{matrix} \,\middle|\, 1\right),
\end{aligned} \tag{2.18}
$$

$$
\begin{aligned}
X_{n,6}(x) = (-i)^n &\left(-n + ix + \frac{(A + B + C)}{2} - \frac{5D}{4} + \frac{3}{2}\right)_n\\
&\cdot {}_3F_2\left(\begin{matrix} -n - D + A + 1, -n - D + B + 1, -n - D + C + 1 \\ -2n + 2 - D, -n + ix + \frac{1}{2}(3 + A + B + C) - \frac{5}{4}D \end{matrix} \,\middle|\, 1\right),
\end{aligned} \tag{2.19}
$$

These can be shown to be minimal solutions as $n \longrightarrow -\infty$ for $\operatorname{Im} x > 0$ and $\operatorname{Im} x < 0$, respectively. Although these last two solutions are not required for the present problem they are needed for calculating the resolvent and continued fractions associated with the corresponding doubly infinite Jacobi matrix (eg. see [15]).

Note that (2.18) reduces to a polynomial solution if A, B or $C = 1$. In particular, with $C = 1$, we have an alternative expression for the monic continuous Hahn polynomial (1.4) given by

$$
\begin{aligned}
P_n(x) = i^n &\left(-n - ix - \frac{(A + B)}{2} + \frac{D}{4} + 1\right)_n\\
&\cdot {}_3F_2\left(\begin{matrix} -n, -n - A + 1, -n - B + 1 \\ -2n + 2 - D, -n - ix + \frac{1}{2}(2 - A - B) + \frac{1}{4}D \end{matrix} \,\middle|\, 1\right).
\end{aligned} \tag{2.20}
$$

We summarize the above discussion in the following theorem.

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THEOREM 1. *The associated continuous Hahn recurrence relation (2.1) has solutions* $X_{n,k}(x)$, $k = 1, 2, \ldots, 6$, *given by (2.7), (2.9), (2.14), (2.16), (2.18) and (2.19), respectively. Any two of these are linearly independent. Except for renormalization, the minimal solution for* $n \longrightarrow \infty$ *are given by* $X_{n,3}(x)$ *if* $\operatorname{Im} x < 0$, *and* $X_{n,4}(x)$ *if* $\operatorname{Im} x > 0$, *while minimal solutions for* $n \longrightarrow -\infty$ *are given by* $X_{n,5}(x)$ *if* $\operatorname{Im} x > 0$ *and* $X_{n,6}(x)$ *if* $\operatorname{Im} x < 0$. *No minimal solutions exist if* $\operatorname{Im} x = 0$.

PROOF. The first part of the theorem has been established in constructing the solutions $X_{n,k}(x)$, $k = 1, \ldots, 6$. It remains to prove the minimality of solutions $X_{n,k}(x)$, $k = 3, 4, 5, 6$ in different parts of the complex plane. This follows from the large $|n|$ asymptotics which we calculate next.

Recall the Thomae transformation [4, 3.2(1)]

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix} \,\middle|\, 1\right) = \frac{\Gamma(d)\Gamma(e)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)} {}_3F_2\left(\begin{matrix} d-a, e-a, s \\ s+b, s+c \end{matrix} \,\middle|\, 1\right), \tag{2.21} $$

where s is defined by

$$ s = d + e - a - b - c. $$

In (2.14) we apply the Thomae transformation to obtain

(2.22)

$$ \begin{aligned} X_{n,3}(x) = {} & i^n \frac{(B, C, D-A, D-B, D-C)_n \Gamma(D)}{\Gamma(A)(D-1)_{2n}} \\ & \cdot \mathbf{\Gamma}\left(\begin{matrix} ix - (A+B+C-1)/2 + 3D/4,\ ix + (1+A+B+C)/2 - D/4 \\ n + ix - (B-C+A-1)/2 + 3D/4,\ n + ix - (A+C-B-1)/2 + 3D/4 \end{matrix}\right) \\ & \cdot {}_3F_2\left(\begin{matrix} n+D-A,\ ix + \frac{1}{2}(1+B+C-A) - \frac{1}{4}D,\ ix + \frac{1}{2}(1-A-B-C) + \frac{3}{4}D \\ n + ix + \frac{1}{2}(1-A-B+C) + \frac{3}{4}D,\ n + ix + \frac{1}{2}(1+B-A-C) + \frac{3}{4}D \end{matrix} \,\middle|\, 1\right). \end{aligned} $$

Using Stirling's formula and the fact that the ${}_3F_2$ in (2.22) approaches 1 as $n \longrightarrow \infty$, we obtain

(2.23)

$$ \begin{aligned} X_{n,3}(x) &= C_3 \left(\frac{i}{4}\right)^n \Gamma(n) n^{-2ix+(D-1)/2}\left(1 + O\left(\frac{1}{n}\right)\right), \\ C_3 &= \sqrt{\pi}\, 2^{2-D} \\ & \quad \cdot \mathbf{\Gamma}\left(\begin{matrix} D-1, D, ix + \frac{1}{2}(A+B+C) - \frac{1}{4}D + \frac{1}{2}, ix - \frac{1}{2}(A+B+C) + \frac{3}{4}D + \frac{1}{2} \\ A, B, C, D-A, D-B, D-C \end{matrix}\right). \end{aligned} $$

