RECURSIONS FOR COMPOUND DISTRIBUTIONS*

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1. INTRODUCTION

Various methods for developing recursive formulae for compound distributions have been reported recently by PANJER (1980, including discussion), PANJER (1981), SUNDT and JEWELL (1981) and GERBER (1982) for a class of claim frequency distributions and arbitrary claim amount distributions. The recursions are particularly useful for computational purposes since the number of calculations required to obtain the distribution function of total claims and related values such as net stop-loss premiums may be greatly reduced when compared with the usual method based on convolutions.

In this paper a broader class of claims frequency distributions is considered and methods for developing recursions for the corresponding compound distributions are examined. The methods make use of the Laplace transform of the density of the compound distribution.

2. THE CLAIM FREQUENCY DISTRIBUTION

Consider the class of claim frequency distributions which has the property that successive probabilities may be written as the ratio of two polynomials. For convenience we write the polynomials in terms of descending factorial powers. For obvious reasons, only distributions on the non-negative integers are considered. Hence, successive probabilities of claim frequencies are written as

$$(1)\qquad \begin{aligned}(\alpha_0+\alpha_1 n+\alpha_2 n^{(2)}+\cdots)p_n &= (\beta_0+\beta_1(n-1)\\ &\quad +\beta_2(n-1)^{(2)}+\cdots)p_{n-1}, \qquad n=1,2,3,\ldots,\end{aligned}$$

where

$$n^{(k)}=n(n-1)\cdots(n-k+1).$$

Note that the coefficients are only specified up to a multiplication constant.

The class of probability distribution satisfying (1) is very broad and includes the following distributions:

1. Binomial:

$$p_n=\binom{N}{n}p^n q^{N-n}$$

$$\alpha_0=0,\qquad \alpha_1=q,\qquad \beta_0=N-p,\qquad \beta_1=-p.$$

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2. Poisson:

$$p_n = \frac{e^{-\lambda}\lambda^n}{n!}$$

$$\alpha_0 = 0, \quad \alpha_1 = 1, \quad \beta_0 = \lambda, \quad \beta_1 = 0.$$

3. Negative Binomial:

$$p_n = \binom{\alpha + n - 1}{n} p^n q^\alpha$$

$$\alpha_0 = 0, \quad \alpha_1 = 1, \quad \beta_0 = p\alpha, \quad \beta_1 = p.$$

4. Hypergeometric:

$$p_n = \frac{\binom{N}{n}\binom{M-N}{m-n}}{\binom{M}{n}}$$

$$\alpha_0 = 0, \quad \alpha_1 = M - N - m + 1, \quad \alpha_2 = 1,$$

$$\beta_0 = Nm, \quad \beta_1 = -(N + m - 1), \quad \beta_2 = 1.$$

5. Hyper-Poisson:

$$p_n = C\frac{\theta^n}{(\lambda + n - 1)^{(n)}}$$

$$\alpha_0 = \lambda - 1, \quad \alpha_1 = 1, \quad \beta_0 = \theta, \quad \beta_1 = 0.$$

6. Logarithmic:

$$p_n = \frac{-\theta^{n+1}}{(n+1)ln(1-\theta)}$$

$$\alpha_0 = 1, \quad \alpha_1 = 1, \quad \beta_0 = \theta, \quad \beta_1 = \theta.$$

7. Waring:

$$p_n = \frac{(\lambda - a)(a + n - 1)^{(n)}}{(\lambda + n)^{(n+1)}}$$

$$\alpha_0 = \lambda, \quad \alpha_1 = 1, \quad \beta_0 = a, \quad \beta_1 = 1.$$

8. Polya–Eggenberger (Negative Hypergeometric):

$$p_n = \frac{\binom{-a}{n}\binom{-b}{N-n}}{\binom{-a-b}{N}}$$

$$\alpha_0 = 0, \quad \alpha_1 = N + b - 1, \quad \alpha_2 = -1$$

$$\beta_0 = aN, \quad \beta_1 = N - a - 1, \quad \beta_2 = -1.$$

9. Generalized Waring Distribution:

$$p_n = \frac{\Gamma(a+n)}{n!\Gamma(a)} \frac{\Gamma(b+\alpha)}{\Gamma(b)\Gamma(\alpha)} \frac{\Gamma(a+b)\Gamma(\alpha+n)}{\Gamma(a+b+\alpha+n)}$$

$$\alpha_0 = 0, \qquad \alpha_1 = a+b+\alpha, \qquad \alpha_2 = 1$$

$$\beta_0 = \alpha(a+1), \qquad \beta_1 = a+\alpha-2, \qquad \beta_2 = 1.$$

The binomial, Poisson and negative binomial distributions have a natural appeal for actuaries in connection with contagion models (cf. BUHLMANN 1970). It is well known that the negative binomial also arises as a mixed Poisson distribution with a gamma mixing function.

