

MULTIPLE QUANTIZATION

A. E. SCHEIDEGGER

1. Introduction. The efforts of most theoretical physicists of this century have been directed towards that branch of the physical science which is commonly called "Quantum Theory." Physically, Quantum Theory was postulated because of a vast amount of physical evidence which led to the postulates of states, observables, superposition, and commutation relations. From these four postulates, all quantum mechanics follows.

It is well known that the differential equations which are obtained in Quantum Theory can be quantized for a second time. There is a strong physical motive for doing this, namely, the necessity for a field description, required both on general theoretical grounds and to explain, for example, pair production. Mathematically the "second" quantization is expressed in exactly the same way as the first one. In this fashion a certain mathematical scheme has been set up for "quantizing" a differential equation. This scheme consists of several stages whereby the original dynamical variables are finally replaced by operators. This scheme has been applied to many examples. Because of the fact that it is mathematically rather involved, this was not an easy task.

Once the mathematical scheme of Quantum Theory has been obtained, it is of considerable interest to investigate the latter even without any further reference to its physical background. It is possible that the mathematical form suggests interesting generalizations which, in the end, might even have a physical application. This is what we propose to do here.

A careful investigation of the mathematical formalism of Quantum Theory shows that it can be regarded as a process which can be applied to almost any differential equation. Thus one might say that quantization is a corrective mathematical process which is applicable to unsatisfactory physical differential equations. This is an entirely new aspect of quantum mechanics which is quite apart from any physical arguments for introducing it. It is suggested by the mathematical form of the equations and it remains to be seen whether it has any physical applications.

If we accept the interpretation of quantization as a corrective mathematical process, the possibility of repeated or multiple quantization presents itself, as it seems to us, quite naturally. There are, of course, no direct physical reasons for such a repetition of the quantization process, except for the fact that it yields physical results when applied the first two times. From the mathematical standpoint, however, a further repetition seems a most natural generalization. In addition, it might be of interest to know, even for a physicist, what the quantization process does to his equations when repeatedly applied. Perhaps one

Received September 20, 1951.

might speculate that each quantization makes the equations describing a physical process more and more accurate.

Thus, we propose to give first a detailed account of the mathematical inducements for the notion of multiple quantization. The idea thus established will be discussed in the rest of the paper.

The results of the present investigations are as follows.

As a first example the Schroedinger-Gordon field is quantized for the third time. To achieve this, it is necessary to obtain a Schroedinger-representation for the Schroedinger-Gordon field, since the customary Heisenberg matrix-representation does not permit a further quantization. Then, the energy eigenvalues of the third quantization of the Schroedinger-Gordon field are obtained; they are found to be $\frac{1}{2} \sum \hbar \omega_k N_k$.

In order to quantize an equation for an infinite number of times, one has to know what the quantization of a Schroedinger equation with an arbitrary Hamiltonian yields (with every equation after a few steps one seems to arrive at a Schroedinger equation). It is found that in this case the $(n + 1)$ th quantization describes a Bose ensemble of systems as described by the n th quantization.

In the case where the quantization is performed using anticommutation, instead of commutation rules, one obtains a Fermi, instead of a Bose ensemble. In this instance it turns out that the repetition of the quantization procedure is, physically, simply an easy way of doing the quantum statistics of fields.

2. Inducements for multiple quantization: notation. First, we should like to show how those ideas which we intend to discuss in this paper suggest themselves quite naturally. In order to do so, we have to summarize the principles of quantization in their mathematical form, emphasizing especially their algebraic features. At the same time, this will serve to explain the notation of the present paper. The physical reasons which require the mathematical steps will not be discussed here, since we are only concerned with the mathematical formalism.

In classical theory one starts with a series of real generalized co-ordinates q_k ($k = 1, \dots, f$) which describe the state of the mechanical system concerned. The co-ordinates q_k are functions of a parameter t which, in applications, may be identified with the physical time. Dots denote derivatives with respect to this parameter. The equations of motion follow from a variational principle

$$(2.1) \quad \delta \int L dt = 0,$$

where $L = L(q_k, \dot{q}_k, t)$ denotes the Lagrangian function of the system. The variational principle is equivalent to the Euler-Lagrangian differential equations

$$(2.2) \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0.$$

Instead of using this representation one may pass to the canonical formalism

in the well-known manner. The algebraic nature of the latter is seen if one introduces Poisson brackets between arbitrary functions F, G, \dots of q_k and $p_k = \partial L / \partial \dot{q}_k$. Then, the equations of motion may be written as follows

$$(2.3) \quad \dot{F} = [F, H],$$

H denoting the Hamiltonian function of the system.

