Generalized forms of the Series of Bessel and Legendre.*

By Rev. F. H. JACKSON.

§ 1.

The object of this paper is to investigate certain series and differential equations, generalizations of the series of Bessel and Legendre.

Throughout the paper

$$[n] \text{ denotes } \frac{p^n - 1}{p - 1}$$

reducing, when $p = 1$, to n.

The series discussed are the following :—

$$y_1 = \mathrm{A}\left\{x^{[n]} - \frac{[n][n-1]}{[2][2n-1]}p^3x^{[n-2]} + \frac{[n][n-1][n-2][n-3]}{[2][4][2n-1][2n-3]}p^8x^{[n-4]} - \dots\right\} \quad (1)$$

a generalized form of $\mathrm{P}_n(x)$,

the relation between coefficients of x in successive terms being

$$\mathrm{A}_{r+1} = -\mathrm{A}_r p^{2r+1}\frac{[n-2r+1][n-2r+2]}{[2r][2n-2r+1]};$$

$$y_2 = \mathrm{A}\left\{x^{[-n-1]} + \frac{[n+1][n+2]}{[2][2n+3]}p^3x^{[-n-3]} + \frac{[n+1][n+2][n+3][n+4]}{[2][4][2n+3][2n+5]}p^6x^{[-n-5]} + \dots\right\} \quad (2)$$

a generalized form of $\mathrm{Q}_n(x)$,

the relation between successive coefficients being

$$\mathrm{A}_{r+1} = \mathrm{A}_r p^3\frac{[n+2r-1][n+2r]}{[2r][2n+2r+1]};$$

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$$y = \mathrm{A}\left\{x^{[n]} + \frac{x^{[n+2]}}{[2][2n+2]} + \frac{x^{[n+4]}}{[2][4][2n+2][2n+4]} + \dots\right\} \quad \text{-} \quad (3)$$

a generalized form of $\mathrm{J}_n(x)$;

$$y = \mathrm{A}\left\{1 + \frac{x^{[1]}}{[1][a_1][a_2][a_3]..[a_n]} + \frac{x^{[2]}}{[1][2][a_1][a_1+1]...[a_n][a_n+1]} + \dots\right\} (4)$$

§ 2.

If
$$y = x^{[n]}$$
$$\frac{dy}{dx} = [n]x^{[n]-1}$$
$$= [n]x^{\nu[n-1]}.$$

Now differentiate, regarding x^ν as the independent variable; denoting the result by $\dfrac{d^2y}{dx^{(2)}}$,

$$\frac{d^2y}{dx^{(2)}} = \frac{d}{d(x^\nu)}\left\{\frac{dy}{dx}\right\} = [n][n-1]x^{\nu^2[n-2]}.$$

Similarly,

$$\frac{d^3y}{dx^{(3)}} = \frac{d}{d(x^{\nu^2})}\left\{\frac{d^2y}{dx^{(2)}}\right\} = [n][n-1][n-2]x^{\nu^3[n-3]};$$

and generally

$$\frac{d^ry}{dx^{(r)}} = [n][n-1]\dots[n-r+1]x^{\nu^r[n-r]},$$

that is
$$x^{[r]}\frac{d^ry}{dx^{(r)}} = [n][n-1]\dots[n-r+1]x^{[n]}.$$

§ 3.

Denote
$$\mathrm{C}_0y + \mathrm{C}_1x\frac{dy}{dx} + \mathrm{C}_2x^{[2]}\frac{d^2y}{dx^{(2)}} + \dots + \mathrm{C}_s x^{[s]}\frac{d^sy}{dx^{(s)}}$$

by $\phi\left[x\dfrac{dy}{dx}\right]$.

Then if
$$y = \mathrm{A}_1x^{[m_1]} + \mathrm{A}_2x^{[m_2]} + \dots + \mathrm{A}_rx^{[m_r]} + \dots$$
and $\phi[m]$ denote

$$C_0 + C_1[m] + C_2[m][m-1] + \dots + C_s[m][m-1]\dots[m-s+1],$$

$$\phi\left[x\frac{dy}{dx}\right] = A_1\phi[m_1]x^{[m_1]} + A_2\phi[m_2]x^{[m_2]} + \dots + A_r\phi[m_r]x^{[m_r]} + \dots .$$

Now choose m_1 so as to make $\phi[m_1] = 0$.

