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NAKAYAMA AUTOMORPHISMS OF FROBENIUS CELLULAR ALGEBRAS

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Abstract

Let A be a finite-dimensional Frobenius cellular algebra with cell datum (Λ, M, C, i). Take a nondegenerate bilinear form f on A. In this paper, we study the relationship among i, f and a certain Nakayama automorphism α. In particular, we prove that the matrix associated with α with respect to the cellular basis is uni-triangular under a certain condition.

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1. Introduction and preliminaries

Cellular algebras were introduced by Graham and Lehrer [2] in 1996. The classical examples of cellular algebras include Hecke algebras of finite type [1], Ariki–Koike algebras, Brauer algebras [2], partition algebras [7], and Birman–Wenzl algebras [8].

It is well known that each symmetric algebra is Frobenius. However, a Frobenius algebra need not be symmetric, even in the cellular case. We refer the reader to [5] for related counterexamples. It is natural to consider how far a Frobenius cellular algebra is from being symmetric. A Frobenius algebra A is symmetric if and only if a certain Nakayama automorphism is an identical mapping. This motivates us to study the so-called Nakayama automorphism. In this paper, we will describe the form of a certain Nakayama automorphism in the cellular background.

More precisely, the main results of this note are as follows. If A is a finite-dimensional Frobenius cellular algebra with cell datum (Λ, M, C, i) and f is a nondegenerate bilinear form on A, then the fact that one of the following three statements holds implies that the others are equivalent: (1) both the left and right dual bases of the cellular basis are cellular; (2) A is symmetric; (3) $f(a, 1) = f(i(a), 1)$ for all $a \in A$. In particular, we prove that the matrix associated with a certain Nakayama automorphism with respect to the cellular basis is uni-triangular when (1) holds.

Let us state some basic facts concerning Frobenius algebras which will be used in later proofs. Let K be a field and let A be a finite-dimensional K-algebra. Suppose that there exists a K-bilinear form $f : A \times A \to K$. We say that f is nondegenerate if the determinant of the matrix $(f(a_i, a_j))_{a_i,a_j \in B}$ is not zero for some basis B of A. We call f associative if $f(ab, c) = f(a, bc)$ for all $a, b, c \in A$.

Definition 1.1. A K-algebra A is called Frobenius if there is a nondegenerate associative bilinear form f on A. We call A a symmetric algebra if, in addition, f is symmetric, that is, $f(a, b) = f(b, a)$ for all $a, b \in A$.

For Frobenius algebras, Holm and Zimmermann proved the following lemma.

Lemma 1.2 [4, Lemma 2.7]. *Let A be a finite-dimensional Frobenius algebra. Then an automorphism α of A is a Nakayama automorphism if and only if*

$$f(a, b) = f(\alpha(b), a).$$

Let A be a Frobenius algebra with a basis $B = \{a_i \mid i = 1, \ldots, n\}$. Let us take a nondegenerate associative bilinear form f. Define a K-linear map $\tau : A \to K$ by

$$\tau(a) = f(a, 1).$$

We call $d = \{d_i \mid i = 1, \ldots, n\}$ the right dual basis of B; it is uniquely determined by the requirement that $\tau(a_i d_j) = \delta_{ij}$ for all $i, j = 1, \ldots, n$. Similarly, the left dual basis $D = \{D_i \mid i = i, \ldots, n\}$ is determined by the requirement that $\tau(D_j a_i) = \delta_{ij}$. Define a K-linear map $\alpha : A \to A$ by

$$\alpha(d_i) = D_i.$$

It follows from Lemma 1.2 that α is a Nakayama automorphism of A. If A is a symmetric algebra, then α is the identity map and the right dual basis coincides with the left dual basis.

For any $i, j, k \in 1, 2, \ldots, n$, let us write

$$a_i a_j = \sum_k r_{ijk} a_k,$$

where $r_{ijk} \in K$. Fixing a τ for A, we have the following lemma about structure constants r_{ijk}.

Lemma 1.3. *Let A be a Frobenius algebra with a basis B and dual bases d and D. Then the following hold.*

(1) $a_i d_j = \sum_k r_{kij} d_k$.
(2) $D_i a_j = \sum_k r_{jki} D_k$.