The Poisson brackets have some remarkable algebraic properties (see, e.g., Dirac's book [1, p. 85-86]), which permit one to reduce any Poisson brackets of arbitrary functions of p_k, q_k to elementary Poisson brackets between p_k and q_k . Thus, if one adds to the algebraic rules of Poisson brackets

$$(2.4) \quad [p_k, q_i] = -\delta_{ki}$$

($\delta =$ Kronecker symbol), then the problem of motion can be solved by *algebra alone*.

This fact makes the quantization of a classical theory *possible*. The dynamical variables are re-interpreted as algebraic entities for which only laws of multiplication need to be defined. If the meaning of "Poisson bracket" is newly fixed in terms of the commutator of the two entities concerned, then the motion of the quantum mechanical variables is entirely determined by the Poisson bracket rules and (2.3).

One can obtain a representation of the quantum mechanical algebraic entities by identifying them with linear Hermitian operators in a Hilbert space. Actually, a proper Hilbert space is not quite sufficient because one has to admit vectors of infinite length. Nevertheless, we shall refer in this paper to such a general space simply by calling it "Hilbert" space. This draws one's attention to the elements of such a generalized Hilbert space. Mathematically, they are a kind of vectors denoted by φ , to each of which a dual exists (φ^*) so that a scalar product $\langle \varphi^* \varphi \rangle$ may be formed. Physically, they are interpreted as representatives of the mechanical system such that

$$(2.5) \quad \bar{A} = \langle \varphi^* (A \varphi) \rangle$$

is the average value for the dynamical observable A in a statistical ensemble of measurements.

The representation of quantum mechanics in this form corresponds to the Heisenberg-representation. By throwing the time-dependence from the operators into the Hilbert vectors φ , one can pass over to the Schroedinger representation. Thus, one arrives at a differential equation for the time-dependence of the Hilbert vector φ :

$$(2.6) \quad \dot{\varphi} = \frac{i}{\hbar} H \varphi.$$

In short, we note that starting from a "vector"

$$(2.7) \quad \vec{q} = (q_1 \dots q_k),$$

which satisfies a differential equation (2.2), one finally arrives through the process of quantization at a new differential equation of a similar type. The vector \vec{q} originally characterizing the state of the system is replaced by the vector φ serving the same purpose.

Thus, quantization is mathematically a correcting process for “classical” equations. But since the results are equations similar to those from which one had started, it must be asked why this correction should be applied only *once* to the classical theory.

3. General theory. We note that in order to make a quantization procedure according to the canonical rules established in the last section, one needs a Hilbert vector $q(k, t)$ which depends on a time-like parameter. The Hilbert vector must be subject to a differential equation of the type (2.2) regarding its time-dependence:

$$(3.1) \quad F(q(k, t), \dot{q}(k, t), \ddot{q}(k, t)) = 0,$$

so that the canonical formalism can be set up. After this has been done, one can proceed to the Heisenberg equations of motion and hence to the Schroedinger equations of motion

$$(3.2) \quad \frac{\hbar}{i} \psi'(x) = H\psi(x),$$

which complete the circle from a Hilbert vector to a Hilbert vector.

If (3.1) is of the type of a Schroedinger equation like (3.2), which will be usually the case if one wants to quantize for the second or third time, there are several difficulties to reckon with. First we note that x in (3.2) is a continuous variable with an infinite range, whereas k in q_k assumes only a finite and discrete set of values. This difficulty, however, has been overcome successfully in the quantum theory of fields.

Furthermore, the ψ 's in (3.2) are complex, and the differential equation (3.2) contains only first-order time-derivatives. The latter fact is insignificant, as (3.1) can be replaced by first-order equations; but the reality of the components of the ψ 's is essential if one wants to maintain the analogy to the ordinary (first) quantization. We shall see, first, how this reality can be obtained by transforming (3.2), although there exists the possibility of abandoning the correspondence between ψ and a canonical variable, which is indeed sometimes done in the quantum theory of fields.

