Canad. Math. Bull. Vol. **52** (4), 2009 pp. 544–554

Intuitionistic Fuzzy γ-Continuity

I. M. Hanafy

Abstract. This paper introduces the concepts of fuzzy γ-open sets and fuzzy γ-continuity in intuitionistic fuzzy topological spaces. After defining the fundamental concepts of intuitionistic fuzzy sets and intuitionistic fuzzy topological spaces, we present intuitionistic fuzzy γ-open sets and intuitionistic fuzzy γ-continuity and other results related topological concepts.

1 Introduction

The concept of fuzzy sets was introduced by Zadeh [7] and later Atanassov [1, 2] generalized this idea to intuitionistic fuzzy sets. On the other hand, Coker [3] introduced the notion of intuitionistic fuzzy topological space, some types of intuitionistic fuzzy continuity, such as fuzzy ($\alpha-$, *semi*$-$, *pre*$-$, β) continuity, and some other related concepts. In this paper , we investigate fuzzy γ-open (γ-closed) sets and fuzzy γ-continuity, introduced by Hanafy [6] in intuitionistic fuzzy topological spaces. A comparison between these concepts and other some known corresponding ones will be discussed. Then we define fuzzy γ-continuity in intuitionistic fuzzy topological spaces and discuss some related properties. Also, throughout this paper, we give several examples to clarify the relationships between these concepts and the relevant concepts in intuitionistic fuzzy topological spaces.

2 Intuitionistic Fuzzy Sets

First we present the fundamental definitions obtained by K. Atanassov and D. Coker.

Definition 2.1 (**[2]**) Let X be a nonempty fixed set. An *intuitionistic fuzzy set* (IFS, for short) A is an object having the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \le \mu_A(x) + \nu_A(x) \le 1$ for each $x \in X$.

Obviously, every fuzzy set A on a nonempty set X is an IFS having the form $A = \{\langle x, \mu_A(x), 1 - \nu_A(x)\rangle : x \in X\}$.

Definition 2.2 (**[2]**) Let X be a nonempty set and the IFS's A and B be in the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}, \quad B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\},$$

Received by the editors September 11, 2006; revised July 3, 2007.
AMS subject classification: Primary: 54A40; secondary: 54A20, 54F99.
Keywords: intuitionistic fuzzy set, intuitionistic fuzzy point, intuitionistic fuzzy topological space, intuitionistic fuzzy γ-open set, intuitionistic fuzzy γ-continuity, intuitionistic fuzzy γ-closure (γ-interior).

and let $A = \{A_j : j \in J\}$ be an arbitrary family of IFS's in X. Then we define

(i) $A \leq B$ if and only if $\forall x \in X\, [\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)]$;
(ii) $\overline{A} = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$;
(iii) $\bigwedge A_j = \{\langle x, \bigwedge \mu_{A_j}(x), \bigvee \nu_{A_j}(x)\rangle : x \in X\}$;
(iv) $\bigvee A_j = \{\langle x, \bigvee \mu_{A_j}(x), \bigwedge \nu_{A_j}(x)\rangle : x \in X\}$;
(v) $\underset{\sim}{1} = \{\langle x, 1, 0\rangle : x \in X\}$ and $\underset{\sim}{0} = \{\langle x, 0, 1\rangle : x \in X\}$;
(vi) $\overline{\overline{A}} = A, \overline{\underset{\sim}{0}} = \underset{\sim}{1}$ and $\overline{\underset{\sim}{1}} = \underset{\sim}{0}$.

Definition 2.3 (**[3]**) Let X and Y be two nonempty sets and $f\colon X \to Y$ be a function.

(i) If $B = \{\langle y, \mu_B(y), \nu_B(y)\rangle : x \in X\}$ is an IFS in Y, then the *preimage* of B under f is denoted and defined by $f^{-1}(B) = \{\langle f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x)\rangle : x \in X\}$;
(ii) If $A = \{\langle x, \lambda_A(x), \upsilon_A(x)\rangle : x \in X\}$ is an IFS in X, then the *image* of A under f is denoted and defined by $f(A) = \{\langle y, f(\lambda_A)(y), f\_(\upsilon_A)(y)\rangle : y \in Y\}$ where

$$f(\lambda_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \lambda_A(x) & \text{if } f^{-1}(y) \neq \underset{\sim}{0}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$f\_(\upsilon_A) = 1 - f(1 - \upsilon_A) = \begin{cases} \inf_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \underset{\sim}{0}, \\ 1 & \text{otherwise.} \end{cases}$$

Corollary 2.4 (**[3]**) *Let $A, A_j (j \in J)$ be IFS's in $X, B, B_j (j \in J)$ be IFS's in Y and $f\colon X \to Y$ be a function. Then*

(i) $A_1 \leq A_2 \Rightarrow f(A_1) \leq f(A_2)$*;*
(ii) $B_1 \leq B_2 \Rightarrow f^{-1}(B_1) \leq f^{-1}(B_2)$*;*
(iii) $A \leq f^{-1}(f(A))$ *(if f is one-to-one, then $A = f^{-1}(f(A))$);*
(iv) $f(f^{-1}(B)) \leq B$ *(if f is onto, then $f(f^{-1}(B)) = B$;*
(v) $f^{-1}(\underset{\sim}{1}) = \underset{\sim}{1}$ *and* $f^{-1}(\underset{\sim}{0}) = \underset{\sim}{0}$*;*
(vi) $f(\underset{\sim}{1}) = \underset{\sim}{1}$*, if f is onto and* $f(\underset{\sim}{0}) = \underset{\sim}{0}$*;*
(vii) $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$*.*

Now we mention the definition of intuitionistic fuzzy points and also some basic results related to it.