Let a, b, c, etc., be roots of $\phi[m_1] = 0$.

Also choose

$$A_2\phi[m_2] = A_1, \qquad m_2 = m_1 + l,$$
$$A_3\phi[m_3] = A_2, \qquad m_3 = m_2 + l,$$
$$\text{etc.} \qquad\qquad \text{etc}$$

Then, giving m_1 the value a,

$$\phi\left[x\frac{dy}{dx}\right] = A_1x^{[a+l]} + A_2x^{[a+2l]} + A_3x^{[a+3l]} + \dots$$
$$= A_1\left\{x^{[a+l]} + \frac{x^{[a+2l]}}{\phi[a+l]} + \frac{x^{[a+3l]}}{\phi[a+l]\phi[a+2l]} + \dots\right\}$$
$$= Ax^{[l]}\left\{x^{p^l[a]} + \frac{x^{p^l[a+l]}}{\phi[a+l]} + \frac{x^{p^l[a+2l]}}{\phi[a+l]\phi[a+2l]} + \dots\right\};$$

and
$$y = A\left\{x^{[a]} + \frac{x^{[a+l]}}{\phi[a+l]} + \frac{x^{[a+2l]}}{\phi[a+l]\phi[a+2l]} + \dots\right\}.$$

Denoting this series by $F(x)$, we have

$$\phi\left[x\frac{d.F(x)}{dx}\right] = x^{[l]}F(x^{p^l}). \qquad - \quad - \quad - \quad (A)$$

In the particular case when $p = 1$ this equation becomes

$$\phi\left[x\frac{dy}{dx}\right] = x^l y.$$

§ 4.

The series
$$y = 1 + \frac{x^{[1]}}{[1][a_1][a_2]\dots[a_n]} + \dots\dots$$

comes under the preceding form.

If we denote

$$\frac{(p^{a_1+m}-1)(p^{a_2+m}-1)(p^{a_3+m}-1)\dots(p^{a_n+m}-1)}{(p-1)^n}$$

by $\Pi[a+m]$,

then

$$\Pi[a+m]=A_0+A_1[m]+A_2[m][m-1]+\ldots+A_n[m][m-1]..[m-n+1],$$

the coefficients A_0, A_1, A_2, etc., being independent of m and given by

$$A_0=\Pi[a]$$

$$A_1=\Pi[a+1]-\Pi[a]$$

$$A_2=\frac{\Pi[a+2]-\dfrac{p^2-1}{p-1}\Pi[a+1]-p\Pi[a]}{\dfrac{p^2-1}{p-1}\cdot\dfrac{p-1}{p-1}}$$

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$$A_r=\frac{\Pi[a+r]}{[r]\,!}-\frac{\Pi[a+r-1]}{[r-1]\,!\,[1]\,!}+p\frac{\Pi[a+r-2]}{[r-2]\,!\,[2]\,!}-\ldots+(-1)^r p^{\frac{1}{2}r\cdot r-1}\frac{\Pi[a]}{[r]\,!},$$

in which $[r]\,!$ denotes $\dfrac{p^r-1\,.\,p^{r-1}-1\,.\,p^{r-2}-1\ldots p^2-1\,.\,p-1}{(p-1)^r}$.

We write

$$A_r=\sum_{s=0}^{s=r}(-1)^s p^{\frac{1}{2}s\cdot s-1}\frac{\Pi[a+r-s]}{[r-s]\,!\,[s]\,!}.\ *$$

Now take $$\phi\left[x\frac{dy}{dx}\right]\equiv\sum_{r=0}^{r=n}A_r x^{[r+1]}\frac{d^{r+1}y}{dx^{(r+1)}},$$

$$A_r \text{ being } \sum_{s=0}^{s=r}(-1)^s p^{\frac{1}{2}s\cdot s-1}\frac{\Pi[a+r-s]}{[r-s]\,!\,[s]\,!}.$$

