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POTENTIAL SYMMETRIES OF INHOMOGENEOUS NONLINEAR DIFFUSION EQUATIONS

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In this paper potential symmetries are sought of the inhomogeneous nonlinear diffusion equations $u_t = x^{1-M}\big[x^{N-1}f(u)u_x\big]_x$. The functional forms of $f(u)$ that admit such symmetries are completely classified. A complete list is presented of the symmetries, which depend on the values of the parameters M and N. We give examples of similarity solutions using potential symmetries. In some cases, the potential symmetries enable us to convert non–invertible mappings of nonlinear partial differential equations to linear ones.

1. INTRODUCTION

We consider generalised radial diffusion equations of the type

$$\frac{\partial u}{\partial t} = \frac{1}{x^{M-1}}\frac{\partial}{\partial x}\left[x^{N-1}f(u)\frac{\partial u}{\partial x}\right], \tag{1}$$

which are of considerable interest in mathematical physics. Some cases have been used to model physical situations in fields involving diffusion processes [**2, 11, 5**]. In particular, when $f(u) = u^n$, (1) has a large number of applications for both $n > 0$ ("slow diffusion") and $n < 0$ ("fast diffusion") [**7, 12**]. There is a continuing interest in finding exact similarity solutions to these equations [**15, 8, 9, 17**]. In [**16**] a complete classification of Lie point [**10**] and Lie-Bäcklund [**3, 1**] symmetries is presented.

Bluman, Kumei and Reid [**4, 3**] introduced a method for finding a new class of symmetries for a system of partial differential equations (PDEs) $\Delta(x,u)$, in the case when at least one of the PDEs can be written in conserved form. If we introduce potential variables v for the PDEs so written as further unknown functions, we obtain a system $Z(x,u,v)$. Any Lie group of transformations for $Z(x,u,v)$ induces a symmetry for $\Delta(x,u)$. When at least one of the generators corresponding to the variables x and u depends explicitly on the potential v, then the local symmetry of $Z(x,u,v)$ induces a nonlocal symmetry of $\Delta(x,u)$. These nonlocal symmetries are called *potential symmetries*.

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In this paper we search for potential symmetries for (1). We classify all the functions $f(u)$ that admit such symmetries. We present a generalisation for the results obtained for the well-known nonlinear diffusion equation $u_t = \left[f(u)u_x\right]_x$ [3]. If we introduce the potential v, (1) can be written as a system

$$v_x = x^{M-1}u, \qquad v_t = x^{N-1}f(u)u_x \tag{2}$$

of two PDEs. We determine infinitesimal transformations of the form

$$\begin{aligned} x' &= x + \varepsilon X(x,t,u,v) + O\left(\varepsilon^2\right), \\ t' &= t + \varepsilon T(x,t,u,v) + O\left(\varepsilon^2\right), \\ u' &= u + \varepsilon U(x,t,u,v) + O\left(\varepsilon^2\right), \\ v' &= v + \varepsilon V(x,t,u,v) + O\left(\varepsilon^2\right) \end{aligned} \tag{3}$$

admitted by (2). These transformations induce potential and point symmetries for (1) and point symmetries for the integrated form

$$v_t = x^{N-1}f\left(x^{1-M}v_x\right)\left[x^{1-M}v_x\right]_x \tag{4}$$

of (1), where $u = x^{1-M}v_x$.

We note also that the second PDE in (2) can be written in conserved form. With the introduction of a potential w, (2) yields the system

$$v_x = x^{M-1}u, \qquad w_x = x^{1-N}v, \qquad w_t = G(u), \tag{5}$$

where $F = \mathrm{d}G/\mathrm{d}u$. The subsystems

$$w_x = x^{1-N}v, \qquad w_t = G\left(x^{1-M}v_x\right), \tag{6}$$

$$w_{xx} - (N-1)x^{-1}w_x = x^{M-N}u, \qquad w_t = G(u), \tag{7}$$

$$w_t = G\left(x^{N-M}w_{xx} - (N-1)x^{N-M-1}w_x\right) \tag{8}$$

arise from (5). We classify all Lie infinitesimal group of transformations of the form

$$\begin{aligned} x' &= x + \varepsilon X(x,t,u,v,w) + O\left(\varepsilon^2\right), \\ t' &= t + \varepsilon T(x,t,u,v,w) + O\left(\varepsilon^2\right), \\ u' &= u + \varepsilon U(x,t,u,v,w) + O\left(\varepsilon^2\right), \\ v' &= v + \varepsilon V(x,t,u,v,w) + O\left(\varepsilon^2\right), \\ w' &= w + \varepsilon W(x,t,u,v,w) + O\left(\varepsilon^2\right) \end{aligned} \tag{9}$$

admitted by (5). Lie point symmetries of (5) induce potential symmetries for (1) and (2), nonlocal symmetries for (4), Lie–Bäcklund symmetries for (6)–(8) and Lie point symmetries for (1), (2), (4) and (6)–(8).

The symmetry analysis is carried out in Section 2. In Section 3 we give examples of similarity solutions while in the final section we introduce the infinite-parameter Lie groups of (1) to derive non-invertible mappings from non-linear PDEs to linear PDEs.