The Polya-Eggenberger (Negative Hypergeometric) arises as a mixed binomial with the parameter p mixed in accordance with a beta mixing function. This can be seen from the following (cf. SKELLAM (1948)):

$$p_n = \int_0^1 \binom{N}{n} p^n q^{N-n} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1} q^{b-1}\, dp$$

$$= \frac{\binom{a+n-1}{n}\binom{b+N-n-1}{N-n}}{\binom{a+b+N-1}{N}} = \frac{\binom{-a}{n}\binom{-b}{N-n}}{\binom{-a-b}{N}}.$$

The generalized Waring distribution can arise as a mixed Poisson (cf. IRWIN, 1965, 1968, SEAL, 1978 or GOOVAERTS and VAN WOUWE 1981) or more simply as a mixed negative binomial with a beta mixing function. This can be seen from the following:

$$p_n = \int_0^1 \binom{\alpha+n-1}{n} p^n q^\alpha \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1} q^{b-1}\, dp$$

$$= \frac{\Gamma(a+n)}{n!\Gamma(a)} \frac{\Gamma(b+\alpha)}{\Gamma(b)\Gamma(\alpha)} \frac{\Gamma(a+b)\Gamma(\alpha+n)}{\Gamma(a+b+\alpha+n)}.$$

It is easy to construct other members of this family by appropriately choosing the coefficients in the polynomials in (1). Such claim frequency distributions may have finite range or infinite range. One obtains a finite range when *inter alia* the polynomial on the right-hand side of (1) has its smallest positive root at an integer. Many references about the listed distributions can be found in JOHNSON and KOTZ (1969).

3. TRANSFORM RELATIONSHIPS

It is assumed that claim sizes are positive so that the claim size distribution $F(x)$ has support only on the non-negative real axis. It is further assumed that

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claim frequencies and amounts are mutually independent. Under these assumptions the distribution function of aggregate claims is given by

(2) $$G(x)=\sum_{n=0}^{\infty} p_n F^{*n}(x), \qquad x \geq 0.$$

Let $\tilde{f}(t)$ and $\tilde{g}(t)$ denote the Laplace (or Laplace-Stieltjes) transforms

(3) $$\tilde{f}(t)=E_F[e^{-tX}]=\int e^{-tx}\,dF(x)$$

and

(4) $$\tilde{g}(t)=E_G[e^{-tX}]=\int e^{-tx}\,dG(x).$$

Further let $P(s)$ denote the probability generating function of claim frequencies

(5) $$P(s)=E_p[s^N]=\sum_{n=0}^{\infty} p_n s^n.$$

Then it is easily shown that

(6) $$\tilde{g}(t)=P(\tilde{f}(t)).$$

Successive differentiation of (5) results in

(7) $$\begin{aligned} P'(s)&=\sum n s^{n-1} p_n \\ P''(s)&=\sum n^{(2)} s^{n-2} p_n \\ &\vdots \\ P^{[k]}(s)&=\sum n^{(k)} s^{n-k} p_n \end{aligned}$$

where the square subscript brackets indicate derivatives. If the polynomials in (1) are of order k, then

(8) $$\left[\sum_{i=0}^{k} \alpha_i n^{(i)}\right] p_n=\left[\sum_{i=0}^{k} \beta_i (n-1)^{(i)}\right] p_{n-1}, \qquad n=1,2,\ldots.$$

Multiplying the nth equation of (8) by s^n and summing all equation results in

(9) $$\begin{aligned} &\alpha_0(P(s)-p_0)+\alpha_1 sP'(s)+\alpha_2 s^2 P''(s)+\cdots+\alpha_k s^k P^{[k]}(s) \\ &\quad = s[\beta_0 P(s)+\beta_1 sP'(s)+\beta_2 s^2 P''(s)+\cdots+\beta_k s^k P^{[k]}(s)]. \end{aligned}$$