Definition 2.5 (**[4]**) Let X be a nonempty set and $c \in X$ a fixed element in X. If $a \in (0, 1]$ and $b \in [0, 1)$ are two fixed real numbers such that $a + b \leq 1$, then the IFS $c(a, b) = \langle x, c_a, 1 - c_{1-b}\rangle$ is called an *intuitionistic fuzzy point* (IFP, for short) *in* X, where α denotes the degree of membership of $c(a, b)$, and $c \in X$ the support of $c(a, b)$.

Definition 2.6 (**[4]**) Let $c(a, b)$ be IFP in X and $A = \langle x, \mu_A, \nu_A\rangle$ an IFS in X. Suppose further that $a, b \in (0, 1)$. Then $c(a, b)$ is said to be *properly contained in* A ($c(a, b) \in A$, for short if and only if $a < \mu_A(c)$ and $b > \nu_A(c)$.

Definition 2.7 ([4]) (i) An IFP $c(a,b)$ in X is said to be quasi-coincident with the IFS $A = \langle x, \mu_A, \nu_A \rangle$, denoted by $c(a,b)qA$, if and only if $a > \nu_A(c)$ or $b < \mu_A(c)$.

(ii) Let $A = \langle x, \mu_A, \nu_A \rangle$ and $B = \langle x, \mu_B, \nu_B \rangle$ be two IFS's in X. Then, A and B are said to be quasi-coincident, denoted by AqB, if and only if there exists an element $x \in X$ such that $\mu_A(x) > \nu_B(x)$ or $\nu_A(x) < \mu_B(x)$.

Proposition 2.8 *Let $f\colon X \to Y$ be a function and $c(a,b)$ is IFP in X.*

(i) *If for IFS B in Y we have $f(c(a,b))qB$, then $c(a,b)qf^{-1}(B)$.*

(ii) *If for IFS A in X we have $c(a,b)qA$, then $f(c(a,b))qf(A)$.*

Proof (i) Let $f(c(a,b))qB$ for IFS B in Y. Then $a > \nu_B(f(c))$ or $b < \mu_B(f(c))$ (equivalently, $f(c)_a > \nu_B$ or $1 - f(c)_{1-b} < \mu_B$). This gives that $a > f^{-1}(\nu_B)(c)$ or $b < f^{-1}(\mu_B)(c)$ (equivalently, $c_a > f^{-1}(\nu_B)$ or $1 - c_{1-b} < f^{-1}(\mu_B)$) which implies $c(a,b)qf^{-1}(B)$.

(ii) Let $c(a,b)qA$, for IFS A in X. Then $a > \nu_A(c)$ or $b < \mu_A(x)$. This implies

$$a > \inf_{x \in f^{-1}(f(c))} \nu_A(x) \quad \text{or} \quad b < \sup_{x \in f^{-1}(f(c))} \mu_A(x),$$

which gives

$$a > f\_(\nu_A)(f(c)) \quad \text{or} \quad b < f(\mu_A)(f(c)).$$

Thus we have $f(c(a,b))qf(A)$. ■

Proposition 2.9 *Let A be an IFS in IFTS in X and $c(a,b)$ be IFP in X. If $c(a,b) \in A$, then $c(b,a)qA$.*

Proof Let $c(a,b) \in A$, then $a < \mu_A(c)$ and $b > \nu_A(c)$, which implies $c(b,a)qA$. ■

3 Intuitionistic Fuzzy Topological Spaces

Here we give the definitions of intuitionistic fuzzy topological space and some types of intuitionistic fuzzy continuity introduced by Coker. Also, some of the results are of interest.

Definition 3.1 ([3]) An *intuitionistic fuzzy topology* (IFT, for short) on a nonempty set X is a family Ψ of IFS's in X satisfying the following axioms:

(i) $\underset{\sim}{0}, \underset{\sim}{1} \in \Psi$;

(ii) $A_1 \wedge A_2 \in \Psi$ for any $A_1, A_2 \in \Psi$;

(iii) $\bigvee A_j \in \Psi$ for any $\{A_j : j \in J\} \subseteq \Psi$.

In this case the pair (X, Ψ) is called an *intuitionistic fuzzy topological space* (IFTS, for short) and each IFS in Ψ is known as an *intuitionistic fuzzy open set* (IFOS, for short) in X.

Definition 3.2 ([3]) The complement $\overline{A}$ of IFOS A in IFTS (X, Ψ) is called an *intuitionistic fuzzy closed set* (IFCS, for short).

https://doi.org/10.4153/CMB-2009-055-0 Published online by Cambridge University Press

Definition 3.3 **([3])** Let (X,Ψ) be an IFTS and $A=\langle x,\mu_A(x),\nu_A(x)\rangle$ be an IFS in X. Then the *fuzzy interior* and *fuzzy closure* of A are denoted and defined by

$$\mathrm{cl}\,(A)=\bigwedge\{K : K \text{ is an } \mathit{IFCS} \text{ in } X \text{ and } A\le K\} \text{ and}$$
$$\mathrm{int}\,(A)=\bigvee\{G : G \text{ is an IFOS in } X \text{ and } G\le K\}.$$

Definition 3.4 An IFTS (X,Ψ) is said to be *extremely disconnected* (IFED, for short) if the closure of every IFOS in X is IFOS.