Similarly we obtain

(2.24)

$$ \begin{aligned} X_{n,4}(x) &= C_4 \left(\frac{-i}{4}\right)^n \Gamma(n) n^{2ix+(D-1)/2}\left(1 + O\left(\frac{1}{n}\right)\right), \\ C_4 &= \sqrt{\pi}\, 2^{2-D} \\ & \quad \cdot \mathbf{\Gamma}\left(\begin{matrix} D-1, D, -ix - \frac{1}{2}(A+B+C) + \frac{5}{4}D + \frac{1}{2}, -ix + \frac{1}{2}(A+B+C) - \frac{3}{4}D + \frac{1}{2} \\ A, B, C, D-A, D-B, D-C \end{matrix}\right) \end{aligned} $$

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and as $n \longrightarrow -\infty$, we find

(2.25)

$$X_{n,5}(x) = C_5\Big(\frac{i}{4}\Big)^n \frac{(-n)^{2ix+(D-3)/2}}{\Gamma(-n)}\Big(1+O\Big(\frac{1}{-n}\Big)\Big),$$
$$C_5 = \frac{2^{1-D}}{\sqrt{\pi}}\mathbf{\Gamma}\left(-ix+\frac{(A+B+C)}{2}-\frac{3D}{4}+\frac{1}{2}\right)\mathbf{\Gamma}\left(-ix-\frac{(A+B+C)}{2}+\frac{D}{4}+\frac{3}{2}\right);$$

(2.26)

$$X_{n,6}(x) = C_6\Big(\frac{-i}{4}\Big)^n \frac{(-n)^{-2ix+(D-3)/2}}{\Gamma(-n)}\Big(1+O\Big(\frac{1}{-n}\Big)\Big),$$
$$C_6 = \frac{2^{1-D}}{\sqrt{\pi}}\mathbf{\Gamma}\left(ix+\frac{(A+B+C)}{2}-\frac{5D}{4}+\frac{3}{2}\right)\mathbf{\Gamma}\left(ix-\frac{(A+B+C)}{2}+\frac{3D}{4}-\frac{1}{2}\right).$$

The above asymptotics demonstrate the minimality properties of these solutions. Thus, except for when the constants C_3 or C_4 become zero or infinite, one has $X_{n,3}(x)/X_{n,4}(x) \longrightarrow 0$ as $n \longrightarrow \infty$ if $\operatorname{Im} x < 0$ and $X_{n,4}(x)/X_{n,3}(x) \longrightarrow 0$ as $n \longrightarrow \infty$ if $\operatorname{Im} x > 0$; and corresponding conclusions for $X_{n,5}(x)$, $X_{n,6}(x)$ as $n \longrightarrow -\infty$.

We note finally that connecting formulas between any three solutions can be obtained from the three term transformation [4, 3.2(2)]

(2.27)

$$ {}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix}\;\Big|\;1\right) = \frac{\Gamma(1-a)\Gamma(d)\Gamma(e)\Gamma(c-b)}{\Gamma(d-b)\Gamma(e-b)\Gamma(1+b-a)\Gamma(c)}\,{}_3F_2\left(\begin{matrix}b,b-d+1,b-e+1\\ 1+b-c,1+b-a\end{matrix}\;\Big|\;1\right)$$
$$+ \text{ a similar expression with } b \leftrightarrow c,$$

together with the identities (2.21) and the transformation [4, p. 98]

$$(2.28)\qquad {}_3F_2\left(\begin{matrix}a,b,c\\ d,e\end{matrix}\;\Big|\;1\right) = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-c)\Gamma(d+e-a-b)}\,{}_3F_2\left(\begin{matrix}d-a,d-b,c\\ d,d+e-a-b\end{matrix}\;\Big|\;1\right).$$

Gasper and Rahman [7] refer to (2.28) as the Kummer-Thomae-Whipple transformation, [7, (3.2.8)].

3. **The continued fraction and measure.**

Pincherle's theorem [8] gives a connection between the large n minimal solution $X_n^{(s)}(x)$ to (2.1) and the corresponding infinite continued fraction

$$(3.1)\quad CF(x) = x - a_0 + \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{-b_n^2}{x-a_n}\right) = x - a_0 - \cfrac{b_1^2}{x-a_1-\cfrac{b_2^2}{x-a_2\cdots}},\quad b_n^2 \neq 0, n > 0,$$

through the formula [12]

$$(3.2)\qquad \frac{1}{\mathrm{CF}(x)} = \frac{X_0^{(s)}(x)}{b_0^2 X_{-1}^{(s)}(x)}.$$

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From Theorem 1 and (2.14), (2.16), we therefore obtain the following continued fraction representation

$$\frac{1}{CF(x)} = \frac{i}{\left(ix + \frac{(A+B+C)}{2} - \frac{D}{4} - \frac{1}{2}\right)} \frac{{}_3F_2\left(\begin{matrix} A,B,C \\ D, ix+\frac{1}{2}(1+A+B+C)-\frac{1}{4}D \end{matrix} \middle| 1\right)}{{}_3F_2\left(\begin{matrix} A-1,B-1,C-1 \\ D-2, ix+\frac{1}{2}(A+B+C-1)-\frac{1}{4}D \end{matrix} \middle| 1\right)} \tag{3.3}$$

$$\operatorname{Im} x < 0,$$

and

$$\frac{1}{CF(x)} = \frac{-i}{\left(-ix - \frac{(A+B+C)}{2} + \frac{5D}{4} - \frac{1}{2}\right)} \frac{{}_3F_2\left(\begin{matrix} D-A,D-B,D-C \\ D, -ix+\frac{1}{2}(1-A-B-C)+\frac{5}{4}D \end{matrix} \middle| 1\right)}{{}_3F_2\left(\begin{matrix} D-A-1,D-B-1,D-C-1 \\ D-2, ix-\frac{1}{2}(1+A+B+C)+\frac{5}{4}D \end{matrix} \middle| 1\right)} \tag{3.4}$$

$$\operatorname{Im} x > 0.$$

Observe that the infinite continued fraction in (3.3) and (3.4) will diverge by oscillation when $\operatorname{Im} x = 0$; see (2.23), (2.24).