2. Potential symmetries

Equations (2) admit Lie transformations of the form (3) if and only if

$$\Gamma^{(1)}\left\{v_x - x^{M-1}u\right\} = 0, \qquad \Gamma^{(1)}\left\{v_t - x^{N-1}f(u)u_x\right\} = 0, \tag{10}$$

where $\Gamma^{(1)}$ is the first extended generator of

$$\Gamma = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v},$$

which is given by the relation

$$\begin{aligned}\Gamma^{(1)} = \Gamma &+ \left[D_xU - (D_xX)u_x - (D_xT)u_t\right]\frac{\partial}{\partial u_x} + \left[D_tU - (D_tX)u_x\right.\\ &\left.-(D_tT)u_t\right]\frac{\partial}{\partial u_t} + \left[D_xV - (D_xX)v_x - (D_xT)v_t\right]\frac{\partial}{\partial v_x}\\ &+\left[D_tV - (D_tX)v_x - (D_tT)v_t\right]\frac{\partial}{\partial v_t}.\end{aligned}$$

Here D_x and D_t are total derivatives with respect to x and t, respectively. Equations (10) lead to a set of determining equations that enable us to find the functional forms of $f(u)$ and the generators X, T, U and V and consequently the desired transformations can be derived.

We omit the calculations, which have been greatly facilitated by the computer algebraic package REDUCE [**6**]. The procedure for determining Lie point symmetries is well explained in [**3**] and [**14**]. In the following analysis let Γ_i $(i = 1, 2, \ldots, m)$ be m linearly independent infinitesimal generators associated with an m–dimensional Lie infinitesimal transformation group, that is,

$$\Gamma = X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + V\frac{\partial}{\partial v} = \sum_{i=1}^{m} c_i\Gamma_i.$$

The point symmetries for (2) can be summarised as follows.

Case 1. $f(u)$ arbitrary. Here the system (2) admits a three–parameter Lie group when $N \neq 2 - M$ and a four–parameter one when $N = 2 - M$. The infinitesimal generators are given by

(i) $N \neq 2 - M$, $\Gamma_1 = \dfrac{\partial}{\partial t}$, $\Gamma_2 = \dfrac{\partial}{\partial v}$ and
$$\Gamma_3 = x\frac{\partial}{\partial x} + (M - N + 2)t\frac{\partial}{\partial t} + Mv\frac{\partial}{\partial v};$$

(ii) $N = 2 - M$, $M \neq 0$, Γ_1, Γ_2, Γ_3, $\Gamma_4 = x^{1-M}\dfrac{\partial}{\partial x}$;

(iii) $N = 2$, $M = 0$, Γ_1, Γ_2, $\Gamma_3' = x\ln x\dfrac{\partial}{\partial x} + 2t\dfrac{\partial}{\partial t} + v\dfrac{\partial}{\partial v}$, $\Gamma_4 = x\dfrac{\partial}{\partial x}$.

CASE 2. $f = e^{\lambda u}$. In this case we obtain a four-parameter Lie group when $N \neq 2 - M$ and a five-parameter one when $N = 2 - M$, with infinitesimal generators

(i) $N \neq 2-M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_5 = \lambda t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} - \frac{1}{M}x^M\frac{\partial}{\partial v}$;

(ii) $N = 2-M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_5$;

(iii) $N \neq 2,\ M = 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_5' = \lambda t\frac{\partial}{\partial t} - \frac{\partial}{\partial u} - \ln x\frac{\partial}{\partial v}$;

(iv) $N = 2,\ M = 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3',\ \Gamma_4,\ \Gamma_5'$.

CASE 3. $f = p(u+q)^n$, where $n \neq 0, -2$. Here we have the following infinitesimal generators in each subcase:

(i) $N \neq 2-M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3$ and

$$\Gamma_6 = nt\frac{\partial}{\partial t} - (u+q)\frac{\partial}{\partial u} - \left(v + \frac{q}{M}x^M\right)\frac{\partial}{\partial v};$$

(ii) $N = 2-M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_6$;

(iii) $N \neq 2,\ M = 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3$ and

$$\Gamma_6' = nt\frac{\partial}{\partial t} - (u+q)\frac{\partial}{\partial u} - (v + q\ln x)\frac{\partial}{\partial v};$$

(iv) $N = 2,\ M = 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3',\ \Gamma_4,\ \Gamma_6'$.

CASE 4. $f = p(u+q)^{-2}$. In this case we distinguish six different subcases:

(i) $N \neq 2-M,\ N \neq 2+3M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_6\ (n = -2)$;

(ii) $N = 3M+2,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_6$ and

$$\begin{aligned}\Gamma_7 = x\left(v + \frac{q}{M}x^M\right)\frac{\partial}{\partial x} + (u+q)\left(Mv - ux^M\right)\frac{\partial}{\partial u}\\ +Mv\left(v + \frac{q}{M}x^M\right)\frac{\partial}{\partial v};\end{aligned}$$

(iii) $N = 2-M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_6$ and

$$\begin{aligned}\Gamma_8 &= -x\left(v + \frac{q}{M}x^M\right)\frac{\partial}{\partial x} + (u+q)\left[Mv + (u+2q)x^M\right]\frac{\partial}{\partial u}\\ &\quad + \left[qx^M\left(v + \frac{q}{M}x^M\right) + 2pMt\right]\frac{\partial}{\partial v},\\ \Gamma_9 &= 4pMt^2\frac{\partial}{\partial t} - x\left[\left(v + \frac{q}{M}x^M\right)^2 + 2pt\right]\frac{\partial}{\partial x}\\ &\quad +(q+u)\left[6pMt + M\left(v + \frac{q}{M}x^M\right)^2 + 2x^M\left(v + \frac{q}{M}x^M\right)(u+q)\right]\frac{\partial}{\partial u}\\ &\quad + \left[qx^M\left(v + \frac{q}{M}x^M\right)^2 + 2pMt\left(2v + 3\frac{q}{M}x^M\right)\right]\frac{\partial}{\partial v},\end{aligned}$$