In order to obtain real equations of the form of (3.1) from (3.2), one has to split ψ into its real and imaginary parts

$$(3.3) \quad \psi = \psi_1 + i\psi_2$$

where both ψ_1 and ψ_2 are real. Then one has to try to set up equations for ψ_1 and ψ_2 alone. Splitting (3.2) into its real and imaginary parts yields, if H is assumed to be real (which will be ordinarily the case),

$$(3.4) \quad \begin{aligned} \hbar\dot{\psi}_1 &= -H\psi_2, \\ \hbar\dot{\psi}_2 &= H\psi_1. \end{aligned}$$

From the first of these equations one obtains

$$(3.5) \quad \psi_2 = -\hbar H^{-1}\dot{\psi}_1,$$

and from the second

$$(3.6) \quad \psi_1 = \hbar H^{-1}\dot{\psi}_2$$

which, inserted into either equation (3.4), yields

$$(3.7) \quad -\hbar^2\ddot{\psi}_1 = H H \psi_1$$

and an identical equation for ψ_2 . It maybe noted that the Schroedinger-Gordon equation is of this form (3.7).

However, as it was said above, it is not necessary to transform (3.2) into real equations. If one keeps equation (3.2), one has to abandon the direct analogy between ψ and ordinary dynamical variables. Thus it may be possible to derive (3.2) from a Lagrangian principle wherein the Lagrangian function will depend on ψ and ψ^* as well as on their derivatives. Then one can formally define the canonical conjugates π and π^* and write down the Hamiltonian. Doing this it should be kept in mind that π and ψ are no longer canonically conjugate in the *classical* sense; it would be difficult to define Poisson brackets between functionals depending on them. One may, however, assume the same commutation relations as if π and ψ were conjugate.

This treatment is very useful if one is only interested in the matrix formulation of the quantized theory. Then, π and ψ are simply represented by non-Hermitian matrices. This is sometimes so convenient that even in the case where one starts with equation (3.7), one transforms the latter into an equation where the variables are no longer real; this is well known from the transition to momentum space in the Schroedinger-Gordon case. Using complex variables, one is, however, generally at a loss if one wants to set up the Schroedinger formulation of the theory in order to proceed with further quantizations. In this case, one has to transform back to variables which are canonical.

In the next section, we shall demonstrate these ideas in a model example.

4. Example: Third quantization¹ of the Schroedinger-Gordon equation. Assuming the Hamiltonian

$$(4.1) \quad H = c\sqrt{p^2 + m^2c^2}$$

leads, after the *first* quantization, to the Schroedinger-Gordon equation (see [6]):

$$(4.2) \quad \left(\Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \mu^2 \right) \psi = 0$$

¹The name "third quantization" has been used by Nambu [4] in quite a different context. Our paper has nothing to do with the work of Nambu.

which can be written in the following Hamiltonian form:

$$(4.3) \quad H = \int \mathfrak{S} d^3x; \quad \mathfrak{S} = \frac{1}{2} \{ \pi^2 + c^2 |\text{grad } \psi|^2 + c^2 \mu^2 \psi^2 \}.$$

If π, ψ are taken as Hermitian operators, (4.3) represents the *second* quantization. A Fourier transform of the variables leads to

$$(4.4) \quad H = \frac{1}{2} \sum_k \{ p_k^* p_k + \omega_k^2 q_k^* q_k \}.$$

This equation is usually solved by the matrix method. If we want to carry on our quantization procedure for the *third* time, we must establish first a Schrodinger equation for the second quantization, instead of (4.4). The transition from a Heisenberg to a Schrodinger formulation is usually done in such a way that one of the canonical variables is taken as the argument of the new Schrodinger function and the conjugate one represented as a differential operator. Unfortunately, this procedure cannot be applied to (4.4) because p and q are not canonical since they are not real. They satisfy, however, the following relations (see [6]):

$$(4.5) \quad p_k^* = p_{-k}; \quad q_k^* = q_{-k}.$$

The ordinary transition from ψ, π to q, p is thus not a canonical transformation. Therefore, one has to introduce for each "dimension" k a variable ξ_k which is canonical. If one finds a representation of p_k, q_k by differential operators acting upon the ξ_k 's such that the commutation relations and the conditions (4.5) are satisfied, then one has obtained the *Schrodinger representation of the second quantization*. It is easy to verify that this can be done by setting

$$(4.6a) \quad q_k = \frac{1}{2} \left(\frac{\hbar}{\omega_k} \right)^{\frac{1}{2}} \left\{ \xi_k + \xi_{-k} - \frac{\partial}{\partial \xi_k} + \frac{\partial}{\partial \xi_{-k}} \right\}$$

$$(4.6b) \quad p_k = \frac{1}{2} (\hbar \omega_k)^{\frac{1}{2}} i \left\{ \xi_k - \xi_{-k} + \frac{\partial}{\partial \xi_k} + \frac{\partial}{\partial \xi_{-k}} \right\}$$