Note that the case of real orthogonality, a_n, b_{n+1} real, $b_{n+1} \neq 0$, $n = 0, 1, \ldots$, implies that the numerator and denominator ${}_3F_2$'s in (3.3) and (3.4) can have no zeros in the closure of the indicated half plane. More generally,

$${}_3F_2\left(\begin{matrix} n+A, n+B, n+C \\ D+2n, n+ix+(1+A+B+C-D/2)/2 \end{matrix} \middle| 1\right) \neq 0,$$
$$\operatorname{Im} x \leq 0, \quad a_m \text{ real}, \quad b_{m+1}^2 > 0, \quad m \geq n+1,$$

and similarly

$${}_3F_2\left(\begin{matrix} n+D-A, n+D-B, n+D-C \\ D+2n, n-ix-(1+A+B+C-5D/2)/2 \end{matrix} \middle| 1\right) \neq 0,$$
$$\operatorname{Im} x \geq 0, \quad a_m \text{ real}, \quad b_{m+1}^2 > 0, \quad m \geq n+1.$$

The proof for the open half plane is trivial since, $d\alpha(t) \geq 0$, which implies that $\int_{\mathbf{R}} (x-t)^{-1} d\alpha(t)$ can have only real zeros or poles. To include the boundary of the half planes we use the fact that $\{\operatorname{Re} X_{n,3}\}$, $\{\operatorname{Im} X_{n,3}\}$ are real linearly independent solutions of (2.1) if x, a_n, b_n are real and $b_n \neq 0$ for $n = 0, 1, \ldots$. Having $X_{n,3} = 0$ for some $n \geq -1$ would contradict linear independence. Similarly for $X_{n,4}$.

In the above we must assume that D is not an integer ≤ 2. However, an exceptional case occurs if $C = 1$ and $D = 2$. In this case $a_n = 0$, $n \geq 1$ but an indeterminacy exists for a_0. This case must be defined by a limiting procedure. Let us choose to first put $C = 1$ and then take the limit as $D \longrightarrow 2$. In the continued fraction (3.1) this yields $a_0 = -i(1-A)(1-B)/2$ and (3.3) becomes

$$\frac{1}{x + \frac{i(1-A)(1-B)}{2} + \mathbf{K}_{n=1}^{\infty}\left(\frac{-[n^2-(1-A)^2][n^2-(1-B)^2/4]}{(2n+1)x}\right)}$$
$$= \frac{i}{ix+(A+B-1)/2} {}_3F_2\left(\begin{matrix} A,B,1 \\ 2, ix+(A+B+1)/2 \end{matrix} \middle| 1\right), \quad \operatorname{Im} x < 0. \tag{3.5}$$

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The ${}_3F_2$ in (3.5) may be summed in terms of gamma functions using the identity (2.27) to obtain the relationship

(3.6)
$$ {}_3F_2\left(\begin{matrix}1,b,c\\2,e\end{matrix}\,\middle|\,1\right) = \frac{(1-e)}{(b-1)(c-1)} + \frac{\Gamma(1-b)\Gamma(e)}{(1-c)\Gamma(e-c)\Gamma(1+c-b)}\,{}_2F_1\left(\begin{matrix}c-1,c-e+1\\1+c-b\end{matrix}\,\middle|\,1\right)$$

and then using the Gauss summation theorem [4, 1.3(1)] we find

$$ {}_3F_2\left(\begin{matrix}1,b,c\\2,e\end{matrix}\,\middle|\,1\right) = \frac{(1-e)}{(b-1)(c-1)}\left[1-\frac{\Gamma(e-1)\Gamma(1+e-b-c)}{\Gamma(e-c)\Gamma(e-b)}\right]. \tag{3.7}$$

Thus we have proved the continued fraction evaluation

(3.8a)
$$\frac{i}{(A-1)(1-B)}\left(1-\frac{\Gamma[ix+(A+B-1)/2]\Gamma[ix+(3-A-B)/2]}{\Gamma[ix+(1+B-A)/2]\Gamma[ix+(1+A-B)/2]}\right) = \frac{1}{x+\frac{1}{2}i(1-A)(1-B)+\mathbf{K}_{n=1}^{\infty}\left(\frac{-[n^2-(A-1)^2][n^2-(B-1)^2]/4}{(2n+1)x}\right)},\quad \operatorname{Im} x<0,$$

or equivalently,

(3.8b)
$$\frac{2i}{(A-1)(1-B)}\left(\frac{1-T(x)}{1+T(x)}\right) = \frac{1}{x+\mathbf{K}_{n=1}^{\infty}\left(\frac{-[n^2-(A-1)^2][n^2-(B-1)^2]/4}{(2n+1)x}\right)},\quad \operatorname{Im} x<0,$$

where $T(x)$ is defined by the following expression

$$T(x) := \mathbf{\Gamma}\left(\begin{matrix}ix+(A+B-1)/2,\, ix-(A+B-3)/2\\ ix+(B-A+1)/2,\, ix+(A-B+1)/2\end{matrix}\right).$$

When $\operatorname{Im} x > 0$, we may repeat the above calculations starting with (3.4). The result, namely

(3.8c)
$$\frac{2i}{(A-1)(B-1)}\left(\frac{1-T(-x)}{1+T(-x)}\right) = \frac{1}{x+\mathbf{K}_{n=1}^{\infty}\left(\frac{-[n^2-(A-1)^2][n^2-(B-1)^2]/4}{(2n+1)x}\right)},\quad \operatorname{Im} x>0,$$

is obviously equivalent to (3.8b).