(11) $$\Gamma_\infty = x^{1-M}\Psi(\eta,t)\frac{\partial}{\partial x} - (u+q)^2\frac{\partial\Psi(\eta,t)}{\partial\eta}\frac{\partial}{\partial u} - q\Psi(\eta,t)\frac{\partial}{\partial v},$$

where $\eta = v + (q/M)x^M$ and $y = \Psi(\eta, t)$ is an arbitrary solution of the linear heat equation

$$p\frac{\partial^2 y}{\partial \eta^2} - \frac{\partial y}{\partial t} = 0; \tag{12}$$

(iv) $N \neq 2,\ M = 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_6'$;

(v) $N = 2,\ M = 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3',\ \Gamma_4,\ \Gamma_6'$,

$$\Gamma_8' = -x\ln x(v + q\ln x)\frac{\partial}{\partial x} + (u+q)\big[v + \ln x(u+2q)\big]\frac{\partial}{\partial u} + \big[2pt + q\ln x(v + q\ln x)\big]\frac{\partial}{\partial v},$$

$$\Gamma_9' = 4pt^2\frac{\partial}{\partial t} - x\ln x\big[2pt + (v+q\ln x)^2\big]\frac{\partial}{\partial x} + (u+q)\big[6pt + (v+q\ln x)^2 + 2\ln x(u+q)(v+q\ln x)\big]\frac{\partial}{\partial u} + \big[q\ln x(v+qx)^2 + 2pt(2v + 3q\ln x)\big]\frac{\partial}{\partial v}$$

and Γ_∞ (M=0), where $\eta = v + q\ln x$ and $y = \Psi(\eta, t)$ is an arbitrary solution of the linear heat equation (12).

CASE 5. $f = \big(s/(u^2+pu+q)\big)\exp\big[r\int (du)/(u^2+pu+q)\big]$, where p, q and r are arbitrary constants. In this final case we have the following results:

(i) $N \neq 2 - M,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3$;

(ii) $N = 2 - M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4$ and

$$\Gamma_{10} = vx^{1-M}\frac{\partial}{\partial x} + (r-p)t\frac{\partial}{\partial t} - \big(u^2 + pu + q\big)\frac{\partial}{\partial u} - \Big(pv + \frac{q}{M}x^M\Big)\frac{\partial}{\partial v}; \tag{13}$$

(iii) $N = 2,\ M = 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3',\ \Gamma_4$ and

$$\Gamma_{10}' = vx\frac{\partial}{\partial x} + (r-p)t\frac{\partial}{\partial t} - \big(u^2+pu+q\big)\frac{\partial}{\partial u} - (pv + q\ln x)\frac{\partial}{\partial v}.$$

Symmetries $\Gamma_1 - \Gamma_{10}$ project to point symmetries of (4) and symmetries $\Gamma_1 - \Gamma_6$ to point symmetries of (1), while symmetries $\Gamma_7 - \Gamma_{10}$ induce potential symmetries admitted by (1).

Now we classify the point symmetries of the form (9) admitted by (5). Here, depending on the form of $f(u)$, we have ten cases. Each case has a number of subcases depending on the values of M and N. As before, we summarise the results without presenting any calculations.

CASE 1. $f(u)$ arbitrary. We have four different subcases:

(i) $N \neq 2 - M,\ N \neq 2,\ \Gamma_1 = \dfrac{\partial}{\partial t},\ \Gamma_2 = \dfrac{\partial}{\partial w}$,

$$\Gamma_3 = (M - N + 2)\,t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + Mv\frac{\partial}{\partial v} + (M-N+2)w\frac{\partial}{\partial w},$$

$$\Gamma_4 = (2-N)\frac{\partial}{\partial v} + x^{2-N}\frac{\partial}{\partial w};$$

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(ii) $N=2-M,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_5=x^{1-M}\dfrac{\partial}{\partial x};$

(iii) $N=2,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4'=\dfrac{\partial}{\partial v}+\ln x\dfrac{\partial}{\partial w};$

(iv) $N=2,\ M=0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3'=2t\dfrac{\partial}{\partial t}+x\ln x\dfrac{\partial}{\partial x}+v\dfrac{\partial}{\partial v}+2w\dfrac{\partial}{\partial w},\ \Gamma_4',\ \Gamma_5.$

Symmetries $\Gamma_1-\Gamma_5$ project to point symmetries for each of (1), (2), (4) and (6)–(8).