A straightforward check shows that the commutation relations for p, q are satisfied as well as (4.5). For the latter condition one has to remember that the star indicates the Hermitian adjoint of an operator and that the Hermitian adjoint of $\partial/\partial \xi$ is $-\partial/\partial \xi$. The procedure used here has been investigated by Infeld and Hull [2, §6] to which the reader is referred for the details. The total Hamiltonian of the field may be represented by

$$(4.7) \quad H = \sum \frac{1}{2} \hbar \omega_k (\xi_k^2 - \partial^2 / \partial \xi_k^2).$$

Since we can split up the eigenfunctions for every "dimension" k , it is possible to make the following "ansatz":

$$(4.8) \quad \phi = \prod_k \exp \left(\frac{i}{\hbar} E_k t \right) \varphi_k(\xi_k).$$

Then, the Schrodinger equation

$$(4.9) \quad H \phi = \frac{\hbar}{i} \dot{\phi}$$

yields the following condition for the φ_k 's:

$$(4.10) \quad \frac{1}{2} \hbar \omega_k (\xi_k^2 - \partial^2 / \partial \xi_k^2) \varphi_k = E_k \varphi_k.$$

As is well known, equation (4.10) is soluble only for

$$(4.11) \quad E_k = \frac{1}{2} \hbar \omega_k (2N_k + 1),$$

which is the same result as obtained by the matrix method [2, p. 40]. A general state vector is obtained by a linear superposition of the particular ones in (4.8).

The *third* quantization of the Schroedinger-Gordon equation is performed if we are able to quantize (4.9), or (4.10). If we keep the time dependence in the φ 's, (4.10) may be written

$$(4.12) \quad \frac{1}{2} \hbar \omega_k (\xi_k^2 - \partial^2 / \partial \xi_k^2) \varphi_k = \frac{\hbar}{i} \dot{\varphi}_k.$$

The task is to quantize (4.12). We note that this equation is of the form of a non-relativistic Schroedinger equation, if we set

$$(4.13) \quad m = \hbar / \omega_k; \quad U = \frac{1}{2} \xi_k^2 \hbar \omega_k.$$

The Hamiltonian formulation of the non-relativistic Schroedinger field has been set up long ago [5, p. 338]. Using those earlier results, we can write down the Hamiltonian which refers to the "dimension" k :

$$(4.14) \quad H_k = \frac{1}{2} i \omega_k \int \{ \text{grad } \pi_k \text{ grad } \varphi_k + \xi_k^2 \pi_k \varphi_k \} d\xi_k.$$

Thus, the *total* Hamiltonian is simply the sum of the expressions (4.14) taken over all the "dimensions" k :

$$(4.15) \quad H = \frac{1}{2} i \sum_k \omega_k \int (\text{grad } \pi_k \text{ grad } \varphi_k + \xi_k^2 \pi_k \varphi_k) d\xi_k.$$

The eigenvalues of the total Hamiltonian are of particular interest. We note that the eigenvalues of each H_k have been calculated; they are

$$(4.16) \quad (H_k) \dots = \sum N_n^{(k)} E_n^{(k)},$$

where the N_n 's range from 0 to ∞ , and $E_n^{(k)}$ is eigenvalue of the equation

$$(4.17) \quad \left(-\frac{1}{2} \hbar \omega_k \frac{\partial^2}{\partial \xi_k^2} + \frac{1}{2} \hbar \omega_k \xi_k^2 \right) \Psi_n(\xi_k) = E_n^{(k)} \Psi_n(\xi_k).$$

Thus we have

$$(4.18) \quad E_n^{(k)} = \frac{1}{2} \hbar \omega_k (2n + 1).$$

Therefore, we find the following eigenvalues of H_k :

$$(4.19) \quad (H_k) \dots = \frac{1}{2} \hbar \omega_k \sum_n N_n^{(k)} (2n + 1),$$

and of H :

$$(4.20) \quad (H) \dots = \sum_{k,n} \frac{1}{2} \hbar \omega_k N_n^{(k)} (2n + 1).$$