If we now replace ix by $x/2$, $A-1$ by m, and $B-1$ by n then (3.8b) can be expressed in the form

(3.8d)
$$\frac{\Gamma\left[\frac{1}{2}(x+m+n+1)\right]\Gamma\left[\frac{1}{2}(x-m-n+1)\right]-\Gamma\left[\frac{1}{2}(x+m-n+1)\right]\Gamma\left[\frac{1}{2}(x-m+n+1)\right]}{\Gamma\left[\frac{1}{2}(x+m+n+1)\right]\Gamma\left[\frac{1}{2}(x-m-n+1)\right]+\Gamma\left[\frac{1}{2}(x+m-n+1)\right]\Gamma\left[\frac{1}{2}(x-m+n+1)\right]} = \frac{mn}{x+\mathbf{K}_{k=1}^{\infty}\left(\frac{(k^2-m^2)(k^2-n^2)}{(2k+1)x}\right)},\quad \operatorname{Re} x>0,$$

which is precisely entry 33 in Chapter 12 of Ramanujan's second notebook; see [6].

Associated with $C = 1, D = 1$ there is indeterminancy in b_1^2 in (2.1). Thus a second exceptional case may be obtained by putting $C = 1 + u, D = 1 + 2u$ and taking the limit as $u \longrightarrow 0$. This yields

$$(3.9)\qquad \begin{aligned} a_n &= \frac{i(1-2A)(1-2B)}{4(2n-1)(2n+1)}, \\ b_n^2 &= \frac{[(n-1/2)^2-(A-1/2)^2][(n-1/2)^2-(B-1/2)^2]}{4(2n-1)^2}. \end{aligned}$$

Applying this limit to (3.3) and (3.4) then yields the continued fraction evaluation

$$(3.10)\qquad \begin{aligned} &\frac{1}{x-a_0+\mathbf{K}_{n=1}^{\infty}\left(\frac{-b_n^2}{x-a_n}\right)} \\ &= \begin{cases} \left[2\left(x-a_0-i\mathbf{\Gamma}\left(\begin{matrix} ix+(3+2A-2B)/4, ix+(3+2B-2A)/4 \\ ix+(2A+2B-1)/4, ix+(3-2A-2B)/4 \end{matrix}\right)\right)\right]^{-1}, & \operatorname{Im} x < 0 \\ \left[2\left(x-a_0+i\mathbf{\Gamma}\left(\begin{matrix} -ix+(5-2A-2B)/4, -ix+(1+2A+2B)/4 \\ -ix+(1+2A-2B)/4, -ix+(1-2A+2B)/4 \end{matrix}\right)\right)\right]^{-1}, & \operatorname{Im} x > 0 \end{cases} \end{aligned}$$

The case $C = 1$ and $D = 1 + u$, $u \longrightarrow 0$ is another way of removing the indeterminacy and one then has

$$(3.11)\qquad b_1^2 = \frac{1}{2}AB(1-A)(1-B).$$

Therefore

$$(3.12)\qquad \begin{aligned} &\frac{1}{x-a_0-\dfrac{AB(1-A)(1-B)/2}{x-a_1+\mathbf{K}_{n=2}^{\infty}\left(\frac{-b_n^2}{x-a_n}\right)}} \\ &= \begin{cases} i\mathbf{\Gamma}\left(\begin{matrix} ix+(2A+2B-1)/4, ix+(3-2A-2B)/4 \\ ix+(3+2A-2B)/4, ix+(3+2B-2A)/4 \end{matrix}\right), & \operatorname{Im} x < 0 \\ -i\mathbf{\Gamma}\left(\begin{matrix} -ix+(1+2A-2B)/4, -ix+(1+2B-2A)/4 \\ -ix+(2A+2B+1)/4, -ix+(5-2A-2B)/4 \end{matrix}\right), & \operatorname{Im} x > 0 \end{cases} \end{aligned}$$

where the a_n's, $n \geq 1$, are as in (3.9) and the b_n's, $n \geq 2$, are the same as in (3.9). It is clear that the continued fractions in the two cases, when we only change b_1, can be obtained from each other. The corresponding spectral measures, however, as we shall see later, will be very different. It is worth pointing out that the further special case $B = 1/2$ is equivalent to entry 25 of the aforementioned notebook of Ramanujan, [6],[13].

In summary we have established the following theorem.

THEOREM 2. *If D is not an integer ≤ 2, one has associated with the recurrence relation (2.1) the continued fraction representations (3.3) and (3.4). The exceptional case $C = 1, D = 2$ yields Ramanujan's entry 33 as given by (3.8d). Furthermore Ramanujan's (3.8d) holds if and only if either* $\operatorname{Re} x > 0$ *with m and n arbitrary, or x is arbitrary with*

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m or n an integer. Another exceptional case, $C = 1$, $D = 1$, gives (3.10) which is a generalization of entry 25 of Chapter 12 in the second notebook of Ramanujan.