Then if we operate with $\phi\left[x\dfrac{dy}{dx}\right]$ on a series of the form

$$y=C_1x^{[m_1]}+C_2x^{[m_2]}+\ldots\ldots$$

$\phi[m]$ will be $A_0[m]+A_1[m][m-1]+\ldots\ldots$ to $n+1$ terms

$$\equiv[m]\left\{A_0+A_1[m-1]+\ldots\ldots+A_n[m-1][m-2]\ldots[m-n]\right\}$$

$$\equiv[m]\Pi[a+m-1].$$

$\phi[m_1]$ vanishes for the following values of m_1:

$$\begin{array}{l}0\ ,\\ 1-a_1,\\ 1-a_2,\\ \quad\vdots\\ 1-a_n.\end{array}$$

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By taking $m_1 = 0$, $l = 1$, we have

$$\sum_{r=0}^{r=n} A_r x^{[r+1]} \frac{d^{r+1}y}{dx^{(r+1)}} = Ax\left\{1 + \frac{x^{p[1]}}{[1][a_1][a_2]\ldots[a_n]} + \ldots\ldots\right\} \quad \text{(B)}$$

$$\text{and } y = A\left\{1 + \frac{x^{[1]}}{[1][a_1][a_2]\ldots\ldots[a_n]} + \ldots\ldots\right\},$$

and similar relations for the other values of m_1, viz., for

$$1 - a_1, \ 1 - a_2, \text{ etc.}:$$

$$\sum_{r=0}^{r=n} A_r x^{[r+1]} \frac{d^{r+1}y}{dx^{(r+1)}}$$

$$= Ax^{[2-a_1]}\left\{1 + \frac{x^{p^{2-a_1}[1]}}{[1].[2-a_1][a_2-a_1+1][a_3-a_1+1]\ldots[a_n-a_1+1]}\right.$$

$$+ \frac{x^{p^{2-a_1}[2]}}{[1][2][2-a_1][3-a_1].[a_2-a_1+1][a_2-a_1+2]\ldots[a_n-a_1+2]}$$

$$\left. + \ldots\ldots\right\},$$

$$y = Ax^{[1-a_1]}\left\{1 + \frac{x^{p^{1-a_1}[1]}}{[1][2-a_1][a_2-a_1+1]\ldots[a_n-a_1+1]} + \ldots\ldots\right\},$$

and $n - 1$ similar equations for the values $1 - a_2$, $1 - a_3$, etc.

§ 5.

Two interesting special cases of the equation (B) are obtained by substituting

(1) $a = a_1 = a_2 = a_3 = a_4 = \ldots = a_n$;

(2) $a = a_1 = a_2 + 1 = a_3 + 2 = a_4 + 3 = \ldots = a_n + n - 1$.

The series $F(x)$ is in case (1)

$$A\left\{1 + \frac{x^{[1]}}{[1][a]^n} + \frac{x^{[2]}}{[1][2][a]^n[a+1]^n} + \ldots\ldots\right\} \quad (1)$$

and in case (2)

$$A\left\{1 + \frac{x^{[1]}}{[1][a]_n} + \frac{x^{[2]}}{[1][2][a]_n[a+1]_n} + \ldots\ldots\right\}; \quad (2)$$

the differential equation being

$$\sum_{r=0}^{r=n} A_r x^{[r+1]} \frac{d^{r+1}y}{dx^{(r+1)}} = xF(x^p). \quad \text{(C)}$$

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In case (1) $$\mathrm{A}_r \equiv \sum_{s=0}^{s=r} (-1)^s p^{\frac{1}{2}s.\overline{s-1}} \frac{[a+r-s]^n}{[r-s]!\,[s]!};$$

in case (2) $$\mathrm{A}_r \equiv \sum_{s=0}^{s=r} (-1)^s p^{\frac{1}{2}s.\overline{s-1}} \frac{[a+r-s]_n}{[r-s]!\,[s]!},$$

where $[a+r-s]_n \equiv [a+r-s][a+r-s-1][a+r-s-2]\ldots$ to n factors.