We may observe that the $N_n^{(k)}$'s are arbitrary integer numbers ranging from 0 to ∞ . But then also

$$(4.21) \quad \mathfrak{N}_k = \sum_n (2n + 1) N_n^{(k)}$$

are arbitrary integer numbers ranging from 0 to ∞ , and we might as well represent the eigenvalues of H in the following form

$$(4.22) \quad (H) \dots = \sum \frac{1}{2} \hbar \omega_k \mathfrak{N}_k.$$

These eigenvalues are very similar to those which were obtained after the second quantization (cf. (4.11)). Speaking in terms of particles we can say that our equations describe an ensemble of \mathfrak{N}_k particles of the energy $\frac{1}{2} \hbar \omega_k$, k ranging over all the lattice points in momentum space.

However, it is interesting to observe that the "particles" have only half the energy of those of the Schroedinger-Gordon equation. Furthermore, the infinite term in the energy has disappeared, since the lowest state is $\mathfrak{N}_k = 0$ for every k . Therefore, it seems better to interpret (4.22) physically in a different way so that the usual physical interpretation of the second quantization can be retained. It is easily seen that the present scheme can be taken as describing a Bose ensemble of fields (instead of an ensemble of particles) each of which satisfies the Schroedinger-Gordon equation. In this instance the new quantization is equivalent to considering Bose statistics of the meson fields.

It will be seen in the next section that quantizing a Schroedinger equation can always be interpreted as taking a Bose ensemble of systems as present before the new quantization, and doing statistics with it. Thus our last remark is in agreement with more general principles.

5. Infinite quantization. After having seen that it should always be possible to perform a quantization of a first- or second-order differential equation, one may wonder how often this process can be repeated.

We may note that after a very few steps, one always arrives at a Schroedinger type of equation; i.e.,

$$(5.1) \quad \phi = \frac{i}{\hbar} {}^n H \phi.$$

The arguments of the ϕ -function are the parameter t and the canonical variables before the quantization. This is due to the fact that one has to set up the Hamiltonian formalism for the equation from which one wants to start, and the Hamiltonian formalism immediately leads to (5.1). One may check that with the Schroedinger-Gordon case where we arrived at a Schroedinger equation after the third step, as was seen in the last section.

Therefore, if we are able to find a recursion formula which connects the eigenvalues of nH in (5.1) with the eigenvalues of the Hamiltonian ${}^{n+1}H$ of the next quantization, then we are able to proceed with the quantizations as often as we like.

In order to obtain that (5.1) is the Euler differential equation of a Lagrangian principle, the Lagrangian density must be of the form

$$(5.2) \quad \mathcal{L} = i\hbar \phi^* \dot{\phi} + B(\phi, \phi^*)$$

so that the canonically conjugate π of ϕ is

$$(5.3) \quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = i\hbar \phi^*$$

It would be rather difficult to find all the terms in the Lagrangian (5.2) explicitly so that such a general equation as (5.1) would follow from it. Nevertheless, we can find the Hamiltonian ${}^{n+1}H$ in matrix representation, in terms of the matrix elements of nH , provided that we assume the existence of such a Hamiltonian formalism and the corresponding quantization.

For, if such a Hamiltonian formalism and the corresponding quantization exists, then the consistency of (5.1) with the quantum formalism requires

$$(5.4) \quad \dot{\phi} = \frac{i}{\hbar} [\phi, {}^{n+1}H] = \frac{i}{\hbar} {}^nH \phi$$

or

$$(5.5) \quad [\phi, {}^{n+1}H] = {}^nH \phi.$$

Expanding ϕ into the complete orthogonal set of eigenfunctions of nH yields

$$(5.6) \quad \phi = \sum a_m u_m(x),$$

$$(5.7) \quad \pi = i\hbar \phi = i\hbar \sum a_m^* u_m^*(x).$$

The canonical commutation relations for π and ϕ are satisfied if

$$(5.8) \quad [a_m, a_{m'}] = [a_m^*, a_{m'}^*] = 0; [a_m, a_{m'}^*] = \delta_{mm'}$$

holds. Inserting the expansions for π and ϕ into (5.5) yields

$$(5.9) \quad \sum a_m u_m(x) {}^{n+1}H - {}^{n+1}H \sum a_m u_m(x) = \sum {}^nE_m u_m(x) a_m,$$