Definition 3.5 **([5])** Let A be an IFS in an IFTS (X,Ψ). A is called an *intuitionistic fuzzy α-open* (*α-closed*) (resp.*semiopen* (*semiclosed*), *preopen* (*preclosed*), *β-open* (*β-closed*), *regular open* (*regular closed*)) *set*, (IFαOS (IFαCS), (resp. IFSOS (IFSCS), IFPOS (IFPCS), IFβOS (IFβCS), IFROS (IFRCS), for short), if

$$A\le \mathrm{int\ cl\ int}\,(A)(A\ge \mathrm{cl\ int\ cl}\,(A)), (\text{ resp. } A\le \mathrm{cl\ int}\,(A)(A\ge \mathrm{int\ cl}\,(A)),$$
$$A\le \mathrm{int\ cl}\,(A)(A\ge \mathrm{cl\ int}\,(A)), A\le \mathrm{cl\ int\ cl}\,(A)(A\ge \mathrm{int\ cl\ int}\,(A)),$$
$$A=\mathrm{int\ cl}\,(A)(A=\mathrm{cl\ int}\,(A)).$$

Definition 3.6 Let A be an IFS in an IFTS (X,Ψ). The *intuitionistic fuzzy semi-closure* (*semi-interior*) (resp. *pre-closure* (*pre-interior*)) of A is denoted by $\mathrm{cl}_s(A)$ $(\mathrm{int}_s(A))$, (resp. $\mathrm{cl}_p(A)$ $(\mathrm{int}_p(A))$ and defined as follows:

(i) $\mathrm{cl}_s(A)(\mathrm{cl}_p(A))=\bigwedge\{K : K\in$ IFSCS (resp. IFPCS) in X and $A\le K\}$
(ii) $\mathrm{int}_s(A)(\mathrm{int}_p(A))=\bigvee\{G : G\in$ IFSOS (resp. IFPOS) in $X$and $G\le K\}$.

It is clear that A is an IFPCS (resp. IFSCS, IFPOS, IFSOS) if and only if $A=\mathrm{cl}_p(A)$ (resp. $A=\mathrm{cl}_s(A), A=\mathrm{int}_p(A), A=\mathrm{int}_s(A)$).

Theorem 3.7 *Let A be an IFS in an IFTS (X,Ψ), then*

(i) $\mathrm{cl}_p(A)\ge A\vee \mathrm{cl\ int}\,(A), \mathrm{int}_p(A)\le A\wedge \mathrm{int\ cl}\,(A)$.
(ii) $\mathrm{cl}_s(A)\ge A\vee \mathrm{int\ cl}\,(A), \mathrm{int}_s(A)\le A\wedge \mathrm{cl\ int}\,(A)$.

Proof We will prove only the first statement of (i), and the others are similar. Since $\mathrm{cl}_p(A)$ is IFPCS, we have; $\mathrm{cl\ int}\,(A)\le \mathrm{cl\ int\ cl}_p(A)\le \mathrm{cl}_p(A)$. Thus $A\vee \mathrm{cl\ int}\,(A)\le \mathrm{cl}_p(A)$. ■

Remark 3.8 In ordinary topological spaces we have the equality of the above relations. The following example shows that the equality (i) may not be true in IFTS's.

Example 3.9 Let $X=\{a,b,c\}$, and

$$G_1=\Big\langle x,\Big(\frac{a}{0.3},\frac{b}{0.2},\frac{c}{0.7}\Big),\Big(\frac{a}{0.6},\frac{b}{0.5},\frac{c}{0.2}\Big)\Big\rangle;$$
$$G_2=\Big\langle x,\Big(\frac{a}{0.8},\frac{b}{0.8},\frac{c}{0.4}\Big),\Big(\frac{a}{0.1},\frac{b}{0.1},\frac{c}{0.5}\Big)\Big\rangle;$$
$$G_3=\Big\langle x,\Big(\frac{a}{0.8},\frac{b}{0.7},\frac{c}{0.6}\Big),\Big(\frac{a}{0.1},\frac{b}{0.2},\frac{c}{0.3}\Big)\Big\rangle.$$

Then the family $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, G_1, G_2, G_1 \wedge G_2, G_1 \vee G_2\}$ of IFS's in X is an IFT on X. It is easily to verify that $\mathrm{cl}_p(G_3) = \underset{\sim}{1} > G_3 = G_3 \vee \mathrm{cl}\ \mathrm{int}\,(G_3)$ and $\mathrm{int}_p(\overline{G}_3) = \underset{\sim}{0} < \overline{G}_3 = \overline{G}_3 \wedge \mathrm{int}\ \mathrm{cl}\,(\overline{G}_3)$.

Definition 3.10 Let (X, Ψ) and (Y, Φ) be IFTS's. A function $f\colon (X, \Psi) \to (Y, \Phi)$ is called *intuitionistic fuzzy continuous* [3]; (resp. α-*continuous* [5], *semicontinuous* [5], *precontinuous* [5], β-*continuous*) (IF-*continuous* (resp. IFα-*continuous*, IFS-*continuous*, IFP-*continuous*, IFβ-*continuous*), for short) if $f^{-1}(B)$ is an IFOS (resp. IFαOS, IFSOS, IFPOS, IFβOS) for every $B \in \Phi$.

4 Intuitionistic Fuzzy γ-Open Sets

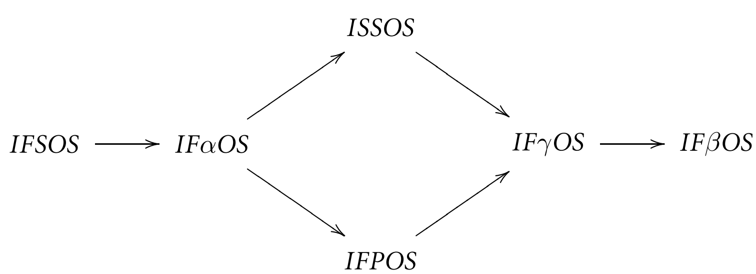
In the sequel, we introduce and study, in IFTS's , the concepts of fuzzy γ-open (closed) sets which generalized the concepts of IFOS's (IFCS's).

