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A NOTE ON DEGREE SEQUENCES OF GRAPHS

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ABSTRACT. Sufficient conditions for a sequence of numbers to be the degree sequence of a graph are derived from the Erdos–Gallai theorem on degree sequences of graphs.

Let $d_1, \ldots, d_n$ be a sequence of non-negative integers which is the degree sequence of a graph G. It then follows that $n-1-d_1, \ldots, n-1-d_n$ is also the degree sequence of a graph (the complement of G works). Hence there exists a smallest positive integer $\Delta(d_1, \ldots, d_n)$ such that

$$\Delta(d_1, \ldots, d_n)-d_1, \ldots, \Delta(d_1, \ldots, d_n)-d_n$$

is the degree sequence of a graph where

$$\max_{1\le i\le n} d_i \le \Delta(d_1, \ldots, d_n) \le n-1. \tag{1}$$

We extend the domain of definition of $\Delta(d_1, \ldots, d_n)$ to include every sequence of n non-negative integers by defining $\Delta(d_1, \ldots, d_n)$ to be ∞ when there is no graph whose degree sequence is $s-d_1, \ldots, s-d_n$ for any non-negative integer s.

The purpose of this note is to derive the following theorem and its corollaries which give some information about $\Delta(d_1, \ldots, d_n)$. Lest there be some misunderstanding we remark that in a graph an edge joins distinct vertices and there is at most one edge joining a pair of vertices.

THEOREM. *Let $n>1$. Let $d_1, \ldots, d_n$ be a sequence of positive integers and let $m=d_1+\cdots+d_n$. Suppose Δ is an integer satisfying*

$$n-1+d_i \ge \Delta \ge d_i \qquad (i=1, \ldots, n), \tag{2}$$

$$\sum_{i=1}^{n} (\Delta-d_i) \equiv 0 \pmod 2, \tag{3}$$

$$\Delta^2-\Delta(n+1)+m \le 0 \tag{4}$$

Then $\Delta-d_1, \ldots, \Delta-d_n$ is the degree sequence of a graph.

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REMARK. Let $e_i = \Delta - d_i$ $(i=1,\ldots,n)$ A reformulation of the theorem is: If (2′) $n-1 \geq e_i \geq 0$ $(i=1,\ldots,n)$, (3′) $e_1+\cdots+e_n \equiv 0 \pmod 2$, and (4) hold, then $e_1,\ldots,e_n$ is the degree sequence of a graph.

Proof. There is no loss in generality in assuming that $d_1 \geq \cdots \geq d_n$. Since (2) is satisfied, to prove the theorem it suffices to show that the 'complementary' sequence $n-1-\Delta+d_1,\ldots,n-1-\Delta+d_n$ is the degree sequence of a graph. Applying the Erdös–Gallai theorem [2, p. 59] we see that $n-1-\Delta+d_1,\ldots,n-1-\Delta+d_n$ is the degree sequence of a graph if and only if (3) is satisfied and

$$(5)\quad \sum_{i=1}^{k}(n-1-\Delta+d_i) \leq k(k-1)+\sum_{j=k+1}^{n}\min\{k, n-1-\Delta+d_j\} \qquad (k=1,\ldots,n).$$

The inequalities (5) are equivalent to the inequalites

$$(6)\quad \sum_{i=1}^{k} d_i \leq k(\Delta+k-n)+\sum_{j=k+1}^{n}\min\{k, n-1-\Delta+d_j\} \qquad (k=1,\ldots,n\}.$$

Let k be a fixed integer with $1 \leq k \leq n$. Suppose t is an integer with $k+1 \leq t \leq n$ such that

$$(7)\quad \begin{aligned} &d_{n-t+1},\ldots,d_n \leq k+\Delta-n \\ &d_{k+1},\ldots,d_{n-t} \geq k+\Delta-n+1. \end{aligned}$$

Then inequality (6) for k is equivalent to

$$(8)\quad \sum_{i=1}^{k} d_i \leq \sum_{j=n-t+1}^{n} d_j + t(n-1-\Delta)+(\Delta-t)k.$$

Now using (7) we see that

$$\sum_{i=1}^{k} d_i = m - \sum_{j=k+1}^{n} d_j \leq m - \left(\sum_{j=n-t+1}^{n} d_j + (n-k-t)(k+\Delta-n+1)\right).$$