Proof. (1) Suppose that $a_i d_j = \sum_k r_k d_k$. Left multiplying by a_{k_0} on both sides of the equation and applying τ, where $k_0 \in 1, \ldots, n$, we get $\tau(a_{k_0} a_i d_j) = r_{k_0}$. Then $r_{k_0} = r_{k_0,i,j}$.

(2) is proved similarly. □

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2. Frobenius cellular algebras

First, let us recall the definition of a cellular algebra given in [2] by Graham and Lehrer.

Definition 2.1 [2]. Let R be a commutative ring with identity. An associative unital R-algebra is called a cellular algebra with cell datum (Λ, M, C, i) if the following conditions are satisfied.

(C1) The finite set Λ is a poset. Associated with each $\lambda \in \Lambda$, there is a finite set $M(\lambda)$. The algebra A has an R-basis $\{C^{\lambda}_{S,T} \mid S, T \in M(\lambda), \lambda \in \Lambda\}$.

(C2) The map i is an R-linear anti-automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) If $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, then for any element $a \in A$,

$$aC^{\lambda}_{S,T} \equiv \sum_{S' \in M(\lambda)} r_a(S', S)C^{\lambda}_{S',T} \pmod{A(< \lambda)},$$

where $r_a(S', S) \in R$ is independent of T and where $A(<\lambda)$ is the R-submodule of A generated by $\{C^{\mu}_{S'',T''} \mid \mu < \lambda, S'', T'' \in M(\mu)\}$.

If we apply i to the equation in (C3),

$$\text{(C3}'\text{)} \qquad C^{\lambda}_{T,S}\, i(a) \equiv \sum_{S' \in M(\lambda)} r_a(S', S)C^{\lambda}_{T,S'} \pmod{A(< \lambda)}.$$

Let A be a finite-dimensional Frobenius cellular K-algebra with cell datum (Λ, M, C, i) and a nondegenerate associative bilinear form f. Denote the right dual basis by $d = \{d^{\lambda}_{S,T} \mid S, T \in M(\lambda), \lambda \in \Lambda\}$, which satisfies

$$\tau(C^{\lambda}_{S,T} d^{\mu}_{U,V}) = \delta_{\lambda\mu}\delta_{S,V}\delta_{T,U}.$$

Denote the left dual basis by $D = \{D^{\lambda}_{S,T} \mid S, T \in M(\lambda), \lambda \in \Lambda\}$, which satisfies

$$\tau(D^{\mu}_{U,V} C^{\lambda}_{S,T}) = \delta_{\lambda\mu}\delta_{S,V}\delta_{T,U}.$$

For $\mu \in \Lambda$, let $A_d(>\mu)$ be the R-submodule of A generated by

$$\{d^{\eta}_{P,Q} \mid P, Q \in M(\eta), \mu < \eta\},$$

and let $A_D(>\mu)$ be the R-submodule of A generated by

$$\{D^{\eta}_{P,Q} \mid P, Q \in M(\eta), \mu < \eta\}.$$

For $\lambda, \mu \in \Lambda$, $S, T \in M(\lambda)$, $U, V \in M(\mu)$, write

$$C^{\lambda}_{S,T} C^{\mu}_{U,V} = \sum_{\epsilon \in \Lambda, X, Y \in M(\epsilon)} r_{(S,T,\lambda),(U,V,\mu),(X,Y,\epsilon)} C^{\epsilon}_{X,Y}.$$

The following lemma is a corollary of Lemma 1.3 and Definition 2.1. It plays an important role in this note.

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Lemma 2.2. *Let A be a finite-dimensional Frobenius cellular algebra with cell datum (Λ, M, C, i). Let d be the right dual basis and D the left dual basis determined by a given τ. Then for arbitrary $\lambda, \mu \in \Lambda$ and $S, T, P, Q \in M(\lambda)$, $U, V \in M(\mu)$, the following equations hold.*

(1) $$D^{\mu}_{U,V} C^{\lambda}_{S,T} = \sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} r_{(S,T,\lambda),(Y,X\epsilon),(V,U,\mu)} D^{\epsilon}_{X,Y}.$$

(2) $$C^{\lambda}_{S,T} d^{\mu}_{U,V} = \sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} r_{(Y,X,\epsilon),(S,T,\lambda),(V,U,\mu)} d^{\epsilon}_{X,Y}.$$