In case (2) A_r simplifies, for the $r+1$ terms of the summation are

$$\equiv p^{r(a-n+r)} \frac{\overline{p^n-1}\,.\,\overline{p^{n-1}-1}\ldots\overline{p^{n-r+1}-1}}{\overline{p-1}\,.\,\overline{p^2-1}\ldots\overline{p^r-1}}[a]_{n-r}\,.$$

The differential equation for series (2) may be written

$$\sum_{r=0}^{r=n} p^{r(a-n+r)} \frac{\overline{p^n-1}\,.\,\overline{p^{n-1}-1}\ldots\overline{p^{n-r+1}-1}}{\overline{p-1}\,.\,\overline{p^2-1}\ldots\ldots\overline{p^r-1}}[a]_{n-r}\, x^{[r+1]}\frac{d^{r+1}y}{dx^{(r+1)}} = x\mathrm{F}(x^p). \qquad \text{(D)}$$

§ 6.

Consider the equation

$$px^{[2]}\frac{d^2y}{dx^{(2)}} + \{1-[n]-[-n]\}x\frac{dy}{dx} + [n][-n]y = x^{[2]}\mathrm{F}(x^{p^2}). \qquad \text{(E)}$$

The form of the series $\mathrm{F}(x)$ is to be determined.

Assuming

$$y = \mathrm{A}_1 x^{[m_1]} + \mathrm{A}_2 x^{[m_2]} + \ldots\ldots$$

as a possible form of solution,

$$\phi[m] \equiv [n][-n] + \{1-[n]-[-n]\}[m] + p[m][m-1].$$

Now $$p[m][m-1] \equiv [m]\{[m]-1\};$$

$$\therefore\quad \phi[m] \equiv \{[m]-[n]\}\{[m]-[-n]\}.$$

The values of m_1 for which $\phi[m_1]$ vanishes are

$$m_1 = +n \text{ and } m_1 = -n.$$

Also $$\mathrm{A}_{r+1}\phi[m_{r+1}] = \mathrm{A}_r, \text{ and } m_{r+1} = m_r + 2.$$

Therefore, for $m_1 = +n$,

$$\mathrm{A}_{r+1}\{[m_{r+1}]-[n]\}\{[m_{r+1}]-[-n]\} = \mathrm{A}_r,$$

that is $$\mathrm{A}_{r+1}\{[n+2r]-[n]\}\{[n+2r]-[-n]\} = \mathrm{A}_r;$$

$$\therefore\quad \mathrm{A}_{r+1} = \frac{\mathrm{A}_r}{[2r][2n+2r]},$$

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and $y = \mathrm{F}(x) = \mathrm{A}\left\{x^{[n]} + \frac{x^{[n+2]}}{[2][2n+2]} + \frac{x^{[n+4]}}{[2][4][2n+2][2n+4]} + \dots\right\}$
$= \mathrm{J}_{[n]}(x)$.

We may write the differential equation

$$px^{[2]}\frac{d^2y}{dx^{(2)}} + \left\{1 - [n] - [-n]\right\}x\frac{dy}{dx} + [n][-n]y = x^{[2]}\mathrm{J}_{[n]}(x^{p^2}), \qquad (\mathrm{F})$$

which reduces to Bessel's Equation when $p = 1$.

§ 7.

Consider the expression

$$\mathrm{C}_0 y + \mathrm{C}_1 x\frac{dy}{dx} + \mathrm{C}_2 x^{[2]}\frac{d^2y}{dx^{(2)}} - \frac{d^2y}{dx^{(2)}}, \qquad (1)$$

denoted by $\phi\left[x\frac{dy}{dx}\right] - \frac{d^2y}{dx^{(2)}}$.