Definition 4.1 Let A be an IFS in an IFTS (X, Ψ). Then A is called

(i) an *intuitionistic fuzzy γ-open set* (IFγOS, for short) in X if $A \leq \mathrm{cl}\ \mathrm{int}\,(A) \vee \mathrm{int}\ \mathrm{cl}\,(A)$, and

(ii) an *intuitionistic fuzzy γ-closed set* (IFγCS, for short) in X if $A \geq \mathrm{cl}\ \mathrm{int}\,(A) \wedge \mathrm{int}\ \mathrm{cl}\,(A)$.

Remark 4.2 From the above definition and some types of IFOS's, we have the following diagram:



The converse of the above implications need not be true in general, as shown by the following examples.

Example 4.3 Let $X = \{a, b\}$,

$$G = \left\langle x, \left(\frac{a}{0.3}, \frac{b}{0.2}\right), \left(\frac{a}{0.5}, \frac{b}{0.5}\right) \right\rangle;$$

$$H = \left\langle x, \left(\frac{a}{0.3}, \frac{b}{0.5}\right), \left(\frac{a}{0.7}, \frac{b}{0.2}\right) \right\rangle.$$

Then the family $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, G\}$ is an IFT on X. Since $H \leq \mathrm{cl}\ \mathrm{int}\,(H) = \overline{G}$, then H is an IFβOS in X, but not IFγOS since $H \nleq \mathrm{cl}\ \mathrm{int}\,(H) \vee \mathrm{int}\ \mathrm{cl}\,(H) = \underset{\sim}{0} \vee G = G$.

https://doi.org/10.4153/CMB-2009-055-0 Published online by Cambridge University Press

Example 4.4 In an intuitionistic fuzzy indiscrete topological space (X, I), each intuitionistic fuzzy subset of X is an IFγOS but not an IFSOS.

Example 4.5 Let $X = \{a, b\}$,

$$G = \left\langle x, \left(\frac{a}{0.2}, \frac{b}{0.1}\right), \left(\frac{a}{0.7}, \frac{b}{0.5}\right) \right\rangle;$$

$$H = \left\langle x, \left(\frac{a}{0.3}, \frac{b}{0.2}\right), \left(\frac{a}{0.7}, \frac{b}{0.5}\right) \right\rangle.$$

Then the family $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, G\}$ is an IFT on X. Since $H \leq \operatorname{cl}\operatorname{int}(H) \vee \operatorname{int}\operatorname{cl}(H) = G \vee \overline{G} = \overline{G}$, H is an IFγOS but not IFPOS, since $H \nleq \operatorname{int}\operatorname{cl}(H) = G$.

Remark 4.6 (i) It is clear that the union of any family of IFγOS's is IFγOS.

(ii) The intersection of two IFγOS's need not be IFγOS, as illustrated by the following example.

Example 4.7 Referring to Example 4.5, H is an IFγOS and

$$K = \left\langle x, \left(\frac{a}{0.1}, \frac{b}{0.2}\right), \left(\frac{a}{0.1}, \frac{b}{0.1}\right) \right\rangle$$

is an IFγOS, since $K \leq \operatorname{cl}\operatorname{int}(K) \vee \operatorname{int}\operatorname{cl}(K) = \underset{\sim}{0} \vee \underset{\sim}{1} = \underset{\sim}{1}$. But

$$H \wedge K = \left\langle x, \left(\frac{a}{0.1}, \frac{b}{0.2}\right), \left(\frac{a}{0.7}, \frac{b}{0.5}\right) \right\rangle$$

is not IFγOS, since $H \wedge K \nleq \operatorname{cl}\operatorname{int}(H \wedge K) \vee \operatorname{int}\operatorname{cl}(H \wedge K) = \underset{\sim}{0} \vee G = G$.

Proposition 4.8 *Let A be an IFS in an IFTS (X, Ψ).*

(i) *If A is an IFγOS and* $\operatorname{int}(A) = \underset{\sim}{0}$, *then A is an IFPOS.*
(ii) *If A is an IFγOS and IFCS, then A is an IFSOS.*
(iii) *If A is an IFβOS and IFCS, then A is an IFγOS.*
(iv) *If A is an IFγOS and X is an IFED, then A is an IFPOS.*

Proof (i) Let A be an IFγOS, that is $A \leq \operatorname{cl}\operatorname{int}(A) \vee intcl(A) = \underset{\sim}{0} \vee intcl(A) = \operatorname{int}\operatorname{cl}(A)$. Hence A is an IFPOS.

(ii) Let A be an IFγOS and IFCS, that is $A \leq \operatorname{cl}\operatorname{int}(A) \vee \operatorname{int}\operatorname{cl}(A) = \operatorname{cl}\operatorname{int}(A) \vee \operatorname{int}(A) = \operatorname{cl}\operatorname{int}(A)$. Hence A is an IFSOS.

(iii) Let A be an IFβOS and IFCS, that is $A \leq \operatorname{cl}\operatorname{int}(A) = \operatorname{cl}\operatorname{int}(A) \leq \operatorname{cl}\operatorname{int}(A) \vee \operatorname{int}\operatorname{cl}(A)$. Hence A is an IFγOS.