(3) $$C^{\lambda}_{S,T} d^{\lambda}_{P,Q} = 0 \quad \text{if } T \neq P.$$

(4) $$D^{\lambda}_{P,Q} C^{\lambda}_{S,T} = 0 \quad \text{if } Q \neq S.$$

(5) $$C^{\lambda}_{S,T} d^{\mu}_{U,V} = 0 \quad \text{if } \mu \nleq \lambda.$$

(6) $$D^{\mu}_{U,V} C^{\lambda}_{S,T} = 0 \quad \text{if } \mu \nleq \lambda.$$

Proof. Clearly, (1), (2) are corollaries of Lemma 1.3; (3)–(6) are corollaries of (1) and (2). □

Now we are ready to study dual bases of the cellular basis of a Frobenius cellular algebra.

Lemma 2.3. *Let A be a Frobenius cellular algebra. For $\mu \in \Lambda$ and $U, V \in M(\mu)$ and an element $a \in A$,*

(1) $$a d^{\mu}_{U,V} \equiv \sum_{U'\in M(\mu)} r_{i(a)}(U, U') d^{\mu}_{U',V} \pmod{A_d(>\mu)}.$$

(2) $$d^{\mu}_{U,V} a \equiv \sum_{V'\in M(\mu)} r_{a}(V, V') d^{\mu}_{U,V'} \pmod{A_d(>\mu)}.$$

(3) $$a D^{\mu}_{U,V} \equiv \sum_{U'\in M(\mu)} r_{i(a)}(U, U') D^{\mu}_{U',V} \pmod{A_D(>\mu)}.$$

(4) $$D^{\mu}_{U,V} a \equiv \sum_{V'\in M(\mu)} r_{a}(V, V') D^{\mu}_{U,V'} \pmod{A_D(>\mu)}.$$

Proof. (1) For arbitrary $C^{\lambda}_{S,T}$, it follows from (2) of Lemma 2.2 that

$$C^{\lambda}_{S,T} d^{\mu}_{U,V} = \sum_{\epsilon\in\Lambda, X,Y\in M(\epsilon)} r_{(Y,X,\epsilon),(S,T,\lambda),(V,U,\mu)} d^{\epsilon}_{X,Y}.$$

By (C3) of Definition 2.1, if $\epsilon < \mu$, then $r_{(Y,X,\epsilon),(S,T,\lambda),(V,U,\mu)} = 0$. Therefore,

$$C^{\lambda}_{S,T} d^{\mu}_{U,V} \equiv \sum_{X,Y\in M(\mu)} r_{(Y,X,\mu),(S,T,\lambda),(V,U,\mu)} d^{\mu}_{X,Y} \pmod{A_d(>\mu)}.$$

By (C3′) of Definition 2.1, if $Y \neq V$, then $r_{(Y,X,\mu),(S,T,\lambda),(V,U,\mu)} = 0$. So

$$C^{\lambda}_{S,T} d^{\mu}_{U,V} \equiv \sum_{X\in M(\mu)} r_{(V,X,\mu),(S,T,\lambda),(V,U,\mu)} d^{\mu}_{X,V} \pmod{A_D(>\mu)}.$$

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Clearly, for arbitrary $X \in M(\mu)$,

$$r_{(V,X,\mu),(S,T,\lambda),(V,U,\mu)} = r_{C^{\lambda}_{T,S}}(U, X),$$

which is independent of V. Since $C^{\lambda}_{S,T}$ is arbitrary,

$$ad^{\mu}_{U,V} \equiv \sum_{U' \in M(\mu)} r_{i(a)}(U, U')d^{\mu}_{U',V} \pmod{A_d(> \mu)}$$

for all $a \in A$. By Definition 2.1, $r_{i(a)}(U, U')$ is independent of V.

(4) is proved similarly.

Applying α on both sides of (1), we get (3). Similarly, applying α^{-1} on both sides of (4), we get (2). □

Lemma 2.3 implies that dual bases of the cellular basis satisfy Definition 2.1 (C3) with respect to the opposite order on Λ. However, (C2) does not hold, that is, either $i(d^{\lambda}_{S,T}) = d^{\lambda}_{T,S}$ or $i(D^{\lambda}_{S,T}) = D^{\lambda}_{T,S}$ need not be true in general. We give an example which was constructed by Koenig and Xi in [5].