The absolutely continuous component $\alpha'(x)$ of the measure that the associated continuous Hahn polynomials are orthogonal with respect to may now be obtained from (3.2) since

$$\frac{1}{\mathrm{CF}(x)} = \int_{-\infty}^{\infty} \frac{\alpha'(t)}{x-t}\,dt + \text{ a possible sum over poles,} \tag{3.13}$$

where the contour of integration is deformed above or below any real zeros of the denominator ${}_3F_2$'s in (3.3) and (3.4), respectively. Such a deformation is always possible since the linear independence of $\{X_{n,3}\}$ and $\{X_{n,4}\}$ prevents these denominator ${}_3F_2$'s having coinciding zeros. In the case of real orthogonality no deformations are necessary and there is no contribution from any pole terms. Thus from (3.2), (3.5) and the Stieltjes inversion formula we can compute $\alpha'(x)$ using [12]. The answer is

$$\alpha'(x) = \frac{1}{2\pi i}\frac{b_0^2 W\big(X_{-1}^{(s)}(x+i0), X_{-1}^{(s)}(x-i0)\big)}{b_0^4 X_{-1}^{(s)}(x+i0) X_{-1}^{(s)}(x-i0)}, \quad -\infty < x < \infty, \tag{3.14}$$

where $W(X_n, Y_n)$ is the Casorati determinant $X_n Y_{n+1} - Y_n X_{n+1}$. The Casorati determinant is the discrete analog of the Wronskian.

The numerator in (3.14) may be calculated using $n \longrightarrow \infty$ asymptotics since from (2.1)

$$b_0^2 W\big(X_{-1}^{(s)}(x+i0), X_{-1}^{(s)}(x-i0)\big) = \lim_{n\to\infty} \frac{W\big(X_n^{(s)}(x+i0), X_n^{(s)}(x-i0)\big)}{b_1^2 b_2^2 \cdots b_n^2}. \tag{3.15}$$

Thus from (3.15), Theorem 1 and (2.23), (2.24) we obtain the Wronskian evaluation

(3.16)
$$\begin{aligned} &b_0^2 W(X_{-1,4}(x+i0), X_{-1,3}(x-i0)) \\ &\quad = i\boldsymbol{\Gamma}\left(ix + \frac{(1+A+B+C)}{2} - \frac{D}{4}\right)\boldsymbol{\Gamma}\left(-ix + \frac{(1-A-B-C)}{2} + \frac{5D}{4}\right) \\ &\quad \cdot \boldsymbol{\Gamma}\left(\begin{matrix} D-1, D, ix + \frac{1}{2}(1-A-B-C) + \frac{3}{4}D, -ix + \frac{1}{2}(1+A+B+C) - \frac{3}{4}D \\ A, B, C, D-A, D-B, D-C \end{matrix}\right) \end{aligned}$$

so that (3.14) becomes

(3.17)
$$\begin{aligned} \alpha'(x) &= \frac{1}{2\pi}\boldsymbol{\Gamma}\left(\begin{matrix} D-1, D, ix + \frac{1}{2}(1-A-B-C) + \frac{3}{4}D, -ix + \frac{1}{2}(1+A+B+C) - \frac{3}{4}D \\ A, B, C, D-A, D-B, D-C \end{matrix}\right) \\ &\cdot \frac{\boldsymbol{\Gamma}\left(ix + \frac{1}{2}(A+B+C-1) - \frac{1}{4}D, -ix - \frac{1}{2}(A+B+C+1) + \frac{5}{4}D\right)}{{}_3F_2\left(\begin{matrix} A-1, B-1, C-1 \\ D-2, ix+\frac{1}{2}(A+B+C-1)-\frac{1}{4}D \end{matrix}\middle| 1\right){}_3F_2\left(\begin{matrix} D-A-1, D-B-1, D-C-1 \\ D-2, -ix-\frac{1}{2}(A+B+C+1)+\frac{5}{4}D \end{matrix}\middle| 1\right)}. \end{aligned}$$

This reduces to (1.7) when $C = 1$ since one ${}_3F_2 = 1$ while the other ${}_3F_2$ becomes

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} -1+D-A, -1+D-B \\ -ix-1+(5D-2A-2B)/4 \end{matrix}\middle| 1\right) \\ &\quad = \boldsymbol{\Gamma}\left(\begin{matrix} -ix-1+(5D-2A-2B)/4, -ix+1+(2A+2B-3D)/4 \\ -ix+(D+2A-2B)/4, -ix+(D+2B-2A)/4 \end{matrix}\right), \end{aligned} \tag{3.18}$$

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using the Gauss sum of a ${}_2F_1$ of unit argument.

The representation (3.17) does not exhibit the fact that α' is real valued for certain values of the parameters. For example it is clear from (2.1) that the three term recurrence relation has real coefficients if B, C, D and $i(D-2A)$ are real, so we expect a real valued measure in this case. To see this set $A = i\lambda + D/2$ and apply the Kummer-Thomae-Whipple transformation (2.28) to the second ${}_3F_2$ in (3.17) with, $c = D - A - 1$, and $d = D - 2$. The result is

(3.19)
$$\alpha'(x) = \frac{1}{2\pi}\mathbf{\Gamma}\left(\begin{matrix} D-1, D \\ i\lambda + D/2, -i\lambda + D/2, B, C, D-B, D-C\end{matrix}\right)\Delta(x,\lambda)\Delta(-x,-\lambda),$$

where

$$A = i\lambda + D/2,$$

(3.19a)
$$\Delta(x,\lambda) = \frac{\Gamma\big(ix + (1 - i\lambda + D - B - C)/2, ix + (B + C + i\lambda - 1)/2\big)}{{}_3F_2\left(\begin{matrix} i\lambda - 1 + D/2, B-1, C-1 \\ D-2, ix + (B+C+i\lambda-1)/2\end{matrix}\,\middle|\, 1\right)}.$$

Similar representations can be found in the other cases of real orthogonality.