(iv) Since A is IFγOS and X is an IFED, then

$$A \leq \operatorname{cl}\operatorname{int}(A) \vee \operatorname{int}\operatorname{cl}(A) \leq \operatorname{int}\operatorname{cl}\operatorname{int}(A) \vee \operatorname{int}\operatorname{cl}(A) = \operatorname{int}\operatorname{cl}(A).$$

Therefore A is an IFPOS. ■

Proposition 4.9 *If A is an IFγOS in an IFTS (X, Ψ), then* $\operatorname{int}_s(A) \vee \operatorname{int}_p(A) \leq A$.

https://doi.org/10.4153/CMB-2009-055-0 Published online by Cambridge University Press

Proof Let A be an IFγOS, then $A \le \mathrm{cl}\ \mathrm{int}\,(A) \vee \mathrm{int}\ \mathrm{cl}\,(A)$. From Theorem 3.7, since $\mathrm{int}_s(A) \le A \wedge \mathrm{cl}\ \mathrm{int}\,(A)$ and $\mathrm{int}_p(A) \le A \wedge \mathrm{int}\ \mathrm{cl}\,(A)$, then $\mathrm{int}_s(A) \vee \mathrm{int}_p(A) \le A \wedge (\mathrm{cl}\ \mathrm{int}\,(A) \vee \mathrm{int}\ \mathrm{cl}\,(A)) = A$. ■

Theorem 4.10 *Each IFγOS and IFαCS in an IFTS (X, Ψ) is IFRCS.*

Proof Let A be an IFγOS and IFαCS in an IFTS (X, Ψ), then $\mathrm{cl}\ \mathrm{int}\ \mathrm{cl}\,(A) \le A \le \mathrm{cl}\ \mathrm{int}\,(A) \vee \mathrm{int}\ \mathrm{cl}\,(A) \le \mathrm{cl}\ \mathrm{int}\ \mathrm{cl}\,(A)$. Hence $A = \mathrm{cl}\ \mathrm{int}\ \mathrm{cl}\,(A)$ and also closed. Therefore $A = \mathrm{cl}\ \mathrm{int}\,(A)$, which implies A is IFRCS. ■

Corollary 4.11 *Each IFγCS and IFαOS in an IFTS (X, Ψ) is an IFROS.*

Theorem 4.12 *In an IFTS (X, Ψ), if A is an IFγOS and IFαCS, then* $A = \mathrm{cl}\ \mathrm{int}\,(A) \vee \mathrm{int}\ \mathrm{cl}\,(A)$.

Proof Since A is an IF γOS, then $A \le \mathrm{cl}\ \mathrm{int}\,(A) \vee \mathrm{int}\ \mathrm{cl}\,(A)$.

The other direction, since A is an IFαOS then A is an IFPCS and IFSCS *i.e.,* $\mathrm{cl}\ \mathrm{int}\,(A) \le A$ and $\mathrm{int}\ \mathrm{cl}\,(A) \le A$. Therefore $\mathrm{cl}\ \mathrm{int}\,(A) \vee \mathrm{int}\ \mathrm{cl}\,(A) \le A$. Hence the result. ■

Definition 4.13 Let (X, Ψ) be an IFTS and $A = \langle x, \mu_A, \nu_A\rangle$ be an IFS in X. Then the *intuitionistic fuzzy γ-interior* and *intuitionistic fuzzy γ-closure* are defined and denoted by

$$\mathrm{cl}_\gamma(A) = \bigwedge\{K : K \text{ is an } IF\gamma CS \text{ in } X \text{ and } A \le K\} \text{ and}$$
$$\mathrm{int}_\gamma(A) = \bigvee\{G : G \text{ is an } IF\gamma OS \text{ in } X \text{ and } G \le K\}.$$

It is clear that A is an IFγCS (IFγOS) in X if and only if $A = \mathrm{cl}_\gamma(A)$ $(A = \mathrm{int}_\gamma(A))$.

Proposition 4.14 *For any IFS A in an IFTS (X, Ψ), we have:*

(i) $\mathrm{cl}_\gamma(\overline{A}) = \overline{\mathrm{int}_\gamma(A)}$, $\mathrm{int}_\gamma(\overline{A}) = \overline{\mathrm{cl}_\gamma(A)}$.
(ii) $\mathrm{cl}_\gamma(A \vee B) \ge \mathrm{cl}_\gamma(A) \vee \mathrm{cl}_\gamma(B)$, $\mathrm{int}_\gamma(A \vee B) \ge \mathrm{int}_\gamma(A) \vee \mathrm{int}_\gamma(B)$.
(iii) $\mathrm{cl}_\gamma(A \wedge B) \le \mathrm{cl}_\gamma(A) \wedge \mathrm{cl}_\gamma(B)$, $\mathrm{int}_\gamma(A \wedge B) \le \mathrm{int}_\gamma(A) \wedge \mathrm{int}_\gamma(B)$.

Remark 4.15 The inclusion of results (ii) and (iii) in the above proposition cannot be replaced by equality. In the following example we shall show one of them.

Example 4.16 Let $X = \{a, b, c, d\}$ and

$$G_1 = \left\langle x, \left(\frac{a}{1.0}, \frac{b}{0.0}, \frac{c}{0.2}, \frac{d}{0.0}\right), \left(\frac{a}{0.0}, \frac{b}{0.1}, \frac{c}{0.7}, \frac{d}{1.0}\right)\right\rangle;$$
$$G_2 = \langle x, \left(\frac{a}{0.0}, \frac{b}{1.0}, \frac{c}{0.0}, \frac{d}{0.0}\right), \left(\frac{a}{1.0}, \frac{b}{0.0}, \frac{c}{1.0}, \frac{d}{0.1}\right)\right\rangle;$$
$$G_3 = \left\langle x, \left(\frac{a}{1.0}, \frac{b}{0.0}, \frac{c}{0.0}, \frac{d}{1.0}\right), \left(\frac{a}{0.0}, \frac{b}{0.2}, \frac{c}{0.0}, \frac{d}{0.0}\right)\right\rangle;$$
$$G_4 = \left\langle x, \left(\frac{a}{0.0}, \frac{b}{0.9}, \frac{c}{0.3}, \frac{d}{1.0}\right), \left(\frac{a}{1.0}, \frac{b}{0.0}, \frac{c}{0.2}, \frac{d}{0.0}\right)\right\rangle.$$

https://doi.org/10.4153/CMB-2009-055-0 Published online by Cambridge University Press