Successive differentiation of (6) results in

(10) $$\begin{aligned} \tilde{g}(t)&=P(\tilde{f}(t)) \\ \tilde{g}'(t)&=\tilde{f}'(t)P'(\tilde{f}(t)) \\ \tilde{g}''(t)&=\tilde{f}''(t)P'(\tilde{f}(t))+[\tilde{f}'(t)]^2 P''(\tilde{f}(t)) \\ \tilde{g}'''(t)&=\tilde{f}'''(t)P'(\tilde{f}(t))+3[\tilde{f}''(t)][\tilde{f}'(t)]P''(\tilde{f}(t))+[\tilde{f}'(t)]^3 P'''(\tilde{f}(t)) \end{aligned}$$

$$\tilde{g}^{[4]}(t)=\tilde{f}^{[4]}(t)P'(\tilde{f}(t))+\{4[\tilde{f}'''(t)][\tilde{f}(t)]+3[\tilde{f}''(t)]^2\}P''(\tilde{f}(t))$$
$$+6[\tilde{f}''(t)][\tilde{f}'(t)]^2P'''(\tilde{f}(t))+[\tilde{f}'(t)]^4P^{[4]}(t).$$

Solving the successive equations for P, P', P'', P''' and $P^{[4]}$ respectively yields (dropping the argument t)

(11) $[\tilde{f}']\cdot P'(f)=\tilde{g}'$

$$[\tilde{f}']^2\cdot P''(f)=\tilde{g}''-\tilde{h}_{21}\tilde{g}'$$
$$[\tilde{f}']^3\cdot P'''(f)=\tilde{g}'''-3\tilde{h}_{21}\tilde{g}''-[\tilde{h}_{31}-3\tilde{h}_{21}^2]\tilde{g}'$$
$$[\tilde{f}']^4\cdot P^{[4]}(f)=\tilde{g}^{[4]}-6\tilde{h}_{21}\tilde{g}'''-[4\tilde{h}_{31}-15\tilde{h}_{21}^2]\tilde{g}''-[\tilde{h}_{41}-10\tilde{h}_{21}\tilde{h}_{31}+15\tilde{h}_{21}^3]\tilde{g}'$$

where

$$\tilde{h}_{ij}(t)=\frac{\tilde{f}^{[i]}(t)}{\tilde{f}^{[j]}(t)}.$$

To obtain the desired relationship of the various transforms, the equations (11) are substituted into (9) with $s=\tilde{f}(t)$. This results in (for $k=4$):

(12)
$$\alpha_0(\tilde{g}-p_0)+\alpha_1\tilde{h}_{01}[\tilde{g}']+\alpha_2\tilde{h}_{01}^2[\tilde{g}''-\tilde{h}_{21}\tilde{g}']$$
$$+\alpha_3\tilde{h}_{01}^3\{\tilde{g}'''-3\tilde{h}_{21}\tilde{g}''-[\tilde{h}_{31}-3\tilde{h}_{21}^2]\tilde{g}'\}+\alpha_4\tilde{h}_{01}^4\{\tilde{g}^{[4]}-6\tilde{h}_{21}\tilde{g}'''$$
$$-[4\tilde{h}_{31}-15\tilde{h}_{21}^2]\tilde{g}''-[\tilde{h}_{41}-10_{21}\tilde{h}_{31}+15\tilde{h}_{21}^3]\tilde{g}'\}$$
$$=\tilde{f}\{\beta_0\tilde{g}+\beta_1\tilde{h}_{01}[\tilde{g}']+\beta_2\tilde{h}_{01}^2[\tilde{g}''-\tilde{h}_{21}\tilde{g}']$$
$$+\text{terms similar to the above in } \beta_3, \beta_4\}.$$

Equation (12) will be used to develop recursions in the next section. The corresponding results for $k=1, 2$ or 3 are obtained by setting the appropriate coefficients α_i and β_i to zero.

4. DEVELOPING RECURSIONS

It is assumed that the claim size distribution $F(x)$, with support on $(0,\infty)$, is of either the continuous type or the discrete type with jumps on the positive integers. In order to distinguish the two cases, the measure μ will refer to Lebesgue or counting measure on $(0,\infty)$ as is done by SUNDT and JEWELL (1981). To set up a recursion, equation (12) (or similar equation for $k<4$) is used. One of $\tilde{g}$, $\tilde{g}'$, $\tilde{g}''$, $\tilde{g}'''$ or $\tilde{g}^{[4]}$ from the left-hand side of (12) is isolated and the resulting equation is inverted. Note that these exists one higher power of $\tilde{f}$ on the right-hand side. Upon inversion, the result may involve an auxiliary function $h(x)$ whose transform $\tilde{h}(t)$ is a function of the transforms $\tilde{h}(t)$ is a function of the transforms $\tilde{h}_{ij}(t)$. The success of any of the resultant recursions depends on the ability to identify these auxiliary functions. Different possible values of k are now considered separately. Special attention is paid to special cases when certain of the coefficients α_i are zero.