Note that $\alpha'(x)$ is a meromorphic function of x when continued into the complex plane.

The measure associated with the exceptional symmetric case requires a separate limiting calculation. Denote the continued fraction on the right-hand side of (3.8b), (3.8c) by $1/\,\mathrm{CF}_R(x)$ and the associated measure by $d\alpha_R(x)$. Then a repetition of the calculation for $\alpha'(x)$ will yield an expression for $\alpha_R'(x)$ where in the right-hand side numerator of (3.17) one puts $C = 1$ and $D = 2$ but the ${}_3F_2$'s in the denominator are replaced by

(3.20)
$$\lim_{n\to 0} {}_3F_2\left(\begin{matrix} A-1, B-1, n \\ 2n, ix + (A+B-1)/2\end{matrix}\,\middle|\, 1\right) = \frac{1}{2}\left[1 + {}_2F_1\left(\begin{matrix} A-1, B-1 \\ ix + (A+B-1)/2\end{matrix}\,\middle|\, 1\right)\right]$$

(3.21)
$$\lim_{n\to 0} {}_3F_2\left(\begin{matrix} 1-A, 1-B, n \\ 2n, -ix + (3-A-B)/2\end{matrix}\,\middle|\, 1\right) = \frac{1}{2}\left[1 + {}_2F_1\left(\begin{matrix} 1-A, 1-B \\ -ix + (3-A-B)/2\end{matrix}\,\middle|\, 1\right)\right],$$

since this limit will yield

$$a_0 = \lim_{n\to 0}(a_n|_{C=1,D=2}) = 0.$$

Thus for the exceptional symmetric case we obtain, using the limits above and the Gauss summation,

(3.22)
$$\alpha_R'(x) = \frac{2}{\pi^3}\frac{\sin(\pi A)\sin(\pi B)}{(A-1)(B-1)} \cdot \frac{\Gamma\left(ix + \frac{3-A-B}{2}, -ix + \frac{A+B-1}{2}, ix + \frac{A+B-1}{2}, -ix + \frac{3-A-B}{2}\right)}{\left[1 + \mathbf{\Gamma}\left(\begin{matrix} ix+(A+B-1)/2, ix+(3-A-B)/2 \\ ix+(1+A-B)/2, ix+(1+B-A)/2\end{matrix}\right)\right]\left[1 + \mathbf{\Gamma}\left(\begin{matrix} -ix+(A+B-1)/2, -ix+(3-A-B)/2 \\ -ix+(1+A-B)/2, -ix+(1+B-A)/2\end{matrix}\right)\right]}$$

where the identity $\Gamma(1-z)\Gamma(z) = \pi/\sin \pi z$ has also been used.

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This can be compared with the corresponding nonsymmetric case having $a_0 = -i(1-A)(1-B)/2$ where

$$\tag{3.23} \begin{aligned} \alpha'(x) &= \frac{1}{2\pi^3}\frac{\sin \pi A \sin \pi B}{(A-1)(B-1)} \\ &\cdot \boldsymbol{\Gamma}\left(ix + \frac{A+B-1}{2}, ix - \frac{A+B-3}{2}, -ix + \frac{A-B+1}{2}, -ix + \frac{B-A+1}{2}\right) \end{aligned}$$

which is obtained from (1.7) by putting $D = 2$ or from (3.17) by taking $C = 1$ and then the limit as $D \longrightarrow 2$.

The spectral measure associated with the case $D = 1 + \delta u$, $C = 1 + u$, $u \longrightarrow 0$ can be obtained by applying this limit to (3.19). The result is

$$\tag{3.24} \alpha'_\delta(x) = \frac{(\delta - 1)\sin(\pi B)\Delta_1(x, \lambda, \delta)\Delta_1(-x, -\lambda, \delta)}{2\delta\pi^2\Gamma(i\lambda + 1/2, -i\lambda + 1/2)},$$

where

(3.24a)

$$\lambda := i(-A + 1/2),$$

$$\Delta(x, \lambda, \delta) := \frac{\delta\Gamma\big(ix + (1 - i\lambda - B)/2, ix + (B + i\lambda)/2\big)}{\delta - 1 + [ix + 1 - (1 + i\lambda)(B + \frac{1}{2})]\boldsymbol{\Gamma}\left(\begin{matrix} ix+(B+i\lambda)/2, ix+(1-B-i\lambda)/2 \\ ix+(1+i\lambda-B)/2, ix+1+(i\lambda-B)/2 \end{matrix}\right)}.$$

In calculating the above limits we have used the limiting relationship

$$\lim_{u\to 0} {}_3F_2\left(\begin{matrix} a, b, u \\ \delta u - 1, e \end{matrix}\,\Big|\, 1\right) = \frac{1}{\delta}\left(\delta - 1 + {}_2F_1\left(\begin{matrix} a, b \\ e \end{matrix}\,\Big|\, 1\right) - \frac{ab}{e}\,{}_2F_1\left(\begin{matrix} a+1, b+1 \\ e+1 \end{matrix}\,\Big|\, 1\right)\right)$$

and Gauss's summation theorem.

The above exceptional cases contain two interesting cases corresponding to $\delta = 2$ and $\delta \longrightarrow \infty$.