Hence (8) will be satisfied if

$$m - \left(\sum_{j=n-t+1}^{n} d_j + (n-k-t)(k+\Delta-n+1)\right) \leq \sum_{j=n-t+1}^{n} d_j + t(n-1-\Delta)+(\Delta-t)k,$$

or, equivalently,

$$(9)\quad 2\sum_{j=n-t+1}^{n}(k+\Delta+1-n-d_j) \leq (n-k)(k+\Delta-n+1)+\Delta k - m.$$

From the assumption that $d_1,\ldots,d_n$ are positive integers it follows that

$$(10)\quad k+\Delta+1-n-d_j \leq k+\Delta-n \qquad (j=n-t+1,\ldots,n).$$

Thus since $t \leq n-k$, it follows that (5) is satisfied if

$$2(n-k)(k+\Delta-n) \leq (n-k)(k+\Delta-n+1)+\Delta k - m \qquad (k=1,\ldots,n),$$

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or, equivalently,

(11) $$k^2-(2n-2\Delta+1)k+(n^2+n-n\Delta-m)\geq 0 \qquad (k=1,\ldots,n).$$

Let

$$f(x)=x^2-(2n-2\Delta+1)x+(n^2+n-n\Delta-m).$$

Then the minimum value of $f(x)$ for integer x occurs at $x=n-\Delta$ and equals $-\Delta^2+\Delta(n+1)-m$. Hence (11) is satisfied provided

(12) $$-\Delta^2+\Delta(n+1)-m\geq 0,$$

which is equivalent to (4). The theorem now follows.

Suppose we replace the assumption in the statement of the theorem that $d_1,\ldots,d_n$ are positive integers by the assumption $d_1,\ldots,d_n$ are non-negative integers. Then the following changes occur. Inequality (10) is replaced by

$$k+\Delta+1-n-d_j\leq k+\Delta+1-n \qquad (j=n-t+1,\ldots,n);$$

inequality (11) is replaced by

$$k^2-(2n-2\Delta-1)k+(n^2-n-n\Delta-m)\geq 0 \qquad (k=1,\ldots,n);$$

and inequality (12) is replaced by

$$-\Delta^2+\Delta(n-1)-m\geq 0.$$

Hence we have the following.

Corollary 1. *Let $n>1$. Let $d_1,\ldots,d_n$ be a sequence of non-negative integers and let $m=d_1+\cdots+d_n$. Suppose Δ is an integer satisfying* (2), (3) *and*

(13) $$\Delta^2-\Delta(n-1)+m\leq 0.$$

Then $\Delta-d_1,\ldots,\Delta-d_n$ is the degree sequence of a graph.

Corollary 2. *Let $d_1,\ldots,d_n$ be a sequence of positive integers such that $m=d_1+\cdots+d_n\leq 2n$. Let Δ be an integer with $\Delta\leq n-1$. Then $\Delta-d_1,\ldots,\Delta-d_n$ is the degree sequence of a graph if and only if $\Delta\geq d_i$ $(i=1,\ldots,n)$ and* (3) *holds.*

Proof. If $\Delta-d_1,\ldots,\Delta-d_n$ is the degree sequence of a graph, then surely $\Delta\geq d_i$ $(i=1,\ldots,n)$ and (3) holds. Now suppose that $\Delta\geq d_i$ $(i=1,\ldots,n)$ and (3) holds. First suppose that $\Delta=1$. It then follows that $d_i=1$ $(i=1,\ldots,n)$ and hence $\Delta-d_i=0$ $(i=1,\ldots,n)$. The conclusion follows trivially in this case. Next suppose $\Delta=2$. Then $\Delta-d_1,\ldots,\Delta-d_n$ is a sequence of 0's and 1's with an even number of terms equal to 1. It follows easily that $\Delta-d_1,\ldots,\Delta-d_n$ is the degree sequence of a graph. Now suppose $\Delta=n-1$. From (3) we see that

$$n(n-1)-\sum_{i=1}^{n} d_i\equiv 0 \qquad (\text{mod } 2);$$

hence it follows that $\sum_{i=1}^{n} d_i \equiv 0 \pmod 2$. It now follows readily by induction that $d_1, \ldots, d_n$ is the degree sequence of a graph. Hence $\Delta - d_1, \ldots, \Delta - d_n$ is the degree sequence of a graph. Thus we may assume that $3 \le \Delta \le n-2$. Let $m = 2(n-1)+p$ where $-(n-2) \le p \le 2$. To complete the proof it suffices by the theorem to show that (4) is satisfied. We calculate that

$$\Delta^2 - \Delta(n+1) + m = (\Delta-2)(\Delta-(n-1)) + p.$$

Since $p \le 2$ and $3 \le \Delta \le n-2$, (4) is satisfied unless $\Delta = 3$, $\Delta = n-2$, and $p = 2$ all hold. If the latter conditions hold, then $n = 5$, $m = 10$ and

$$\sum_{i=1}^{5} (\Delta - d_i) = 5,$$

contradicting (3). Thus (4) holds and $\Delta - d_1, \ldots, \Delta - d_n$ is the degree sequence of a graph.