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Case I: $k=1$
Equation (12) reduces to

$$\alpha_0(\tilde{g}-p_0)+\alpha_1\tilde{h}_{01}\tilde{g}'=\tilde{f}\{\beta_0\tilde{g}+\beta_1\tilde{h}_{01}\tilde{g}'\}. \tag{13}$$

Special Case I(a): $\alpha_0=0$
Isolating $\tilde{g}'$ from the left-hand side of (13) results in

$$\begin{aligned}\tilde{g}&=\{\beta_0\tilde{f}'\tilde{g}+\beta_1\tilde{f}\tilde{g}'\}/\alpha_1\\ &=\{\beta_0p_0\tilde{f}'+\beta_0\tilde{f}'[\tilde{g}-p_0]+\beta_1\tilde{f}\tilde{g}'\}/\alpha_1\end{aligned} \tag{14}$$

Upon inversion and division by $-x$, one obtains the recursion

$$\begin{aligned}g(x)=\Big\{&\beta_0p_0f(x)+\beta_0\int_{(0,x)}\frac{y}{x}f(y)g(x-y)\,d\mu(y)\\ &+\beta_1\int_{(0,x)}\frac{x-y}{x}f(y)g(x-y)\,d\mu(y)\Big\}\Big/\alpha_1\end{aligned} \tag{15}$$

which reduces to

$$g(x)=\frac{\beta_0p_0}{\alpha_1}f(x)+\int_{(0,x)}\frac{\beta_1x+(\beta_0-\beta_1)y}{\alpha_1x}f(y)g(x-y)\,d\mu(y). \tag{16}$$

This is the recursion obtained by PANJER (1981) and applies to the Poisson, binomial, negative binomial and geometric claim frequency distributions.

Special Case I(b): $\alpha_1=0$
In this case, the claim frequencies satisfy

$$\alpha_0p_n=\{\beta_0+\beta_1(n-1)\}p_{n-1}. \tag{17}$$

If there exists a positive integer N such that $\beta_0+\beta_1N=0$, then the claim frequency distribution has finite range. In this case, (17) leads to

$$p_n=\left(\frac{\beta_0}{\alpha_0}\right)^n\left(n+\frac{\beta_1}{\beta_0}\right)^{(n)}p_0,\qquad n=1,2,\ldots,N. \tag{18}$$

If there exists no such N, the condition that $\sum p_n=1$ implies that $|\beta_0+\beta_1n|<|\alpha_0|$ for sufficiently large n; and so $\beta_1=0$ and the geometric distribution follows. The geometric distribution was considered in Special Case I(a) above. In order to develop a recursion for the distribution (18), equation (12) reduces to

$$\alpha_0(\tilde{g}-p_0)=\tilde{f}\{\beta_0(\tilde{g}-p_0)+\beta_0p_0+\beta_1\tilde{h}_{01}\tilde{g}'\}. \tag{19}$$

Upon division by α_0 and inversion one obtains

$$\begin{aligned}g(x)=&\frac{\beta_0p_0}{\alpha_0}f(x)+\frac{\beta_0}{\alpha_0}\int_{(0,x)}f(y)g(x-y)\,d\mu(y)\\ &-\frac{\beta_1}{\alpha_0}\int_{(0,x)}(x-y)h(y)g(x-y)\,d\mu(y)\end{aligned}$$

where $h(y)$ has transform $\tilde{h}=[\tilde{f}]^2/\tilde{f}'$. The problem of obtaining the auxiliary function $h(y)$ is considered in the next section of this paper.