CASE 2. $f=e^{\lambda u}$. We have:

(i) $N\neq 2-M,\ N\neq 2+M/2,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4$ and $\Gamma_6=\lambda(M-N+2)t\dfrac{\partial}{\partial t}-(M-N+2)\dfrac{\partial}{\partial u}-\big((M-N+2)/M\big)\dfrac{\partial}{\partial v}-(1/M)x^{M-N+2}\dfrac{\partial}{\partial w};$

(ii) $N=2+M/2,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_6$ and

$$\Gamma_7=-2\lambda x^{1-M/2}\frac{\partial}{\partial x}+Mx^{-M/2}\frac{\partial}{\partial u}-\frac{1}{2}\left(\lambda M^2w-4x^{M/2}+2\lambda Mvx^{-M/2}\right)\frac{\partial}{\partial v}+\left(2\ln x+\lambda Mwx^{M/2}\right)\frac{\partial}{\partial w};$$

(iii) $N=2-M,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_5,\ \Gamma_6;$

(iv) $N\neq 2,\ M=0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4$ and $\Gamma_6'=\lambda t\dfrac{\partial}{\partial t}-\dfrac{\partial}{\partial u}-\ln x\dfrac{\partial}{\partial v}+\big(1/(N-2)^2\big)x^{2-N}\big[(N-2)\ln x+1\big]\dfrac{\partial}{\partial w};$

(v) $N=2,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4',\ \Gamma_6;$

(vi) $N=2,\ M=0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3',\ \Gamma_4',\ \Gamma_5$ and $\Gamma_6''=\lambda t\dfrac{\partial}{\partial t}-\dfrac{\partial}{\partial u}-\ln x\dfrac{\partial}{\partial v}-(\ln x)^2/2\dfrac{\partial}{\partial w}.$

Symmetry Γ_6 projects to a point symmetry for each of (1), (2), (4) and (6)–(8). Symmetry Γ_7 projects to a point symmetry of (1) and (6)–(8), while it induces a potential symmetry admitted by (2) and a nonlocal symmetry admitted by (4).

CASE 3. $f=p(u+q)^n$, n arbitrary. We have:

(i) $N\neq 2+M,\ N\neq 2-M,\ N\neq 2,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_8=nt\dfrac{\partial}{\partial t}-(u+q)\dfrac{\partial}{\partial u}-\big(v+(q/M)x^M\big)\dfrac{\partial}{\partial v}-\Big(w+\big(q/M(M-N+2)\big)x^{M-N+2}\Big)\dfrac{\partial}{\partial w};$

(ii) $N=2-M,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_5,\ \Gamma_8;$

(iii) $N=2+M,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_8'=nt\dfrac{\partial}{\partial t}-(u+q)\dfrac{\partial}{\partial u}-\big(v+(q/M)x^M\big)\dfrac{\partial}{\partial v}-\big(w+(q/M)\ln x\big)\dfrac{\partial}{\partial w};$

(iv) $N\neq 2,\ M=0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_8''=nt\dfrac{\partial}{\partial t}-(u+q)\dfrac{\partial}{\partial u}-(v+q\ln x)\dfrac{\partial}{\partial v}-\Big[w-\big(q/(N-2)^2\big)\big((N-2)\ln x+1\big)x^{2-N}\Big]\dfrac{\partial}{\partial w};$

(v) $N=2,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4',\ \Gamma_8$;

(vi) $N=2,\ M=0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3',\ \Gamma_4',\ \Gamma_5,\ \Gamma_8''' = nt\frac{\partial}{\partial t}-(u+q)\frac{\partial}{\partial u} - (v+q\ln x)\frac{\partial}{\partial v}-\left(w+(q/2)(\ln x)^2\right)\frac{\partial}{\partial w}$.

Symmetry Γ_8 projects to a point symmetry for each of (1), (2), (4) and (6)-(8).

CASE 4. $f=p(u+q)^n,\ n=(3N-M-6)/(M-2N+4)$. We have:

$$N\neq 2-M,\ N\neq 2+M,\ N\neq 2,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_8$$

and

$$\begin{aligned}\Gamma_9 = x^{3-N}\frac{\partial}{\partial x} &- (M-2N+4)(u+q)x^{2-N}\frac{\partial}{\partial u}\\ &+\left[(N-2)^2 w+(N-2)vx^{2-N}-q\frac{M-2N+4}{M-N+2}x^{M-N+2}\right]\frac{\partial}{\partial v}\\ &-\left[(N-2)x^{2-N}w+\frac{q}{M-N+2}x^{M-2N+4}\right]\frac{\partial}{\partial w}.\end{aligned}$$

Symmetry Γ_9 projects to a point symmetry of (1) and (6)–(8), while it induces a potential symmetry admitted by (2) and a nonlocal symmetry admitted by (4). We note that in the above subcase, if $N=2,\ M\neq 0$, then $n=-1$ (case 6), if $N=2-M,\ M\neq 0$, then $n=-4/3$ (case 5) and if $N=2+M,\ M\neq 0$, then $n=-2$ (case 8). For the subcase $N\neq 2,\ M=0$, we obtain the same symmetries as in case 3(iv) ($n=-3/2$).

CASE 5. $f=p(u+q)^n,\ n=-4/3$. We have:

(i) $N=2-M,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_5,\ \Gamma_8,\ \Gamma_9$;

(ii) $N=2,\ M=0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3',\ \Gamma_4',\ \Gamma_5,\ \Gamma_8'''$ and

$$\begin{aligned}\Gamma_9' = x(\ln x)^2\frac{\partial}{\partial x}-3\ln x(u+q)\frac{\partial}{\partial u}+\left[w-v\ln x-\frac{3q}{2}(\ln x)^2\right]\frac{\partial}{\partial v}\\ +\ln x\left[w-\frac{q}{2}(\ln x)^2\right]\frac{\partial}{\partial w}.\end{aligned}$$

CASE 6. $f=p(u+q)^n,\ n=-1$. We have:

(i) $N\neq 2,\ N\neq 2+M,\ N\neq 2-M,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ pt\frac{\partial}{\partial w}-\Gamma_8$;

(ii) $N=2-M,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_5,\ pt\frac{\partial}{\partial w}-\Gamma_8$;

(iii) $N=2+M,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ pt\frac{\partial}{\partial w}-\Gamma_8'$;

(iv) $N\neq 2,\ M=0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ pt\frac{\partial}{\partial w}-\Gamma_8''$;

(v) $N=2,\ M\neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4',\ pt\frac{\partial}{\partial w}-\Gamma_8$ and

$$\Gamma_{10} = x\ln x\frac{\partial}{\partial x} - (u+q)(2+M\ln x)\frac{\partial}{\partial u} - \left(v + Mpt + qx^M\ln x + \frac{q}{M}x^M\right)\frac{\partial}{\partial v} - \left(2pt + Mpt\ln x + \frac{q}{M}x^M\ln x\right)\frac{\partial}{\partial w};$$

(vi) $N=2,\ M=0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3',\ \Gamma_4',\ \Gamma_5,\ pt\frac{\partial}{\partial w} - \Gamma_8'''$.

Symmetry Γ_{10} projects to a point symmetry for each of (1), (2), (4) and (6)–(8).

CASE 7. $f = p(u+q)^n,\ n = -2/3$. We have:

(i) $N \neq 2,\ N \neq 2-M,\ N \neq 2+M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_8$;

(ii) $N = 2-M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_5,\ \Gamma_8$ and

$$\Gamma_{11} = \left(x^{1-M}w + \frac{q}{2M^2}x^{M+1}\right)\frac{\partial}{\partial x} - 3(u+q)\left(v + \frac{q}{M}x^M\right)\frac{\partial}{\partial u} - \left[qw + \left(v + \frac{q}{M}x^M\right)^2 + \frac{q^2}{2M^2}x^{2M}\right]\frac{\partial}{\partial v} - \frac{q}{M}x^M\left(w + \frac{q}{2M^2}x^{2M}\right)\frac{\partial}{\partial w};$$

(iii) $N = 2+M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_8'$;

(iv) $N \neq 2, M = 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_8''$;

(v) $N = 2,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4',\ \Gamma_8$;

(vi) $N = 2,\ M = 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3',\ \Gamma_4',\ \Gamma_5,\ \Gamma_8'''$ and

$$\Gamma_{11}' = \left[w + \frac{q}{2}(\ln x)^2\right]x\frac{\partial}{\partial x} - 3(u+q)(v + q\ln x)\frac{\partial}{\partial u} - \left[qw + (v+q\ln x)^2 + \frac{q^2}{2}(\ln x)^2\right]\frac{\partial}{\partial v} - q\ln x\left[w + \frac{q}{2}(\ln x)^2\right]\frac{\partial}{\partial w}.$$

Symmetry Γ_{11} projects to a point symmetry of (6) and (8) and induces a potential symmetry admitted by (1) and (2), a nonlocal symmetry of (4) and a Lie–Bäcklund symmetry of (7).

CASE 8. $f = p(u+q)^n,\ n = -2$. We have:

(i) $N \neq 2+M,\ N \neq 2-M,\ N \neq 2,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_8$;

(ii) $N = 2-M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_5,\ \Gamma_8$ and

$$\Gamma_{12} = x^{1-M}\left[M\left(w + \frac{q}{2M^2}x^{2M}\right) - 2x^M\left(v + \frac{q}{M}x^M\right)\right]\frac{\partial}{\partial x} + (u+q)\left[2x^M(u+q) + M\left(v + \frac{q}{M}x^M\right)\right]\frac{\partial}{\partial u} - \left[qM\left(w + \frac{q}{2M^2}x^{2M}\right) - 2qx^M\left(v + \frac{q}{M}x^M\right) - 2pMt\right]\frac{\partial}{\partial v} - x^M\left[q\left(w + \frac{q}{2M^2}x^{2M}\right) + \left(v + \frac{q}{M}x^M\right)\left(v - \frac{q}{M}x^M\right) - 2pt\right]\frac{\partial}{\partial w}$$

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and

(14) $$\Gamma_{1\infty} = x^{1-M} F_\eta \frac{\partial}{\partial x} - (u+q)^2 F_{\eta\eta} \frac{\partial}{\partial u} - qF_\eta \frac{\partial}{\partial v} + (vF_\eta - F)\frac{\partial}{\partial w},$$

where $\eta = v + (q/M)x^M$ and $y = F(t,\eta)$ satisfies (12);

(iii) $N = 2 + M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_8'$,

$$\begin{aligned}\Gamma_{13} = & -x^{1-M}\left[2\left(v + \frac{q}{M}x^M\right) + Mx^M\left(w + \frac{q}{M}\ln x\right)\right]\frac{\partial}{\partial x} \\ & + x^{-M}(u+q)\left[2x^M(u+q) - M\left(v + \frac{q}{M}x^M\right)\right]\frac{\partial}{\partial u} \\ & - Mx^{-M}\left[\left(v + \frac{q}{M}x^M\right)\left(v - \frac{q}{M}x^M\right) + Mx^M v\left(w + \frac{q}{M}\ln x\right)\right]\frac{\partial}{\partial v} \\ & - x^{-2M}\left[\left(v + \frac{q}{M}x^M\right)\left(v - \frac{q}{M}x^M\right) - qx^{2M}\left(w + \frac{q}{M}\ln x\right) - 2px^{2M}t\right]\frac{\partial}{\partial w}\end{aligned}$$