where nE_m denotes the m th eigenvalue of nH . We multiply (5.9) by u_k^* and integrate over x ; one then obtains

$$(5.10) \quad a_k {}^{n+1}H - {}^{n+1}H a_k = {}^nE_k a_k,$$

which is a matrix equation for ${}^{n+1}H$. The solution is obviously

$$(5.11) \quad {}^{n+1}H = \sum_j {}^nE_j a_j^* a_j,$$

because of the following relation which holds in virtue of (5.8):

$$(5.12) \quad [a_k, {}^{n+1}H] = \sum_j {}^n E_j (a_k a_j^* a_j - a_j^* a_j a_k) = {}^n E_k [a_k, a_k^*] a_k = {}^n E_k a_k.$$

A diagonal representation of ${}^{n+1}H$ is obtained if we choose

$$(5.13) \quad a_k = \begin{pmatrix} 0 & \sqrt{1} & & \\ & 0 & \sqrt{2} & \\ & & & \ddots \end{pmatrix}; \quad a_k^* = \begin{pmatrix} 0 & & & \\ \sqrt{1} & 0 & & \\ & \sqrt{2} & & \\ & & & \ddots \end{pmatrix}; \quad N_k = a_k^* a_k = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 2 & \\ & & & \ddots \end{pmatrix}$$

which obviously conforms with (5.8). (The matrices (5.13) refer to the index k only, a_k is the unit matrix with respect to all other indices.) Thus, the diagonal elements of ${}^{n+1}H$ become

$$(5.14) \quad {}^{n+1}E_m = \sum N_k {}^n E_k; \quad m = (N_0 N_1 \dots).$$

This means that the $(n + 1)$ th quantization simply describes a statistical ensemble of systems as described by the n th quantization, wherein the counting prescriptions of Einstein and Bose hold. (Each combination of N_k 's is enumerated just once.²) The lowest eigenvalue is always zero, no matter what the ${}^n E_k$'s are; it never diverges.

If we perform one further quantization, the eigenvalues will be

$$(5.15) \quad {}^{n+2}E_j = \sum_m N_m \sum N'_k {}^n E_k = \sum \mathfrak{N}_k {}^n E_k$$

where \mathfrak{N}_k again may assume any integer values from 0 to ∞ .

Therefore we see that the eigenvalues of the energy do not change any more after one has had a Schroedinger equation and quantized it once more. Thus, even if one repeats quantization an *infinite number of times*, the energy eigenvalues remain final after a few steps. Therefore the eigenvalues of the Schroedinger-Gordon equation will always be as given in (4.22), notwithstanding how many quantizations are performed afterwards.

6. The anticanonical formalism. It seems desirable to find out what becomes of the formalism of the last section in the case of anticanonical quantization.

The anticanonical quantization is distinguished from the canonical procedure by the fact that for π, ϕ anticommutation (rather than commutation) relations are assumed. The equation of motion, however, holds in its canonical form.

In order to find the anticanonical formulae that correspond to what we have done in the last section, one can keep practically all the equations that were developed there, except that one has to find a new representation for the matrices a_k and a_k^* . These representations have been proposed by Jordan and Wigner [3] in connection with the quantization of the Dirac electron field. The same matrices may be used in the *general* formalism as required in the present investigations. Thus we choose

²It has been known before that a Schroedinger equation with an arbitrary *potential* leads to a Bose ensemble of the non-quantized system [5]; but the proof had never been given for an arbitrary *Hamiltonian*, as far as I know.

$$(6.1) \quad a_k = \eta_k \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$(6.2) \quad a_k^* = \eta_k \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The index k labels the state of the system as before. The factors η_k are equal to $+1$ or -1 according to whether the number of occupied states with labels $n < k$ is even or odd.

Using this representation it is easy to see that the Hamiltonian is again given by

$$(6.3) \quad {}^{n+1}H = \sum_j {}^n E_j a_j^* a_j.$$

This satisfies indeed the matrix equation (5.10).

This means that the $(n + 1)$ th anticanonical quantization of a Schrodinger equation describes a statistical ensemble of such systems as are described by the n th Schrodinger equation. But now, since we are using the anticanonical formalism, we see that this is a *Fermi-Dirac ensemble*.

This result holds under the same premises and assumptions as before, that is, the *consistency* of the formalism has to be presumed in each case.

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Imperial Oil Limited
Calgary, Alberta