COROLLARY 3. *Let $d_1, \ldots, d_n$ be a sequence of positive integers such that $d_1 + \cdots + d_n \le 2n$. Let Δ be an integer such that $n-1 \ge \Delta \ge d_i$ $(i = 1, \ldots, n)$. Then there exists a regular graph G of degree Δ with n vertices having a spanning subgraph H with degree sequence $d_1, \ldots, d_n$ if and only if*

$$\sum_{i=1}^{n} d_i \equiv n\Delta \equiv 0 \pmod 2. \tag{14}$$

Proof. If graphs G and H as specified in the corollary exist, then clearly (14) holds. Now suppose (14) is satisfied. It follows easily by induction on n that there exists a graph with degree sequence $d_1, \ldots, d_n$ and hence a graph with degree sequence $n-1-d_1, \ldots, n-1-d_n$. It follows from Corollary 2 that there exists a graph with degree sequence $\Delta - d_1, \ldots, \Delta - d_n$. Since $n-1-d_i-(\Delta - d_i) = n-1-\Delta \ge 0$ $(i = 1, \ldots, n)$, it follows from the 'k-factor theorem' [3, 4] with $k = n-1-\Delta$ that there exists a graph H' with degree sequence $n-1-d_1, \ldots, n-1-d_n$ having a spanning subgraph G' which is regular of degree $n-1-\Delta$. Hence the complement G of G' is a regular graph of degree Δ having the complement H of H' as a spanning subgraph where the degree sequence of H is $d_1, \ldots, d_n$.

COROLLARY 4. *Let n be an even integer and let $d_1, \ldots, d_n$ be a sequence of positive integers with $m = d_1 + \cdots + d_n \le \left(\frac{n+1}{2}\right)^2$. There exists a graph with degree sequence $\frac{n}{2} - d_1, \ldots, \frac{n}{2} - d_n$ $\left[\text{respectively, } \frac{n+2}{2} - d_1, \ldots, \frac{n+2}{2} - d_n\right]$ if and only if m is even and $\frac{n}{2} - d_i \ge 0$ $(i = 1, \ldots, n)$ $\left[\text{respectively, } \frac{n+2}{2} - d_i \ge 0\ (i = 1, \ldots, n)\right]$.*

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Proof. The quadratic polynomial $x^2-x(n+1)+m$ has roots

$$r_1=\frac{n+1+\sqrt{(n+1)^2-4m}}{2}, \qquad r_2=\frac{n+1-\sqrt{(n+1)^2-4m}}{2},$$

which are real since $m\le\left(\frac{n+1}{2}\right)^2$. Since n is even, $(n+1)^2-4m\ge 1$, and hence $r_1\ge\frac{n+2}{2}\ge\frac{n}{2}\ge r_2$. The corollary now follows by applying the theorem with $\Delta=\frac{n}{2}$ and $\Delta=\frac{n+2}{2}$.

Let $d_1,\ldots,d_n$ be a sequence of positive integers where n is an even integer. Let $m=d_1+\cdots+d_n\le\left(\frac{n+1}{2}\right)^2$. Then Corollary 4 has the following consequences. If m is odd, then $\Delta(d_1,\ldots,d_n)=\infty$. If m is even, then $\Delta(d_1,\ldots,d_n)\le\frac{n}{2}$ if $d_i\le\frac{n}{2}$ $(i=1,\ldots,n)$ while $\Delta(d_1,\ldots,d_n)\le\frac{n+2}{2}$ if $d_i\le\frac{n+2}{2}$ $(i=1,\ldots,n)$.