Special Case I(c): $\alpha_0 \neq 0$, $\alpha_1 \neq 0$

The hyper-Poisson, Waring and logarithmic distributions are examples of this special case. It appears that the most convenient way to obtain a recursion from equation (12) is to solve for $\tilde{g}'$ on the left-hand side. This results in

$$\text{(20)} \qquad \tilde{g}' = \{\beta_0 p_0 \tilde{f}' + \beta_0 \tilde{f}'[\tilde{g} - p_0] + \beta_1 \tilde{f}\tilde{g}' - \alpha_0 \tilde{h}_{10}[\tilde{g} - p_0]\}/\alpha_1.$$

Observe that $\tilde{h}_{10} = \tilde{f}'/\tilde{f} = \tilde{h}'$ where $\tilde{h}(t) = \log \tilde{f}(t)$. Making this substitution into (20) and inverting the resultant equation yields

$$\text{(21)} \qquad g(x) = \frac{\beta_0 p_0}{\alpha_1} f(x) + \int_{(0,x)} \frac{\beta_0 x + (\beta_0 - \beta_1) y}{\alpha_1 x} f(y) g(x-y)\, d\mu(y)$$
$$- \frac{\alpha_0}{\alpha_1} \int_{(0,x)} \frac{y}{x} h(y) g(x-y)\, d\mu(y)$$

where the auxiliary function $h(y)$ has transform

$$\text{(22)} \qquad \tilde{h}(t) = \log \tilde{f}(t).$$

Because of possible difficulties in obtaining $h(x)$ defined by (22), it may be more convenient to replace the last term inside the curly brackets of (20) by the quantity

$$\text{(23)} \qquad -[\tilde{h}_{10} - h_{10}(0)][\tilde{g} - p_0] - h_{10}(0)[\tilde{g} - p_0]$$

and the last term in (21) by the quantity

$$+\frac{\alpha_0}{\alpha_1 x} \int_{(0,\infty)} h_{10}(y) g(x-y)\, d\mu(y) + \frac{\alpha_0}{\alpha_1 x} h_{10}(0) g(x).$$

Upon making this substitution and solving for $g(x)$, the following recursion emerges:

$$\text{(24)} \qquad g(x) = \left\{\beta_0 p_0 f(x) + \int_{(0,x)} [\beta_1 x + (\beta_0 - \beta_1) y] f(y) g(x-y) d\mu(y)\right.$$
$$\left. + \int_{(0,x)} h_{10}(y) g(x-y) d\mu(y)\right\} \Big/ \{\alpha_1 x - \alpha_0 h_{10}(0)\}.$$

Case II: $k = 2$

The hypergeometric, Polya–Eggenberger (negative hypergeometric) and generalized Waring distributions are members of the class $k = 2$. Note that in each case $\alpha_0 = 0$. Dropping this first term in (12) and multiplying by $\tilde{h}_{10}$ yields

$$\text{(25)} \qquad \alpha_1 \tilde{g}' + \alpha_2 \tilde{h}_{01}[\tilde{g}'' - \tilde{h}_{21}\tilde{g}'] = \beta_0 \tilde{f}'\tilde{g} + \beta_1 \tilde{f}\tilde{g}' + \beta_2 \tilde{f}\tilde{h}_{01}[\tilde{g}'' - \tilde{h}_{21}\tilde{g}'].$$

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For $\alpha_2 \neq 0$, solving for the $\tilde{g}''$ which appears in the left-hand side of (25) yields

$$\alpha_2\tilde{g}'' = -\alpha_1\tilde{h}_{10}\tilde{g}' + \alpha_2\tilde{h}_{21}\tilde{g}' + \beta_0\tilde{f}'\tilde{h}_{10}\tilde{g} + \beta_1\tilde{f}'\tilde{g}' + \beta_2\tilde{f}\tilde{g}'' - \beta_2\tilde{g}\tilde{h}_{21}\tilde{g}'. \tag{26}$$

Inversion of (26) yields

$$\begin{aligned}
&\{\alpha_2 x^2 - (\alpha_1 h_{10}(0) - \alpha_2 h_{21}(0))x\}g(x) \\
&\quad = \beta_0 \int_{(0,x)} k_1(y) g(x-y)\, d\mu(y) \\
&\qquad + \int_{(0,x)} (x-y)\{\alpha_1 h_{10}(y) - \alpha_2 h_{21}(y) - \beta_1 y f(y) \\
&\qquad - \beta_2 k_2(y)\} g(x-y)\, d\mu(y) + \beta_2 \int_{(0,x)} (x-y)^2 f(y) g(x-y)\, d\mu(y)
\end{aligned} \tag{27}$$

where $\tilde{k}_1 = \tilde{f}'\tilde{h}_{10}$ and $\tilde{k}_2 = \tilde{f}\tilde{h}_{21}$.