Example 2.4. Let K be a field. Let us take $\lambda \in K$ with $\lambda \neq 0$ and $\lambda \neq 1$. Let

$$A = K\langle a, b, c, d\rangle / I,$$

where I is generated by

$$a^2, b^2, c^2, d^2, ab, ac, ba, bd, ca, cd, db, dc, cb - \lambda bc, ad - bc, da - bc.$$

If we define τ by $\tau(1) = \tau(a) = \tau(b) = \tau(c) = \tau(d) = 0$ and $\tau(bc) = 1$ and define an involution i on A to be fixing a and d, but interchanging b and c, then A is a Frobenius cellular algebra with a cellular basis

$$bc; \quad \begin{matrix} a & b \\ c & d \end{matrix}; \quad 1.$$

The right dual basis is

$$1; \quad \begin{matrix} d & c \\ b/\lambda & a \end{matrix}; \quad bc.$$

The left dual basis is

$$1; \quad \begin{matrix} d & c/\lambda \\ b & a \end{matrix}; \quad bc.$$

Clearly, $i(c) = b \neq b/\lambda$, $i(b) = c \neq c/\lambda$.

Example 2.4 implies that for a Frobenius cellular algebra, the dual bases of a cellular basis need not be cellular again. The following result reveals a relation among α, i and τ.

Theorem 2.5. *Let A be a Frobenius cellular algebra with cell datum (Λ, M, C, i). If one of the following three statements holds, then the other two are equivalent.*

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(1) $i(d^{\lambda}_{S,T}) = d^{\lambda}_{T,S}$ *and* $i(D^{\lambda}_{S,T}) = D^{\lambda}_{T,S}$ *for arbitrary* $\lambda \in \Lambda$ *and* $S, T \in M(\lambda)$.
(2) $\alpha = id$, *that is, A is a symmetric algebra.*
(3) $\tau(a) = \tau(i(a))$ *for all* $a \in A$.

Proof. It is enough to prove that if two of the statements hold then the other one is true.

Suppose that (1) and (2) hold. Since A is symmetric, the right dual basis is equal to the left dual basis. Denote the dual basis by $\{D^{\lambda}_{S,T} \mid \lambda \in \Lambda, S, T \in M(\lambda)\}$. Let

$$1 = \sum_{\varepsilon\in\Lambda, X,Y\in M(\varepsilon)} r_{X,Y,\varepsilon} D^{\varepsilon}_{X,Y}.$$

Then

$$\tau(C^{\lambda}_{S,T}) = \tau\Big(C^{\lambda}_{S,T} \sum_{\varepsilon\in\Lambda, X,Y\in M(\varepsilon)} r_{X,Y,\varepsilon} D^{\varepsilon}_{X,Y}\Big) = r_{T,S,\lambda}.$$

On the other hand, it follows from (1) that

$$1 = i(1) = \sum_{\varepsilon\in\Lambda, X,Y\in M(\varepsilon)} r_{X,Y,\varepsilon}\, i(D^{\varepsilon}_{X,Y}) = \sum_{\varepsilon\in\Lambda, X,Y\in M(\varepsilon)} r_{X,Y,\varepsilon} D^{\varepsilon}_{Y,X}.$$

Then

$$\tau(C^{\lambda}_{T,S}) = \tau\Big(\sum_{\varepsilon\in\Lambda, X,Y\in M(\varepsilon)} r_{X,Y,\varepsilon} D^{\varepsilon}_{Y,X} C^{\lambda}_{T,S}\Big) = r_{T,S,\lambda}.$$

Now we obtain $\tau(C^{\lambda}_{S,T}) = \tau(C^{\lambda}_{T,S})$ for any $\lambda \in \Lambda$ and $S, T \in M(\lambda)$. Hence $\tau(a) = \tau(i(a))$ for all $a \in A$.

Assume that (1) and (3) hold. Then

$$\tau(d^{\mu}_{U,V} C^{\lambda}_{T,S}) = \tau(C^{\lambda}_{S,T} d^{\mu}_{V,U}) = \delta_{\lambda\mu}\delta_{TV}\delta_{SU}.$$

This implies that $d^{\mu}_{U,V} = D^{\mu}_{U,V}$ for any $\mu \in \Lambda$ and $U, V \in M(\mu)$ by the definition of dual bases. Hence the algebra A is symmetric.