Corollary 5. *Let n be an odd integer and let $d_1,\ldots,d_n$ be a sequence of positive integers with $m=d_1+\cdots+d_n\le\left(\frac{n+1}{2}\right)^2$.*

(a) *Let $m=\left(\frac{n+1}{2}\right)^2$. There exists a graph with degree sequence $\frac{n+1}{2}-d_1,\ldots,\frac{n+1}{2}-d_n$ if and only if $\frac{n+1}{2}-d_i\ge 0$ $(i=1,\ldots,n)$.*

(b) *Let m be odd with $m<\left(\frac{n+1}{2}\right)^2$. There exists a graph with degree sequence $\frac{n+1}{2}-d_1,\ldots,\frac{n+1}{2}-d_n$ if and only if $n\equiv 1 \pmod 4$ and $\frac{n+1}{2}-d_i\ge 0$ $(i=1,\ldots,n)$. There exists a graph with degree sequence $\frac{n-1}{2}-d_i,\ldots,\frac{n-1}{2}-d_n$ $\left[\text{respectively, } \frac{n+3}{2}-d_1,\ldots,\frac{n+3}{2}-d_n\right]$ if and only if $n\equiv 3 \pmod 4$ and $\frac{n-1}{2}-d_i\ge 0$ $(i=1,\ldots,n)$ $\left[\text{respectively, } \frac{n+3}{2}-d_i\ge 0\ (i=1,\ldots,n)\right]$.*

(c) *Let m be even with $m<\left(\frac{n+1}{2}\right)^2$. There exists a graph with degree sequence $\frac{n+1}{2}-d_1,\ldots,\frac{n+1}{2}-d_n$ if and only if $n\equiv 3 \pmod 4$ and $\frac{n+1}{2}-d_i\ge 0$ $(i=1,\ldots,n)$. There exists a graph with degree sequence*

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$\frac{n-1}{2}-d_1,\ldots,\frac{n-1}{2}-d_n$ $\left[\textit{respectively } \frac{n+3}{2}-d_1,\ldots,\frac{n+3}{2}-d_n\right]$ *if and only if* $n\equiv 1 \pmod 4$ *and* $\frac{n-1}{2}-d_i\geq 0$ $(i=1,\ldots,n)$ $\left[\textit{respectively, } \frac{n+3}{2}-d_i\geq 0 \ (i=1,\ldots,n)\right]$.

Proof. Let r_1 and r_2 be defined as in the proof of Corollary 4.

(a) In this case $r_1=r_2=\frac{n+1}{2}$ and the conclusion follows by applying the theorem with $\Delta=\frac{n+1}{2}$.

(b) Since $m<\left(\frac{n+1}{2}\right)^2$ and n is odd, it follows that $(n+1)^2-4m\geq 4$ and hence that $r_1\geq\frac{n+3}{2}$, $r_2\leq\frac{n-1}{2}$. Thus $\Delta=\frac{n-1}{2},\frac{n+1}{2},\frac{n+3}{2}$ all satisfy (6). Then conclusions now follow from the theorem using the assumptions that n and m are both odd.

(c) Again $\Delta=\frac{n-1}{2},\frac{n+1}{2},\frac{n+3}{2}$ all satisfy (6) and the conclusions follow from the theorem using the assumptions that n is odd and m is even.

Let $d_1,\ldots,d_n$ be a sequence of positive integers where n is an odd integer. Let $m=d_1+\cdots+d_n\leq\left(\frac{n+1}{2}\right)^2$. From corollary 5 we obtain the following. If $m=\left(\frac{n+1}{2}\right)^2$, then $\Delta(d_1,\ldots,d_n)\leq\frac{n+1}{2}$ if $d_i\leq\frac{n+1}{2}$ $(i=1,\ldots,n)$. If m is odd and $m<\left(\frac{n+1}{2}\right)^2$, then $\Delta(d_1,\ldots,d_n)\leq\frac{n+1}{2}$ if $d_i\leq\frac{n+1}{2}$ $(i=1,\ldots,n)$ and $n\equiv 1 \pmod 4$ while $\Delta(d_1,\ldots,d_n)\leq\frac{n-1}{2}$ $\left(\text{respectively, } \frac{n+3}{2}\right)$ if $d_i\leq\frac{n-1}{2}$ $\left(\text{respectively, } \frac{n+3}{2}\right)$ $(i=1,\ldots,n)$ and $n\equiv 3 \pmod 4$. If m is even and $m<\frac{n+1}{2}$, then $\Delta(d_1,\ldots,d_n)\leq\frac{n+1}{2}$ if $d_i\leq\frac{n+1}{2}$ $(i=1,\ldots,n)$ and $n\equiv 3 \pmod 4$ while $\Delta(d_1,\ldots,d_n)\leq\frac{n-1}{2}$ $\left(\text{respectively, } \frac{n+3}{2}\right)$ if $d_i\leq\frac{n-1}{2}$ $\left(\text{respectively, } \frac{n+3}{2}\right)$ $(i=1,\ldots,n)$ and $n\equiv 1 \pmod 4$.

For additional results on degree sequences of graphs, see Chapter 6 of [1] and the references given there.

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