It is also possible to solve for the first $\tilde{g}'$ in the lefthand side and develop a recursion. However, the "divisor" in the recursion which appears in the left-hand side of (27) would be of order x while the integrations (summations) would involve convolutions of $x^2 g(x)$ and some auxiliary function. Division by the largest possible power of x probably leads to more stable integrations (summations). However, further investigation is required here.

For higher values of k, the same general approach can be taken. A variety of recursions can be established. It is not obvious (at least to the authors) which recursion will give more stable results.

5. OBTAINING THE AUXILIARY FUNCTIONS

When the claim size distribution is of some simple functional form, it may be possible to obtain the auxiliary functions analytically. For example, for the gamma claim size distribution,

$$\begin{aligned}
\tilde{f}(t) &= (1 + t/\lambda)^{-\alpha} \\
\tilde{f}'(t) &= -\alpha(1 + t/\lambda)^{-\alpha-1}/\lambda \\
\tilde{f}''(t) &= \alpha(\alpha+1)(1 + t/\lambda)^{-\alpha-2}/\lambda^2.
\end{aligned} \tag{28}$$

The transforms of the auxiliary functions appearing in the recursions of the previous section can be expressed in terms of the quantities in (28). The form of the auxiliary function is obtained by recognizing the form of the resultant transforms. In the case of the gamma claim size distribution, the auxiliary functions are exponential or gamma functions.

In the case of discrete claim sizes, such as binomial, Poisson or negative binomial (each translated so that the origin is at one) a similar approach can be used. The form of the auxiliary function will be related to the form of the claim size distribution.

TABLE 1

x	$f(x)$	$f'(x)$	$f''(x)$	$h_{01}(x)$	$h_{10}(x)$	$h_{21}(x)$	$k_1(x)$	$k_2(x)$
	—	—	—	−1.00	1.00	−1.00	0	0
1	0.50	−0.50	0.50	0.50	−0.50	−1.00	0.5	−0.5
2	0.25	−0.50	1.00	0	−0.25	−0.50	0.75	−0.75
3	0.125	−0.375	1.125	0	−0.125	−0.25	0.75	−0.625
4	0.0625	−0.25	1.00	0	−0.0625	−0.125	0.625	−0.4375
5	0.03125	−0.15625	0.7815	0	−0.03125	−0.0625	0.46875	−0.28125
6	0.015625	−0.09375	0.5625	0	−0.15625	−0.0312	0.32813	−0.17188
7	0.0078125	−0.054688	0.38281	0	−0.0078125	−0.015625	0.21875	−0.10156
8	0.0039063	−0.03125	0.25	0	−0.0039063	−0.0078125	0.14063	−0.058593
9	0.0019531	−0.017578	0.1582	0	−0.0019531	−0.0039063	0.087891	−0.0033203
10	0.00097656	−0.0097656	0.097656	0	−0.00097656	−0.0019531	0.05371	−0.018555
⋮								

Of greatest interest in practical applications is the situation in which an arbitrary discrete claim size distribution, $f(x)$, is specified. Then

$$\tilde{f}(t) = \sum_{x=1}^{\infty} f(x)\, e^{-tx}$$

$$\tilde{f}'(t) = -\sum_{x=1}^{\infty} x f(x)\, e^{-tx}$$

$$\tilde{f}''(t) = \sum_{x=1}^{\infty} x^2 f(x)\, e^{-tx}.$$

The transforms of auxiliary functions such as $\tilde{h}_{10}$, $\tilde{h}_{21}$ and $\tilde{k}_1$ can be expressed as products and/or quotients of the quantities in (28). The quantities in (29) are just polynomials in $z = e^{-tx}$. It is relatively easy to carry out multiplication or divisions of polynomials on modern computers with advanced programming languages such as APL. The coefficients of the powers of $z = e^{-tx}$ in the resultant polynomial are the values of the auxiliary function at integral values. This procedure was carried out for the geometric claim size distribution. The results are displayed in Table 1. The numerical values can be verified analytically for this particular example.

Note that the auxiliary functions do not depend upon the claim frequency distribution. If a claim size distribution is to be used for many calculations, the auxiliary functions need only be calculated once.

6. CONCLUDING REMARKS

The purpose of this paper was to demonstrate the versatility of using transforms in developing recursions in the situation where ratios of successive claim frequencies can be written as ratios of polynomials. A number of questions remain open. A large number of different recursions can be developed by rearranging (12). The questions of which recursion is "best" or "most convenient" or "most stable" remain unanswered.

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