and

(15) $$\begin{aligned}\Gamma_{2\infty} = & \; x^{1-M}F_\xi \frac{\partial}{\partial x} - \frac{x^{-M}}{M}(u+q)\big[(u+q)F_{\xi\xi} - M^2 F_\xi\big]\frac{\partial}{\partial u} \\ & + (M^2 F - qF_\xi)\frac{\partial}{\partial v} + x^{-2M}\big(vF_\xi - Mx^M F\big)\frac{\partial}{\partial w},\end{aligned}$$

where $\xi = w + (v/M)x^{-M} + (q/M)\ln x + (q/M^2)$ and $z = F(t,\xi)$ satisfies the linear equation

$$\frac{p}{M^2}\frac{\partial^2 z}{\partial \xi^2} - \frac{p}{q}\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial t} = 0;$$

(iv) $N = 2,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4',\ \Gamma_8$;

(v) $N = 2,\ M = 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3',\ \Gamma_4',\ \Gamma_5,\ \Gamma_8''$,

$$\begin{aligned}\Gamma_{12}' = & \; x\left[\left(w + \frac{q}{2}(\ln x)^2\right) - 2\ln x(v + q\ln x)\right]\frac{\partial}{\partial x} \\ & + (u+q)\big[2\ln x(u+q) + (v + q\ln x)\big]\frac{\partial}{\partial u} \\ & - \left[q\left(w + \frac{q}{2}(\ln x)^2\right) - 2q\ln x(v + q\ln x) - 2pt\right]\frac{\partial}{\partial v} \\ & - \ln x\left[q\left(w + \frac{q}{2}(\ln x)^2\right) + (v + q\ln x)(v - q\ln x) - 2pt\right]\frac{\partial}{\partial w}\end{aligned}$$

and $\Gamma_{1\infty}$, where $\eta = v + q\ln x$ and $y = F(t,\eta)$, satisfies (12).

Symmetries Γ_{12} and Γ_{13} project to point symmetries of (6) and induce potential symmetries admitted by (1) and (2), nonlocal symmetries admitted by (4), Lie–Bäcklund symmetries of (7) and Lie–Bäcklund symmetries equivalent to contact symmetries of (8).

CASE 9. $f = s/(u^2 + pu + q)$. We have:

(i) $N \neq 2 - M,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4$;

(ii) $N = 2 - M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_5$ and

$$\Gamma_{14} = x^{1-M}\left(2v + \frac{p}{M}x^M\right)\frac{\partial}{\partial x} - 2\left(u^2 + pu + q\right)\frac{\partial}{\partial u} - \left(pv + 2\frac{q}{M}x^M\right)\frac{\partial}{\partial v} + \left(v^2 - 2st - \frac{q}{M^2}x^{2M}\right)\frac{\partial}{\partial w};$$

(iii) $N = 2,\ M = 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3',\ \Gamma_4',\ \Gamma_5$ and

$$\Gamma_{14}' = x(2v + p\ln x)\frac{\partial}{\partial x} - 2\left(u^2 + pu + q\right)\frac{\partial}{\partial u} - (pv + q\ln x)\frac{\partial}{\partial v} + \left(v^2 - 2st - q(\ln x)^2\right)\frac{\partial}{\partial w}.$$

Symmetry Γ_{14} projects to a point symmetry of (2), (4) and (6) and induces a potential symmetry admitted by (1), a Lie–Bäcklund symmetry admitted by (7) and a contact symmetry of (8).

CASE 10. $f = \left(s/(u^2 + pu + q)\right)\exp\left[r\int (du)/(u^2 + pu + q)\right],\ r \neq 0$. We have:

(i) $N \neq 2 - M,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4$;

(ii) $N = 2 - M,\ M \neq 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3,\ \Gamma_4,\ \Gamma_5$ and

$$\Gamma_{15} = (r - p + 4)t\frac{\partial}{\partial t} + \left(vx^{1-M} + \frac{2}{M}x\right)\frac{\partial}{\partial x} - \left(u^2 + pu + q\right)\frac{\partial}{\partial u} + \left(2v - pv - \frac{q}{M}x^M\right)\frac{\partial}{\partial v} + \frac{1}{2}\left(8w - 2pw + v^2 - \frac{q}{M^2}x^{2M}\right)\frac{\partial}{\partial w}; \tag{16}$$

(iii) $N = 2,\ M = 0,\ \Gamma_1,\ \Gamma_2,\ \Gamma_3',\ \Gamma_4',\ \Gamma_5$ and

$$\Gamma_{15}' = (r - p + 4)t\frac{\partial}{\partial t} + (vx + 2x\ln x)\frac{\partial}{\partial x} - \left(u^2 + pu + q\right)\frac{\partial}{\partial u} + (2v - pv - q\ln x)\frac{\partial}{\partial v} + \frac{1}{2}\left[8w - 2pw + v^2 - q(\ln x)^2\right]\frac{\partial}{\partial w}.$$

We note that if $r = 0$ then $\Gamma_{15} \equiv \Gamma_3$ and $\Gamma_{15}' \equiv \Gamma_3'$. Symmetry Γ_{15} projects to a point symmetry of (2), (4) and (6) and induces a potential symmetry admitted by (1), a Lie–Bäcklund symmetry admitted by (7) and a contact symmetry of (8).