A symmetric case occurs if and only if one of the remaining parameters, say B, equals $1/2$. This symmetric case is equivalent to entry 25 of Chapter 12 of Ramanujan's second notebook and is a special case of associated Meixner-Pollazcek polynomials; see [13].

Thus we are led to the following theorem.

THEOREM 3. *If $D \neq 2, 1, 0, \ldots$ then the complex measure of orthogonality for associated continuous Hahn polynomials is $\alpha'(x)\,dx$ where $\alpha'(x)$ is given by (3.17). In the exceptional case $C = 1$, $D = 2$ one has $\alpha'(x)$ given by (3.23) if $a_0 = -i(A-1)(B-1)/2$ and one must replace $\alpha'(x)$ by $\alpha'_R(x)$ given by (3.22) if $a_0 = 0$. In the exceptional case $C = 1$, $D = 1$ the family of measures $\alpha'_\delta(x)\,dx$ given by (3.24) corresponds to $b_1 = \pm AB[(1 - 1/\delta)/2]^{1/2}$. The case $\delta = 2$ is associated with the continued fraction (3.10) while $\delta \longrightarrow \infty$ is associated with the continued fraction (3.12).*

PROOF. It remains to show that for a suitable contour Γ we have

$$\tag{3.25} \int_\Gamma p_n(t; C)\, p_m(t; C)\, d\alpha(t) = b_1^2 b_2^2 \cdots b_n^2 \delta_{m,n}.$$

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Consider $n \leq m$ and the representation

$$\text{(3.26)} \qquad \frac{p_n(x;C)X_m^{(s)}(x)}{b_0^2X_{-1}^{(s)}(x)} = \int_{-\infty}^{\infty} \frac{p_n(t;C)\,p_m(t;C)\,d\alpha(t)}{x-t} + \sum_k \frac{R_k\,p_n(x_k;C)\,p_m(x_k;C)}{x-x_k}$$

where $\{x_k\}$ are the zeros of $b_0^2X_{-1}^{(s)}(x)$, $\operatorname{Im} x \neq 0$ or $b_0^2X_{-1}^{(s)}(x \pm i0)$, $x \in (-\infty, \infty)$, which for convenience we assume are simple for suitable parameter values.

Representation (3.26) follows from the fact that

$$\text{(3.27)} \quad p_m(x;C) = \frac{X_{m,3}(x)X_{-1,4}(x) - X_{m,4}(x)X_{-1,3}(x)}{W\big(X_{-1,4}(x), X_{-1,3}(x)\big)}, p_n(x;C)\frac{X_m^{(s)}(x)}{b_0^2X_{-1}^{(s)}(x)} = O(x^{n-m-1})$$

and

$$\text{(3.28)} \qquad p_n(x;C)\left(\frac{X_m^{(s)}(x-i0)}{b_0^2X_{-1}^{(s)}(x-i0)} - \frac{X_m^{(s)}(x+i0)}{b_0^2X_{-1}^{(s)}(x+i0)}\right) = 2\pi i p_n(x;C)p_m(x;C)\alpha'(x),$$

for real x. If we now consider $|x| \longrightarrow \infty$ in (3.26) we obtain

$$\text{(3.29)} \qquad \begin{aligned} \frac{b_1^2b_2^2\cdots b_m^2}{x^{m-n+1}}\left(1+O\left(\frac{1}{x}\right)\right) &= \frac{1}{x}\left[\int_{-\infty}^{\infty} p_n(t;C)p_m(t;C)\,d\alpha(t) + R_{m,n}\right] + O\left(\frac{1}{x^2}\right), \\ R_{m,n} &= \sum_k R_k p_m(x_k;C)p_n(x_k;C), \end{aligned}$$

where in the left side of (3.26) we have used the fact that

$$p_n(x;C) \approx x^n\left(1+O\left(\frac{1}{x}\right)\right), \text{ as } |x| \longrightarrow \infty \text{ and}$$

$$\text{(3.30)} \qquad \begin{aligned} \frac{X_m^{(s)}(x)}{b_0^2X_{-1}^{(s)}(x)} &= \frac{X_0^{(s)}(x)}{b_0^2X_{-1}^{(s)}(x)} \cdot \frac{X_1^{(s)}(x)}{b_1^2X_0^{(s)}(x)} \cdots \frac{X_m^{(s)}(x)}{b_m^2X_{m-1}^{(s)}(x)} \cdot b_1^2b_2^2\cdots b_m^2 \\ &\sim \left[\frac{1}{x}\left(1+O\left(\frac{1}{x}\right)\right)\right]^{m+1} b_1^2b_2^2\cdots b_{m'}^2 \text{ as } |x| \longrightarrow \infty. \end{aligned}$$

From (3.29) we have

$$\text{(3.31)} \qquad \int_{-\infty}^{\infty} p_n(t;C)p_m(t;C)\,d\alpha(t) + R_{m,n} = b_1^2b_2^2\cdots b_m^2\delta_{m,n}$$

and by using a suitable contour Γ, with the analytic continuation of $\alpha'(t)$ along Γ which avoids the pole contributions, we obtain (3.25).