Then the family $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, G_1, G_2, G_1 \vee G_2\}$ is an IFT on X. Notice that G_3 and G_4 are IFγCS's in X, then $\operatorname{cl}_\gamma(G_3) = G_3$ and $\operatorname{cl}_\gamma(G_4) = G_4$. But $\operatorname{cl}_\gamma(G_3 \vee G_4) = \underset{\sim}{1}$ (obviously, $G_3 \vee G_4$ is not IFγCS). Then $\underset{\sim}{1} = \operatorname{cl}_\gamma(G_3 \vee G_4) \not\leq \operatorname{cl}_\gamma(G_3) \vee \operatorname{cl}_\gamma(G_4) = G_3 \vee G_4$.

Proposition 4.17 *For any IFS A in an IFTS (X, Ψ), we have:*

(i) $\operatorname{cl}_\gamma(A) \geq \operatorname{cl}\operatorname{int}(A) \wedge \operatorname{int}\operatorname{cl}(A)$.
(ii) $\operatorname{int}_\gamma(A) \leq \operatorname{cl}\operatorname{int}(A) \vee \operatorname{int}\operatorname{cl}(A)$.
(iii) $\operatorname{cl}_\gamma \operatorname{int}_\gamma(A) \vee \operatorname{int}_\gamma \operatorname{cl}_\gamma(A) \leq \operatorname{cl}\operatorname{int}(A)$.

Proof (i) $\operatorname{cl}_\gamma(A)$ is an IFγCS and $A \leq \operatorname{cl}_\gamma(A)$, then $\operatorname{cl}_\gamma(A) \geq \operatorname{cl}\operatorname{int}\operatorname{cl}_\gamma(A) \wedge \operatorname{int}\operatorname{cl}\operatorname{cl}_\gamma(A) \geq \operatorname{cl}\operatorname{int}(A) \wedge \operatorname{int}\operatorname{cl}(A)$

(ii) This follows from (i) by taking the complementation.

(iii) Notice that

$$\operatorname{cl}_\gamma \operatorname{int}_\gamma(A) \vee \operatorname{int}_\gamma \operatorname{cl}_\gamma(A) \leq \operatorname{cl}_\gamma \operatorname{int}_\gamma(A) \vee \operatorname{int}_\gamma \operatorname{cl}(A) = \operatorname{cl}_\gamma \operatorname{int}_\gamma(\operatorname{cl}(A)).$$

Since $\operatorname{int}_\gamma \operatorname{cl}(A)$ is IFγOS, then by Proposition 4.17(i):

$$\begin{aligned}\operatorname{cl}_\gamma \operatorname{int}_\gamma(A) \vee \operatorname{int}_\gamma \operatorname{cl}_\gamma(A) &\leq \operatorname{cl}(\operatorname{int}\operatorname{cl}(\operatorname{int}_\gamma \operatorname{cl}(A)) \vee \operatorname{cl}\operatorname{int}(\operatorname{int}_\gamma \operatorname{cl}(A)))\\ &= \operatorname{cl}\operatorname{int}\operatorname{cl}(\operatorname{int}_\gamma \operatorname{cl}(A)) \vee \operatorname{cl}\operatorname{int}(\operatorname{int}_\gamma \operatorname{cl}(A))\\ &\leq \operatorname{cl}\operatorname{int}\operatorname{cl}(\operatorname{cl}(A)) \vee \operatorname{cl}\operatorname{int}(\operatorname{cl}(A)) = \operatorname{cl}\operatorname{int}\operatorname{cl}(A).\end{aligned}$$ ■

Corollary 4.18 *If A is an IFCS in an IFTS (X, Ψ), then*

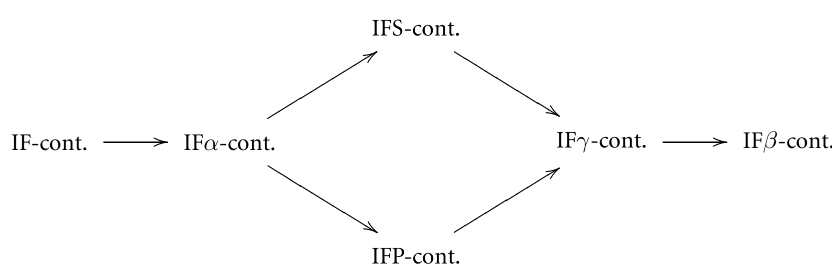
$$\operatorname{cl}_\gamma \operatorname{int}_\gamma(A) \vee \operatorname{int}_\gamma \operatorname{cl}_\gamma(A) \leq \operatorname{cl}\operatorname{int}(A) \vee \operatorname{int}\operatorname{cl}(A).$$

Proof This follows easily from the fact that $\operatorname{cl}\operatorname{int}(A) \vee \operatorname{int}\operatorname{cl}(A) \leq \operatorname{cl}\operatorname{int}\operatorname{cl}(A)$ and Proposition 4.17(iii). ■

5 Intuitionistic Fuzzy γ-Continuity

Definition 5.1 A function $f\colon (X, \Psi) \to (Y, \Phi)$ is called *intuitionistic fuzzy γ-continuous* (IFγ*-continuous*, for short) if $f^{-1}(B)$ is an IFγOS in X, for every $B \in \Phi$.

