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A SHORT COMBINATORIAL PROOF OF THE VAUGHT CONJECTURE

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1. In [5] R. C. Lyndon gave the first proof of the Vaught conjecture: that if a, b, and c are elements of a free group F such that $a^2b^2=c^2$, then $ab=ba$. Lyndon's proof has been followed by many alternative proofs and generalizations [1, 2, 3, 4, 6, 8, 9, 10, 11, 13, 14] all of which involve rather long combinatorial arguments or group theoretical arguments of a noncombinatorial nature. This note provides a short, purely combinatorial proof of the Vaught conjecture.

2. Let F be the free group $\langle x_1, x_2, \ldots ; \varnothing \rangle$ where 1 denotes the empty word. Denote the identical equality of words in F by "$\equiv$" and their equality, modulo insertions and deletions of the words $x_i^\varepsilon x_i^{-\varepsilon}(\varepsilon \in \{-1, +1\})$, by "$=$". A word is *freely reduced* if it contains no subword of the form $x_i^\varepsilon x_i^{-\varepsilon}$ and *cyclically reduced* if every cyclic permutation of it is freely reduced. The reader is referred to Magnus, Karrass, and Solitar [7] for any unexplained notation.

3. Assuming that $a^2b^2=c^2$ in F, we will show that a, b, and c generate a cyclic subgroup of F. We begin with a "change of variables". Let $x=abc^{-1}a^{-1}$, $y=ac^{-1}$, and $z=acb^{-1}a^{-2}$. It is easy to compute that $a=z^{-1}x^{-1}$, $b=xzxz^{-1}x^{-1}y^{-1}z^{-1}x^{-1}$, and $c=y^{-1}z^{-1}x^{-1}$. Using this substitution, the equation $a^2b^2=c^2$ becomes $x^{-1}y^{-1}xy=z^2$; therefore it suffices to show that x, y, and z generate a cyclic subgroup of F. This follows from a result of M. J. Wicks about commutators in free groups.

In [12] Wicks proved that if w is a commutator in F (i.e. a word of the form $x^{-1}y^{-1}xy$), then some cyclic permutation of the cyclically reduced form of w is identically either of the form $X^{-1}Y^{-1}XY$ or $X^{-1}Y^{-1}Z^{-1}XYZ$. (The proof of this is neither long nor difficult and is purely combinatorial in nature.) Starting with the equation $x^{-1}y^{-1}xy=z^2$, we note that z, which we assume w.l.o.g. to be freely reduced, satisfies an identity $z\equiv u^{-1}z_1u$ where z_1 is cyclically reduced and u is possibly empty. Thus our equation can be written $x_1^{-1}y_1^{-1}x_1y_1=z_1^2$ where $x_1=uxu^{-1}$, $y_1=uyu^{-1}$, and z_1^2 is cyclically reduced. Denoting the freely reduced form of the left hand side by w, we arrive at the identity $w\equiv z_1^2$. Since z_1^2 is cyclically reduced, so is w; therefore, using a cyclic permutation if necessary, we have either $X^{-1}Y^{-1}XY\equiv z_2^2$ or $X^{-1}Y^{-1}Z^{-1}XYZ\equiv z_2^2$ where z_2 is a cyclic permutation of z_1. It follows immediately that either $X^{-1}Y^{-1}\equiv XY$ or $X^{-1}Y^{-1}Z^{-1}\equiv XYZ$. Considering lengths of subwords we see that $X\equiv X^{-1}$, $Y\equiv Y^{-1}$, and $Z\equiv Z^{-1}$. It is then clear that $X\equiv Y\equiv Z\equiv 1$ and thus, in either case, that $z_2\equiv 1$. Therefore $z_1\equiv 1$ and $z=1$.

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Our equation has become $x^{-1}y^{-1}xy=1$, or $xy=yx$. It is easy to see that this has solutions if and only if x and y generate a cyclic subgroup of F.

Added in proof: The above technique can also be used, in conjunction with [12], to obtain the result in [14].

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