4. **Remarks.** A birth and death process with birth rates $\{\lambda_n\}$ and death rates $\{\mu_n\}$ gives rise to a family of orthogonal polynomials $\{Q_n(x)\}$ generated by the recurrence relation

$$\text{(4.1)} \qquad -xQ_n(x) = \lambda_nQ_{n+1}(x) + \mu_nQ_{n-1}(x) - (\lambda_n+\mu_n)Q_n(x), \quad n \geq 0,$$

with

$$\text{(4.2)} \qquad Q_{-1}(x) := 0, \quad Q_0(x) := 1.$$

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The birth and death rates satisfy the restriction

$$\lambda_n > 0, \mu_{n+1} > 0, n \geq 0, \text{ and } \mu_0 \geq 0. \tag{4.3}$$

Under the above assumptions the polynomials $\{ Q_n(x)\}$ are always orthogonal on a subset of $[0, \infty)$ and we will refer to the $Q_n(x)$'s as birth and death process polynomials. The monic form of (4.1) is

$$\begin{aligned} xq_n(x) &= q_{n+1}(x) + \lambda_{n-1}\mu_n q_{n-1}(x) + (\lambda_n + \mu_n)q_n(x), \quad n \geq 0, \\ q_{-1}(x) &:= 0, \quad q_0(x) := 1. \end{aligned} \tag{4.4}$$

A symmetric system of orthogonal polynomials $\{ p_n(x)\}$ generates two sets of orthogonal polynomials, say $\{ u_n(x)\}$ and $\{ v_n(x)\}$,

$$u_n(x) = p_{2n}(\sqrt{x}), v_n(x) = x^{-1/2}p_{2n+1}(\sqrt{x}). \tag{4.5}$$

There is no loss of generality in assuming that the p_n's are monic. Assume that the $p_n(x)$'s are orthogonal with repsect to $d\mu(x)$ and

$$\int_{-\infty}^{\infty} p_m(x)\, p_n(x)\, d\mu(x) = \xi_n\delta_{m,m}. \tag{4.6}$$

Both $\{ u_n(x)\}$ and $\{ v_n(x)\}$ are multiples of birth and death process polynomials. To see this start with the recursive definition

$$p_{-1}(x) = 0, \quad p_0(x) = 1, \quad xp_n(x) = p_{n+1}(x) + \beta_n p_{n-1}(x), \quad n \geq 0, \tag{4.7}$$

and find

$$x^2 p_n(x) = p_{n+2}(x) + (\beta_{n+1} + \beta_n)p_n(x) + \beta_n\beta_{n-1}p_{n-2}(x). \tag{4.8}$$

Now (4.4) indicates that

$$\text{if } \lambda_n = \beta_{2n+1}, \mu_n = \beta_{2n}, \text{ then } q_n(x) = u_n(x), \tag{4.9}$$

$$\text{if } \lambda_n = \beta_{2n+2}, \mu_n = \beta_{2n+1}, \text{ then } q_n(x) = v_n(x), \tag{4.10}$$

Observe that if $\mu_0 = 0$ in (4.1) then $Q_n(0) = 1$. Furthermore since $p_n(x)$ does not depend on β_0, there is no loss of generality in assuming $\beta_0 = 0$. Therefore

$$u_n(0) = (-1)^n\beta_1\beta_3\cdots\beta_{2n-1}. \tag{4.11}$$

Assume that the u_n's and v_n's are orthogonal with respect to du and dv respectively. From (4.5) and (4.6) we get

$$du(x) = d\mu(x^2), dv(x) = x^{-1}d\mu(x^2). \tag{4.12}$$

Thus $du(x) = x\, dv(x)$ and the Christoffel formula [16, § 2.5] yields

$$xv_n(x) = [u_{n+1}(x)u_n(0) - u_n(x)u_{n+1}(0)]/\, u_n(0),$$

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that is

$$xv_n(x) = u_{n+1}(x) + \beta_{2n+1}u_n(x). \tag{4.13}$$

Thus starting with the p_n's in (4.7) we generate two new families of orthogonal polynomials $\{u_n(x)\}$ and $\{v_n(x)\}$ via (4.9), and (4.10) or (4.13). Conversely starting with polynomials $\{u_n(x)\}$ associated with a birth and death process with $\mu_0 = 0$ define another family $\{v_n(x)\}$ through (4.13) and a symmetric set of orthogonal polynomials $\{p_n(x)\}$ by (4.5).

We now apply the above procedure to the Wilson polynomials. They are associated with a birth and death process with

$$\lambda_n = \frac{(n+a+b)(n+a+c)(n+a+d)(n+s-1)}{(2n+s)(2n+s-1)}, \quad s := a+b+c+d, \tag{4.14}$$

$$\mu_n = \frac{n(n+b+c-1)(n+b+d-1)(n+c+d-1)}{(2n+s-2)(2n+s-1)}. \tag{4.15}$$

This defines an orthogonal polynomials system $\{u_n(x)\}$ in the above notation. Let

$$W_n(x) := {}_4F_3\left(\begin{matrix} -n, n+s-1, a+(a^2-x^2)^{1/2}, a-(a^2-x^2)^{1/2} \\ a+b, a+c, a+d \end{matrix} \,\middle|\, 1\right). \tag{4.16}$$

Then

$$W_n(x) = (-1)^n\lambda_0\cdots\lambda_{n-1}u_n(x). \tag{4.16a}$$

The v_n's and the p_n's can now be found from (4.9) and (4.13).

Note that the polynomials $\{v_n(x)\}$ and $\{p_n(x)\}$ associated with the Wilson polynomials in the above way are different from the ones discussed in [2].

The polynomials generated by (1.5) will be symmetric if $D = 2B$. In this case they can be our starting polynomials $\{p_n(x)\}$ in (4.7) with

$$\beta_n = \frac{n(n+A-1)(n+2B-A-1)(n+2B-2)}{4(2n+2B-1)(2n+2B-3)}. \tag{4.17}$$

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