From the above definition and some known types of intuitionistic fuzzy continuity, one can show the following diagram:



Now the following examples show that the converses of these implications are not true in general.

Example 5.2 Let $X = \{a, b\}$, and

$$G = \left\langle x, \left(\frac{a}{0.2}, \frac{b}{0.3}\right), \left(\frac{a}{0.7}, \frac{b}{0.6}\right)\right\rangle.$$

Then the family $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, G\}$ is an IFT on X. The identity function from an intuitionistic fuzzy indiscrete topological space (X, I) into IFTS (X, Ψ) is IFγ-continuous but not IFS-continuous, since every IFS in (X, I) is not IFSOS.

Example 5.3 Let $X = \{a, b\}$, and

$$G = \left\langle x, \left(\frac{a}{0.2}, \frac{b}{0.3}\right), \left(\frac{a}{0.7}, \frac{b}{0.7}\right)\right\rangle$$
$$H = \left\langle x, \left(\frac{a}{0.2}, \frac{b}{0.6}\right), \left(\frac{a}{0.6}, \frac{b}{0.4}\right)\right\rangle.$$

Consider the IFT's $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, G\}$ and $\Phi = \{\underset{\sim}{0}, \underset{\sim}{1}, H\}$ on X. Then the identity function $f\colon (X, \Psi) \to (Y, \Phi)$ is IFγ-continuous, but not IFP-continuous (indeed, $H \leq \text{int cl}\,(H) \vee \text{cl int}\,(H) = G \vee \overline{G}$, but $H \nleq \text{int cl}\,(H) = G$).

Example 5.4 Let $X = \{a, b, c\}$, $Y = \{1, 2, 3\}$, and

$$G_1 = \left\langle x, \left(\frac{a}{0.3}, \frac{b}{0.1}, \frac{c}{0.4}\right), \left(\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.4}\right)\right\rangle;$$
$$G_2 = \left\langle x, \left(\frac{a}{0.3}, \frac{b}{0.2}, \frac{c}{0.5}\right), \left(\frac{a}{0.2}, \frac{b}{0.2}, \frac{c}{0.4}\right)\right\rangle;$$
$$H = \left\langle y, \left(\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.3}\right), \left(\frac{a}{0.4}, \frac{b}{0.5}, \frac{c}{0.4}\right)\right\rangle.$$

Now the family $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, G_1, G_2\}$ of IFS's in X is an IFT on X and the family $\Phi = \{\underset{\sim}{0}, \underset{\sim}{1}, H\}$ of IFS's in Y is an IFT on Y. If we define the function $f\colon X \to Y$ by $f(a) = 3$, $f(b) = 1$, $f(c) = 2$, then

$$f^{-1}(H) = \langle y, (\frac{a}{0.3}, \frac{b}{0.4}, \frac{c}{0.4}), (\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.5})\rangle, \text{ and}$$
$$f^{-1}(H) \leq \text{cl int cl}\,(f^{-1}(H)) = \overline{G}_1.$$

But $f^{-1}(H) \nleq \text{cl int}\,(f^{-1}(H)) \vee \text{int cl}\,(f^{-1}(H)) = \underset{\sim}{0} \vee G_1 = G_1$. Thus f is IFβ-continuous but not IFγ-continuous

Definition 5.5 Let (X, Ψ) be an IFTS on X and $c(a, b)$ an IFP in X. An IFS N is called $\varepsilon\gamma - nbd$ ($\varepsilon\gamma q - nbd$) of $c(a, b)$ if there exists an IFγOS G in X such that $c(a, b) \in G \leq N(c(a, b)qG \leq A)$.

Let the family of all $\varepsilon\gamma - nbd$ ($\varepsilon\gamma q - nbd$) of $c(a, b)$ be denoted by $N_\varepsilon^\gamma (N_\varepsilon^{\gamma q})(c(a, b))$.

Theorem 5.6 *An IFS A of an IFTS (X, Ψ) is an IFγOS if and only if for every IFP $c(a, b)qA$, $A \in N_\varepsilon^{\gamma q}(c(a, b))$.*

https://doi.org/10.4153/CMB-2009-055-0 Published online by Cambridge University Press

Proof Let $A = \langle x, \mu_A, \nu_A \rangle$ be an IFγOS of X and $c(a,b)qA$. Then $c(a,b)qA \le A$. Hence $A \in N_\varepsilon^{\gamma q}(c(a,b))$.

Conversely, let $c(a,b) \in A$. This implies $a < \mu_A(c)$ and $b > \gamma_A(c)$. Since $a, b \in (0,1)$ and $a + b \le 1$, we have $c(b,a)qA$ and by hypothesis $A \in N_\varepsilon^{\gamma q}(c(b,a))$, then there exists an IFγOS G such that $c(b,a)qG \le A$ which implies $c(a,b) \in G \le A$. Hence by Remark 4.6(i), we have that A is an IFγOS. ∎

Theorem 5.7 *Let $f\colon (X,\Psi) \to (Y,\Phi)$ be a function. Then the following are equivalent.*

(i) *f is IFγ-continuous.*
(ii) *For every $B \in N_\varepsilon^{\gamma q}(f(c(a,b)))$, then there exists $A \in N_\varepsilon^{\gamma q}(c(a,b))$ such that $f(A) \le B$.*

Proof (i) $\Rightarrow$ (ii): Let $c(a,b)$ be any IFP in X and $B \in N_\varepsilon^{\gamma q}(f(c(a,b)))$. Then there exists an IFOS G of Y such that $f(c(a,b))qG \le B$. Since f is IFγ-continuous, $f^{-1}(G)$ is an IFγOS of X with $c(a,b)qf^{-1}(G)$ (by Proposition 2.8). Let $A = f^{-1}(G)$, then $A \in N_\varepsilon^{\gamma q}(c(a,b))$ such that $f(A) = ff^{-1}(G) \le G \le B$.