Then if $y = \mathrm{A}_1 x^{[m_1]} + \mathrm{A}_2 x^{[m_2]} + \dots + \mathrm{A}_r x^{[m_r]} + \dots$,

performing the operations indicated by (1) we have the expression

$$\begin{aligned} &\mathrm{A}_1\phi[m_1]x^{[m_1]} && - \mathrm{A}_1 x^{p^2[m_1-2]}[m_1][m_1-1] \\ &+ \mathrm{A}_2\phi[m_2]x^{[m_2]} && - \mathrm{A}_2 x^{p^2[m_2-2]}[m_2][m_2-1] \\ &+ \mathrm{A}_3\phi[m_3]x^{[m_3]} && - \mathrm{A}_3 x^{p^2[m_3-2]}[m_3][m_3-1] \\ &+ \quad \dots\dots\dots && - \quad \dots\dots\dots \end{aligned}$$

Choose
$$m_2 = m_1 - 2,$$
$$m_3 = m_2 - 2,$$
etc.,

and $\mathrm{A}_{r+1}\phi[m_{r+1}] = \mathrm{A}_r[m_r][m_r - 1]$.

Write
$$\mathrm{C}_0 = [n][-n-1],$$
$$\mathrm{C}_1 = 1 - [n] - [-n-1],$$
$$\mathrm{C}_2 = p.$$

Then $\phi[m_1] \equiv [n][-n-1] + \{1 - [n] - [-n-1]\}[m_1] + p[m_1][m_1-1]$
$\equiv \{[m_1] - [n]\}\{[m_1] - [-n-1]\}$.

The values of m_1 which make $\phi[m_1]$ vanish are

$$m_1 = n \text{ and } m_1 = -n-1.$$

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Giving m_1 the value n, we have

$$m_{r+1} = n - 2r,$$

and the relation between successive coefficients in the series y is

$$A_{r+1}\{[n-2r]-[n]\}\{[n-2r]-[-n-1]\} = [n-2r+2][n-2r+1]A_r,$$

which reduces to

$$A_{r+1} = -p^{2r+1}\frac{[n-2r+2][n-2r+1]}{[2r][2n-2r+1]}A_r;$$

$$\therefore\ y = A\left\{x^{[n]} - \frac{[n][n-1]}{[2][2n-1]}p^3x^{[n-2]} + \frac{[n][n-1][n-2][n-3]}{[2][4][2n-1][2n-3]}p^8x^{[n-4]} - \ldots\right\}$$

$$= P_{[n]}(x)$$

which is a solution of

$$px^{[2]}\frac{d^2y}{dx^{(2)}} - \frac{d^2y}{dx^{(2)}} + \{1-[n]-[-n-1]\}x\frac{dy}{dx} + [n][-n-1]y$$

$$= P'_{[n-2]}(x) - P'_{[n-2]}(x^{p^2}), \quad \text{(G)}$$

$P'_{[n-2]}(x)$ denoting

$$A[n][n-1]\left\{x^{[n-2]} - p^3\frac{[n-2][n-3]}{[2][2n-1]}x^{[n-4]} + \ldots\ldots\right\}.$$

Similarly, giving m_1 the value $-n-1$, we obtain a series in which the relation between successive coefficients is given by

$$A_{r+1}\{[-n-1-2r]-[n]\}\{[-n-1-2r]-[-n-1]\}$$

$$= [-n-1-2r+2][-n-1-2r+1]A_r;$$

from which $\quad A_{r+1} = \frac{A_r \,.\, [n+2r][n+2r-1]}{[2r][2n+2r+1]}p^3;$

$$\therefore\ y = A\left\{x^{[-n-1]} + \frac{[n+1][n+2]}{[2][2n+3]}p^3x^{[-n-3]}\right.$$

$$\left. + \frac{[n+1][n+2][n+3][n+4]}{[2][4][2n+3][2n+5]}p^6x^{[-n-5]} + \ldots\right\}$$

$$= Q_{[n]}(x).$$

The equation is

$$px^{[2]}\frac{d^2y}{dx^{(2)}} - \frac{d^2y}{dx^{(2)}} + \{1-[-n-1]-[n]\}x\frac{dy}{dx} + [n][-n-1]y$$

$$= Q'_{[n+3]}(x) - Q'_{[n+3]}(x^{p^2}), \quad \text{(H)}$$

$Q'_{[n+3]}(x)$ denoting

$$A[n+1][n+2]\left\{x^{[-n-3]} + \frac{[n+3][n+4]}{[2][2n+3]}p^3x^{[-n-5]} + \ldots\ldots\right\}.$$

The equations (G) and (H) reduce to Legendre's Equation when $p=1$.

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