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WIENER INDEX AND TRACEABLE GRAPHS

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Abstract

In this short paper, we show that, with three exceptions, if the Wiener index of a connected graph of order n is at most $(n + 5)(n - 2)/2$, then it is traceable.

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1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For a graph G, we let $d_G(v)$ be the degree of a vertex v in G and $d_G(u, v)$ be the distance between two vertices u and v in G.

The Wiener number $W(G)$ of a connected graph G is a well-known distance-based graph invariant. Also called the Wiener index, it is defined [10] as the sum of distances between all pairs of vertices in G, namely,

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v) = \frac{1}{2} \sum_{v\in V(G)} D_G(v),$$

where $D_G(v) = \sum_{u\in V(G)} d_G(v, u)$.

The chemistry and mathematics literature has many results and applications on the Wiener index. See, for example, the recent papers [2–9, 11] and the references quoted therein.

A graph is said to be *traceable* if it possesses a Hamiltonian path. In this paper, we will provide a new sufficient condition in terms of the Wiener index for a connected graph to be traceable.

Before proceeding, we introduce some further notation and terminology. Denote by K_n the complete graph on n vertices. Let G and H be two vertex-disjoint graphs. The *join* of G and H, denoted by $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G) \text{ and } v \in V(H)\}$. For other notation and terminology not defined here, the reader is referred to [1].

2. A new sufficient condition for a connected graph to be traceable

We first introduce the following result about traceable graphs.

LEMMA 2.1 [1]. *Let G be a nontrivial graph of order n with degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \le d_2 \le \cdots \le d_n$ and $n \ge 4$. Suppose that there is no integer $k < (n+1)/2$ such that $d_k \le k-1$ and $d_{n-k+1} \le n-k-1$. Then G is traceable.*

For the sake of brevity, we will write d_i and D_i instead of $d_G(v_i)$ and $D_G(v_i)$, respectively, in the proof of the following theorem.

THEOREM 2.2. *Let G be a connected graph of order $n \ge 4$. If*

$$W(G) \le \frac{(n+5)(n-2)}{2},$$

then G is traceable unless $G \cong K_1 + (K_{n-3} \cup 2K_1)$ or $K_2 + (3K_1 \cup K_2)$ or $K_4 + 6K_1$.

PROOF. Suppose that G is a nontraceable connected graph with degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \le d_2 \le \cdots \le d_n$ and $n \ge 4$. By Lemma 2.1, there is an integer $k < (n+1)/2$ such that $d_k \le k-1$ and $d_{n-k+1} \le n-k-1$. Since G is connected and $d_k \le k-1$, we have $k > 1$. Thus,

$$\begin{aligned}
W(G) &= \frac{1}{2}\sum_{i=1}^{n} D_i \\
&\ge \frac{1}{2}\sum_{j=1}^{n}(d_j + 2(n-1-d_j)) && (2.1)\\
&= \frac{1}{2}\sum_{j=1}^{n}(2(n-1) - d_j) \\
&= n(n-1) - \frac{1}{2}\sum_{j=1}^{n} d_j \\
&\ge n(n-1) - \frac{1}{2}(k(k-1) + (n-2k+1)(n-k-1) + (k-1)(n-1)) && (2.2)\\
&= n(n-1) - 2 - \frac{(n-2)(n-3)}{2} + \frac{(k-2)(2n-3k-5)}{2} \\
&\ge n(n-1) - 2 - \frac{(n-2)(n-3)}{2} && (2.3)\\
&= \frac{(n+5)(n-2)}{2}.
\end{aligned}$$

Combining this fact and our assumption that $W(G) \le (n+5)(n-2)/2$, we have $W(G) = (n+5)(n-2)/2$. So, all inequalities in (2.1)–(2.3) should be equalities. Thus:

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(a) from (2.1), we know that all vertices in G have eccentricity no more than 2;
(b) from (2.2), we know that $d_1 = \cdots = d_k = k - 1$, $d_{k+1} = \cdots = d_{n-k+1} = n - k - 1$ and $d_{n-k+2} = \cdots = d_n = n - 1$;
(c) from (2.3), we know that $k = 2$ or $2n = 3k + 5$.

If $k = 2$, then G is a connected graph with $d_1 = d_2 = 1$, $d_3 = \cdots = d_{n-1} = n - 3$ and $d_n = n - 1$, which implies that $G \cong K_1 + (K_{n-3} \cup 2K_1)$.

If $2n = 3k + 5$, then $n \leq 10$, as $k < (n + 1)/2$. Thus $n = 7$, $k = 3$, or $n = 10$, $k = 5$. By (b), we know that G is a connected graph of order seven with $d_1 = d_2 = d_3 = 2$, $d_4 = d_5 = 3$, $d_6 = d_7 = 6$, or G is a connected graph of order 10 with $d_1 = \cdots = d_6 = 4$, $d_7 = \cdots = d_{10} = 9$, which implies that $G \cong K_2 + (3K_1 \cup K_2)$, or $G \cong K_4 + 6K_1$.

It is easy to check that none of the graphs $K_1 + (K_{n-3} \cup 2K_1)$, $K_2 + (3K_1 \cup K_2)$ and $K_4 + 6K_1$ is traceable. This completes the proof. □

Since $W(G) \geq n(n - 1)/2$ for all graphs G, and $W(G) = n(n - 1)/2$ if and only if $G \cong K_n$, and K_n is traceable, we ask: is there an upper bound for $W(G)$ between $n(n - 1)/2$ and $(n + 5)(n - 2)/2$ that guarantees that G is traceable without exceptions?

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References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications* (Macmillan, London and Elsevier, New York, 1976).
[2] N. Cohen, D. Dimitrov, R. Krakovski, R. Skrekovski and V. Vukasinovic, 'On Wiener index of graphs and their line graphs', *MATCH Commun. Math. Comput. Chem.* **64** (2010), 683–698.
[3] K. C. Das and I. Gutman, 'Estimating the Wiener index by means of number of vertices, number of edges, and diameter', *MATCH Commun. Math. Comput. Chem.* **64** (2010), 647–660.
[4] A. Dobrynin, R. Entringer and I. Gutman, 'Wiener index of trees: theory and applications', *Acta Appl. Math.* **66** (2001), 211–249.
[5] A. A. Dobrynin, 'On the Wiener index of fibonacenes', *MATCH Commun. Math. Comput. Chem.* **64** (2010), 707–726.
[6] A. Graovac and T. Pisanski, 'On the Wiener index of a graph', *J. Math. Chem.* **8** (1991), 53–62.
[7] H. Hua, 'Wiener and Schultz molecular topological indices of graphs with specified cut edges', *MATCH Commun. Math. Comput. Chem.* **61** (2009), 643–651.
[8] I. Pesek, M. Rotovnik, D. Vukicevic and J. Zerovnik, 'Wiener number of directed graphs and its relation to the oriented network design problem', *MATCH Commun. Math. Comput. Chem.* **64** (2010), 727–742.
[9] S. Wagner, 'A note on the inverse problem for the Wiener index', *MATCH Commun. Math. Comput. Chem.* **64** (2010), 639–646.

[10] H. Wiener, 'Structural determination of paraffin boiling point', *J. Amer. Chem. Soc.* **69** (1947), 17–20.
[11] B. Wu, 'Wiener index of line graphs', *MATCH Commun. Math. Comput. Chem.* **64** (2010), 699–706.

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