Assume that (2) and (3) hold. Since A is symmetric, we can denote the dual basis by $\{D^{\lambda}_{S,T} \mid \lambda \in \Lambda, S, T \in M(\lambda)\}$. It follows from (3) and $\tau(C^{\lambda}_{S,T} D^{\lambda}_{T,S}) = 1$ that

$$\tau(i(D^{\lambda}_{T,S}) C^{\lambda}_{T,S}) = 1.$$

On the other hand, $\tau(C^{\mu}_{U,V} D^{\lambda}_{T,S}) = 0$ if $(U, V, \mu) \neq (S, T, \lambda)$. This implies that

$$\tau(i(D^{\lambda}_{T,S}) C^{\lambda}_{V,U}) = 0$$

if $(U, V, \mu) \neq (S, T, \lambda)$. Note that the dual basis is uniquely determined by τ. Then

$$i(D^{\lambda}_{S,T}) = D^{\lambda}_{T,S}.$$

The proof is complete. □

Remark 2.6. Graham in [3] showed that for a symmetric cellular algebra A with cell datum (Λ, M, C, i), the dual basis is again cellular with respect to the opposite order on Λ if $\tau(a) = \tau(i(a))$ for all $a \in A$.

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3. Nakayama automorphisms of Frobenius cellular algebras

The cellularity of dual bases and the symmetry of the algebra are connected by Theorem 2.5. In this section, we describe this relation by investigating the Nakayama automorphism α defined by $\alpha(d^{\lambda}_{S,T}) = D^{\lambda}_{S,T}$. For convenience, we extend the poset Λ to a totally ordered set. Then the main result of this section is as follows.

THEOREM 3.1. *Let A be a finite-dimensional Frobenius cellular K-algebra with a cell datum (Λ, M, C, i). Suppose that both the dual bases are cellular with respect to the opposite order on Λ. Then*

$$\alpha(C^{\lambda}_{S,T}) \equiv C^{\lambda}_{S,T} \pmod{A(<\lambda)}.$$

PROOF. Suppose that

$$\alpha(C^{\lambda}_{S,T}) = \sum_{\varepsilon\in\Lambda, X,Y\in M(\varepsilon)} r_{X,Y,\varepsilon} C^{\varepsilon}_{X,Y}.$$

Let us first prove some claims.

Claim 1. Let $\varepsilon > \lambda$ and $X, Y \in M(\varepsilon)$. Then $r_{X,Y,\varepsilon} = 0$.

Suppose that there exist $\mu > \lambda$ and $U, V \in M(\lambda)$ such that $r_{U,V,\mu} \neq 0$. Without loss of generality, assume that $\mu \in \Lambda$ is the maximal element satisfying this condition, that is, if $\mu < \varepsilon$, then $r_{X,Y,\varepsilon} = 0$ for all $X, Y \in M(\varepsilon)$. Then, from Lemma 2.2,

$$\alpha(C^{\lambda}_{S,T})D^{\mu}_{V,U} = \alpha(C^{\lambda}_{S,T}d^{\mu}_{V,U}) = 0.$$

On the other hand, it follows from $i(D^{\mu}_{V,U}) = D^{\mu}_{U,V}$ that

$$\tau(i(\alpha(C^{\lambda}_{S,T})D^{\mu}_{V,U})) = \tau(r_{U,V,\mu}D^{\mu}_{U,V}C^{\mu}_{V,U}) = r_{U,V,\mu} \neq 0.$$

This is a contradiction and the claim is proved.

This claim implies that $\alpha(A(<\lambda)) = A(<\lambda)$. But the form

$$\alpha^{-1}(A(<\lambda)) = A(<\lambda)$$

is more useful to us.

Claim 2. $r_{P,Q,\lambda} = 0$ if $Q \neq T$.