3. SIMILARITY SOLUTIONS

As in the case of point symmetries, potential symmetries may be used to derive similarity transformations (solutions). Such transformations reduce by one the number of independent variables of a system of partial differential equations. We present two examples of similarity solutions which are obtained using potential symmetries. In the first, we use a potential symmetry of (1) derived using (2) and in the second we use a potential symmetry obtained using (5). Similarly, any symmetry which was obtained in the previous section may be employed to derive similarity solutions.

We consider the point symmetry Γ_{10} (13) of (2) (potential symmetry of (1)) with $f = \left(s/(u^2+q^2)\right)\exp\left[r\tan^{-1}\left(u/q\right)\right]$ and $N = 2-M$, $M \neq 0$. The corresponding invariant surface conditions are

$$vx^{1-M}u_x + qrtu_t + u^2 + q^2 = 0,$$
$$vx^{1-M}v_x + qrtv_t + \frac{q^2}{M}x^M = 0,$$

which admit the three integrals

$$c_1 = v^2 + \frac{q^2}{M^2}x^{2M}, \qquad c_2 = \ln t + r\tan^{-1}\left(\frac{u}{q}\right),$$
$$\frac{Mc_1}{q}\sin\left(\frac{1}{r}\ln t + c_0\right) = x^M.$$

From the above relations we derive the similarity solutions

$$u = q\tan\left[F_2(\eta) - \frac{1}{r}\ln t\right], \; v = F_1(\eta)\cos\left[\eta + \frac{1}{r}\ln t\right], \tag{17}$$

where η is the similarity variable and is defined implicitly by

$$F_1(\eta)\sin\left[\eta + \frac{1}{r}\ln t\right] = \frac{q}{M}x^M. \tag{18}$$

Substitution of (17) into (2) provides

$$\frac{dF_1}{d\eta} = F_1\tan\left(F_2+\eta\right), \quad F_1\frac{dF_1}{d\eta} = sr\exp\left(rF_2\right)\frac{dF_2}{d\eta}, \tag{19}$$

where the independent variable η is defined by (18). Employing the solution of (19), (18) and the first relation in (17) will produce a similarity solution of (1) with $f = \left(s/(u^2+q^2)\right)\exp\left[r\tan^{-1}(u/q)\right]$, $N = 2-M$.

It was pointed out in **[13]** that a wider class of similarity solutions may be obtained by the direct introduction of (17) in (1). We can therefore substitute the first relation in (17) into (1). In this way, we obtain a relation involving η, F_1, F_2, derivatives of F_1, F_2 and t which appears as a parameter. That this relation is identically zero for any value of the parameter t leads to the system of ordinary differential equations

$$s\left(\frac{dF_1}{d\eta}\right)^3 + r\left[r\frac{dF_1}{d\eta}\left(\frac{dF_2}{d\eta}\right)^2 + \frac{dF_1}{d\eta}\frac{d^2F_2}{d\eta^2} - \frac{dF_2}{d\eta}\frac{d^2F_1}{d\eta^2} + F_1\frac{dF_2}{d\eta}\right]\sin^2\left(F_2+\eta\right)\exp\left(rF_2\right) = 0,$$

$$s\left[F_1^3+F_1^3\frac{dF_2}{d\eta}\right]+r\left[rF_1\left(\frac{dF_2}{d\eta}\right)^2+F_1\frac{d^2F_2}{d\eta^2}\right.$$
$$\left.-2\frac{dF_1}{d\eta}\frac{dF_2}{d\eta}\right]\cos^2(F_2+\eta)\exp(rF_2)=0,$$
$$\left[3F_1^2\frac{dF_1}{d\eta}+2F_1^2\frac{dF_1}{d\eta}\frac{dF_2}{d\eta}\right]-\left(\frac{dF_1}{d\eta}\right)^3\cot^2(F_2+\eta)$$
$$-2\left[F_1^3+F_1^3\frac{dF_2}{d\eta}\right]\tan(F_2+\eta)=0,$$
$$\left[3F_1\left(\frac{dF_1}{d\eta}\right)^2+F_1\left(\frac{dF_1}{d\eta}\right)^2\frac{dF_2}{d\eta}\right]-2\left(\frac{dF_1}{d\eta}\right)^3\cot(F_2+\eta)$$
$$-\left[F_1^3+F_1^3\frac{dF_2}{d\eta}\right]\tan^2(F_2+\eta)=0.$$

As pointed out in [**13**], the solution of this system will also contain that of (19).

In the second example we consider the point symmetry Γ_{15} (16) of (5) (potential symmetry of (1)) which is in the second part of (10). The corresponding invariant surface conditions are

$$\left(\frac{r}{q}vx^{1-M}+\frac{2x}{M}\right)u_x+\left(r^2+4\right)tu_t+\frac{r}{q}\left(u^2+q^2\right)=0,$$
$$\left(\frac{r}{q}vx^{1-M}+\frac{2x}{M}\right)v_x+\left(r^2+4\right)tv_t-\left(2v-\frac{qr}{M}x^M\right)=0,$$
$$\left(\frac{r}{q}vx^{1-M}+\frac{2x}{M}\right)w_x+\left(r^2+4\right)tw_t-\frac{1}{2}\left(8w+\frac{r}{q}v^2-\frac{qr}{M^2}x^{2M}\right)=0,$$

which admit the integrals

$$c_0=\ln t-\frac{r^2+4}{4}\ln\left(M^2v^2+q^2x^{2M}\right),$$
$$c_1=\ln t+\frac{r^2+4}{r}\tan^{-1}\left(\frac{Mv}{qx^M}\right),$$
$$c_2=\ln t+\frac{r^2+4}{r}\tan^{-1}\left(\frac{u}{q}\right),$$
$$c_3=\ln t-\frac{r^2+4}{4}\ln\left(x^Mv-2Mw\right).$$