Suppose that $r_{P,Q,\lambda} \neq 0$ and $Q \neq T$. Then, by Lemma 2.2, $Q \neq T$ implies that

$$\alpha(C^{\lambda}_{S,T})D^{\lambda}_{Q,P} = \alpha(C^{\lambda}_{S,T}d^{\lambda}_{Q,P}) = 0.$$

However, still by Lemma 2.2,

$$\begin{aligned}
\tau(i(\alpha(C^{\lambda}_{S,T})D^{\lambda}_{Q,P})) &= \tau(D^{\lambda}_{P,Q}i(\alpha(C^{\lambda}_{S,T}))) \\
&= \tau\Big(D^{\lambda}_{P,Q} \sum_{X,Y\in M(\lambda)} r_{X,Y,\lambda}C^{\lambda}_{Y,X}\Big) \\
&= \tau\Big(\sum_{X,Y\in M(\lambda)} r_{X,Y,\lambda}D^{\lambda}_{P,Q}C^{\lambda}_{Y,X}\Big) = r_{P,Q,\lambda} \neq 0.
\end{aligned}$$

Hence $i(\alpha(C^{\lambda}_{S,T})D^{\lambda}_{Q,P}) \neq 0$, or $\alpha(C^{\lambda}_{S,T})D^{\lambda}_{Q,P} \neq 0$. This is a contradiction.

Claim 3. $r_{X,T,\lambda} = 0$ if $X \neq S$.

By Claims 1 and 2,

$$\alpha(C^{\lambda}_{S,T}) \equiv \sum_{X\in M(\lambda)} r_{X,T,\lambda} C^{\lambda}_{X,T} \pmod{(A < \lambda)}.$$

Then it follows from $\alpha^{-1}(A(<\lambda)) = A(<\lambda)$ that

$$C^{\lambda}_{S,T} \equiv \sum_{X\in M(\lambda)} r_{X,T,\lambda}\alpha^{-1}(C^{\lambda}_{X,T}) \pmod{(A < \lambda)}.$$

Left multiplying by $d^{\lambda}_{T,V}$ on both sides, we have from Lemma 2.2 that

$$\begin{aligned} d^{\lambda}_{T,V} C^{\lambda}_{S,T} &= \sum_{X\in M(\lambda)} r_{X,T,\lambda} d^{\lambda}_{T,V}\alpha^{-1}(C^{\lambda}_{X,T}) \\ &= \sum_{X\in M(\lambda)} r_{X,T,\lambda}\alpha^{-1}(D^{\lambda}_{T,V} C^{\lambda}_{X,T}) \\ &= r_{V,T,\lambda}\alpha^{-1}(D^{\lambda}_{T,V} C^{\lambda}_{V,T}). \end{aligned}$$

On the other hand, by Lemma 2.2, $i(d^{\lambda}_{T,V} C^{\lambda}_{S,T}) = C^{\lambda}_{T,S} d^{\lambda}_{V,T} = 0$ if $V \neq S$. Thus $d^{\lambda}_{T,V} C^{\lambda}_{S,T} = 0$ if $V \neq S$. Moreover, the facts that $D^{\lambda}_{T,V} C^{\lambda}_{V,T} \neq 0$ and α is an automorphism of A imply that $\alpha^{-1}(D^{\lambda}_{T,V} C^{\lambda}_{V,T}) \neq 0$. Now we get $r_{V,T\lambda} = 0$ if $V \neq S$. Then the claim is proved.

We have proved that the Nakayama automorphism α is of the form

$$\alpha(C^{\lambda}_{S,T}) \equiv r_{S,T,\lambda} C^{\lambda}_{S,T} \pmod{(A < \lambda)}.$$

It is sufficient to show that $r_{S,T,\lambda} = 1$. We claim that $C^{\eta}_{P,Q} D^{\lambda}_{T,S} = 0$ if $\eta < \lambda$. In fact, if $C^{\eta}_{P,Q} D^{\lambda}_{T,S} \neq 0$, then $i(C^{\eta}_{P,Q} D^{\lambda}_{T,S}) \neq 0$, that is, $D^{\lambda}_{S,T} C^{\eta}_{Q,P} \neq 0$. But Lemma 2.2 tells us that this is impossible. Now it follows from this claim and Lemma 2.2 that

$$\begin{aligned} r_{S,T,\lambda} C^{\lambda}_{S,T} D^{\lambda}_{T,S} &= \alpha(C^{\lambda}_{S,T}) D^{\lambda}_{T,S} \\ &= \alpha(C^{\lambda}_{S,T} d^{\lambda}_{T,S}) \\ &= \alpha\Big(\sum_{\varepsilon\in\Lambda, X,Y\in M(\varepsilon)} r_{(Y,X,\varepsilon),(S,T,\lambda),(S,T,\lambda)} d^{\varepsilon}_{X,Y}\Big) \\ &= \sum_{\varepsilon\in\Lambda, X,Y\in M(\varepsilon)} r_{(Y,X,\varepsilon),(S,T,\lambda),(S,T,\lambda)} D^{\varepsilon}_{X,Y}. \end{aligned}$$