From these integrals we obtain the similarity solutions

$$u=q\tan\left[F_2(\eta)-r_1\ln t\right],\quad v=\frac{q}{M}x^M\tan\left[F_1(\eta)-r_1\ln t\right],$$
$$w=\frac{q}{2M^2}x^{2M}\tan\left[F_1(\eta)-r_1\ln t\right]-\frac{1}{2M}t^{r_2}F_3(\eta),\tag{20}$$

where the similarity variable is defined implicitly by

$$\eta=\frac{q}{M}x^M\sec\left[F_1(\eta)-r_1\ln t\right]t^{-r_2/2},$$

where $r_1 = r/(r^2+4)$ and $r_2 = 4/(r^2+4)$.

Substitution of (20) into (5) provides

$$\eta \frac{dF_1}{d\eta} = \tan(F_2 - F_1), \quad \eta^2 \frac{dF_1}{d\eta} - \frac{q}{M}\frac{dF_3}{d\eta} = 0,$$
$$Mrr_1\eta^2 + qrr_2F_3 + 2sM\exp(rF_2) = 0.$$

Direct introduction of (20) into (1) reduces the latter to a complicated system of six ordinary differential equations.

4. Non-invertible mappings

In [**3**] it is shown that an invertible mapping which transforms a non–linear PDE into a linear PDE does not exist if the nonlinear PDE does not admit an infinite–parameter Lie group of contact transformations. Also such mappings do not exist for a non–linear system of PDEs if the system does not admit an infinite–parameter Lie group of transformations. If such infinite–parameter groups exist then the non–linear PDE (or system of non–linear PDEs) can be transformed into a linear PDE (or system of linear PDEs) provided that these groups satisfy certain criteria [**3**].

Equation (1) does not admit an infinite–parameter Lie group of contact transformations [**16**]. But as we have seen in Section 2, its auxiliary system, given by (2) admits an infinite–parameter Lie group of point transformations in the cases 4(iii) and 4(iv) (11). Similarly, the auxiliary system (5) admits an infinite–parameter Lie group of point transformations in the cases 8(ii), 8(v) (14) and 8(iii) (15). In these cases the infinitesimal generators satisfy the criteria required [**3**] so that the auxiliary systems (2) and (5) can be linearised by invertible mappings. In turn, these mappings lead to non-invertible mappings of (1).

The procedure for determining such invertible mappings is well explained in [**3**]. The infinite symmetry given by (11) leads to the invertible mapping

$$x' = v + \frac{q}{M}x^M, \; t' = t, \; u' = \frac{1}{M}x^M, \; v' = \frac{1}{u+q},$$

which transforms any solution $\big(u'(x',t'), v'(x',t')\big)$ of the linear system of PDEs

(21) $$u'_{x'} = v', \quad u'_{t'} = pv'_{x'}$$

to a solution $\big(u(x,t), v(x,t)\big)$ of the non-linear system (2) (with $f = p(u+q)^{-2}$ and $N = 2 - M$, $M \neq 0$) and hence to a solution $u(x,t)$ of (1). Similarly, it can be shown that the invertible mapping

$$x' = v + q\ln x, \; t' = t, \; u' = \ln x, \; v' = \frac{1}{u+q}$$

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transforms the linear system (21) into the nonlinear system (2) (with $f = p(u+q)^{-2}$and $N = 2,\ M = 0$).

Finally, we employ the appropriate symmetries of the system (5) to derive invertible mappings. Hence, from the infinite symmetry (14) we deduce the mapping

$$x' = v + \frac{q}{M}x^M,\ t' = t,\ u' = -w - \frac{1}{2q}v^2,\ v' = \frac{1}{M}x^M,\ w' = \frac{1}{u+q}$$

which transforms the linear system

$$u'_{x'} = v',\quad v'_{x'} = w',\quad u'_{t'} = pw' \tag{22}$$

into the nonlinear system (5) (with $f = p(u+q)^{-2}$ and $N = 2 - M,\ M \neq 0$). Similarly, it can be shown that the invertible mapping

$$x' = v + q\ln x,\ t' = t,\ u' = -w - \frac{1}{2q}v^2,\ v' = \ln x,\ w' = \frac{1}{u+q}$$

transforms the linear system (22) into the nonlinear system (5) (with $f = p(u+q)^{-2}$ and $N = 2,\ M = 0$).

Symmetry (15) leads us to the mapping

$$x' = w + \frac{v}{M}x^{-M} + \frac{q}{M}\ln x + \frac{q}{M^2},\ t' = t,$$
$$u' = \frac{1}{M}\left(v + \frac{q}{M}x^M\right),\ v' = \frac{1}{M}x^M,\ w' = \frac{Mx^M}{u+q},$$

which transforms the linear system

$$u'_{x'} = v',\quad v'_{x'} = w',\quad u'_{t'} = \frac{p}{M^2}w' - \frac{p}{q}v'$$

into the nonlinear system (5) (with $f = p(u+q)^{-2}$ and $N = 2 + M,\ M \neq 0$).

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