On the other hand,

$$\begin{aligned} r_{S,T,\lambda} C^{\lambda}_{S,T} D^{\lambda}_{T,S} &= r_{S,T,\lambda} i(D^{\lambda}_{S,T} C^{\lambda}_{T,S}) \\ &= i\Big(\sum_{\varepsilon\in\Lambda, X,Y\in M(\varepsilon)} r_{S,T,\lambda} r_{(T,S,\lambda),(Y,X,\varepsilon),(T,S,\lambda)} D^{\varepsilon}_{X,Y}\Big) \end{aligned}$$

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$$= \sum_{\varepsilon\in\Lambda, X,Y\in M(\varepsilon)} r_{S,T,\lambda} r_{(T,S,\lambda),(Y,X,\varepsilon),(T,S,\lambda)} D^{\varepsilon}_{Y,X}$$
$$= \sum_{\varepsilon\in\Lambda, X,Y\in M(\varepsilon)} r_{S,T,\lambda} r_{(T,S,\lambda),(X,Y,\varepsilon),(T,S,\lambda)} D^{\varepsilon}_{X,Y}.$$

Note that α is an automorphism of A; then clearly $r_{S,T,\lambda} \neq 0$. Then it follows from $C^{\lambda}_{S,T} D^{\lambda}_{T,S} \neq 0$ that $r_{S,T,\lambda} C^{\lambda}_{S,T} D^{\lambda}_{T,S} \neq 0$. This implies that there exist $\mu \in \Lambda$ and $U, V \in M(\mu)$ such that $r_{(T,S,\lambda),(U,V,\mu),(T,S,\lambda)} \neq 0$ and

$$r_{S,T,\lambda} r_{(T,S,\lambda),(U,V,\mu),(T,S,\lambda)} = r_{(V,U,\mu),(S,T,\lambda),(S,T,\lambda)}.$$

From (C3)′ of Definition 2.1,

$$r_{(T,S,\lambda),(U,V,\mu),(T,S,\lambda)} = r_{(V,U,\mu),(S,T,\lambda),(S,T,\lambda)}.$$

This implies that $r_{S,T,\lambda} = 1$ and we complete the proof. □

Remark 3.2. In fact, the condition $i(d^{\lambda}_{S,T}) = d^{\lambda}_{T,S}$ could be weakened to

$$i(d^{\lambda}_{S,T}) \equiv d^{\lambda}_{T,S} \pmod{(A_d(>\lambda))}.$$

We omit the details here.

In [6], the author studied the centres of symmetric cellular algebras. We generalise Theorem 1.1 of [6] to the Frobenius case in the present paper.

Let A be a Frobenius cellular algebra. The Higman ideal of $Z(A)$ is defined by

$$H(A) = \Big\{ \sum_{\lambda\in\Lambda, S,T\in M(\lambda)} C^{\lambda}_{S,T} a D^{\lambda}_{S,T} \mid a \in A \Big\}.$$

For any $\lambda \in \Lambda$ and $T \in M(\lambda)$, set $e_{\lambda} = \sum_{S\in M(\lambda)} C^{\lambda}_{S,T} D^{\lambda}_{T,S}$ and

$$L(A) = \Big\{ \sum_{\lambda\in\Lambda} r_{\lambda} e_{\lambda} \mid r_{\lambda} \in R \Big\}.$$

Then we have the following corollary.

Corollary 3.3. *Let A be a Frobenius cellular algebra with cellular datum* (Λ, M, C, i) *and both the right and left dual bases are cellular. Then $L(A)$ is an ideal of $Z(A)$ containing the Higman ideal.*

Proof. The proof is similar to that in [6]. □

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