(ii)$\Rightarrow$ (i): Let B be an IFOS in Y and $c(a,b) \in f^{-1}(B)$. This implies that $f(c(a,b)) \in B$. Thus by Proposition 2.9 $f(c(a,b))qB$, *i.e.*, $B \in N_\varepsilon^{q}(f(c(b,a)))$. So there exists $A \in N_\varepsilon^{\gamma q}(c(b,a))$ such that $f(A) \le B$. Then there exists an IFγOS H of X such that $c(b,a)qH \le A \le f^{-1}(B)$. This implies that $c(a,b) \in H \le f^{-1}(B)$. Hence by Remark 4.6(i), $f^{-1}(B)$ is an IFγOS. ∎

Theorem 5.8 *Let $f\colon (X,\Psi) \to (Y,\Phi)$ be a function. Then the following are equivalent.*

(i) *f is IFγ-continuous.*
(ii) *$f^{-1}(B)$ is an IFγCS in X, for every $B \in \Phi$.*
(iii) *$f(\operatorname{cl}_\gamma(A)) \le \operatorname{cl}(f(A)$, for every IFS A in X.*
(iv) *$\operatorname{cl}_\gamma(f^{-1}(B)) \le f^{-1}(\operatorname{cl}(B))$, for every IFS B in Y.*

Proof (i) $\Rightarrow$ (ii): Obvious.

(ii) $\Rightarrow$ (iii): Let A be an IFS in X. Then $\operatorname{cl}(f(A)$ is an IFCS in Y. By (ii), $f^{-1}(\operatorname{cl}(f(A))$ is an IFγCS in X, and so $f^{-1}(\operatorname{cl}(f(A)) = \operatorname{cl}_\gamma(f^{-1}(\operatorname{cl}(f(A)))$. Since $A \le f^{-1}f(A)$, we have $\operatorname{cl}_\gamma(A) \le \operatorname{cl}_\gamma(f^{-1}f(A)) \le \operatorname{cl}_\gamma(f^{-1}(clf(A))) = f^{-1}(clf(A))$. Hence $f(\operatorname{cl}_\gamma(A)) \le \operatorname{cl}(f(A)$.

(iii) $\Rightarrow$ (iv): Let B be an IFOS in Y. By (iii), we have $f(\operatorname{cl}_\gamma(f^{-1}(B))) \le \operatorname{cl}(ff^{-1}(B))$. Hence $\operatorname{cl}_\gamma(f^{-1}(B)) \le f^{-1}(\operatorname{cl}(ff^{-1}(B))) \le f^{-1}(\operatorname{cl}(B))$.

(iv) $\Rightarrow$ (i): Let B be an IFOS in Y. Then $\overline{B}$ is an IFCS. By (iv) we have $\operatorname{cl}_\gamma(f^{-1}(B)) \le f^{-1}(\operatorname{cl}(\overline{B})) = f^{-1}(\overline{B})$, which implies $f^{-1}(\overline{B}) \ge \operatorname{cl}_\gamma(f^{-1}(\overline{B})) = \overline{\operatorname{int}_\gamma(f^{-1}(B))}$. Hence $f^{-1}(B)$ is an IFγOS in X. ∎

Theorem 5.9 *Let $f\colon (X,\Psi) \to (Y,\Phi)$ be a function. Then the following are equivalent.*

(i) *f is IFγ-continuous.*
(ii) $\operatorname{cl}\operatorname{int}(f^{-1}(B)) \wedge \operatorname{int}\operatorname{cl}(f^{-1}(B)) \le f^{-1}(\operatorname{cl}(B))$, *for every IFS B in Y.*

https://doi.org/10.4153/CMB-2009-055-0 Published online by Cambridge University Press

Proof (i) ⇒ (ii): Let B be an IFS in Y. Then $\mathrm{cl}\,(B)$ is an IFCS. By (i) and using Theorem 5.8, we have $f^{-1}(\mathrm{cl}\,(B))$ is an IFγCS in X. Hence

$$\begin{aligned} f^{-1}(\mathrm{cl}\,(B)) &\geq \mathrm{cl}\ \mathrm{int}\,(f^{-1}(\mathrm{cl}\,(B))) \wedge \mathrm{int}\ \mathrm{cl}\,(f^{-1}(\mathrm{cl}\,(B))) \\ &\geq \mathrm{cl}\ \mathrm{int}\,(f^{-1}(B)) \wedge \mathrm{int}\ \mathrm{cl}\,(f^{-1}(B)). \end{aligned}$$

(ii) ⇒ (i): Let B be an IFCS in Y. Then by (ii)

$$\mathrm{cl}\ \mathrm{int}\,(f^{-1}(B)) \wedge \mathrm{int}\ \mathrm{cl}\,(f^{-1}(B)) \leq f^{-1}(\mathrm{cl}\,(B)) = f^{-1}(B),$$

which implies $f^{-1}(B)$ is an IFγCS in X. Hence the result. ■

Theorem 5.10 *If $f\colon (X,\Psi) \to (Y,\Phi)$ be an IFγ-continuous function and X is an IFED, then f is IFP-continuous.*

Proof Let f be an IFγ-continuous function and $B \in \Phi$, then $f^{-1}(B)$ is an IFγOS in X. Since X is an IFED, then, by Theorem 4.12, $f^{-1}(B)$ is an IFPOS in X. Hence the result. ■

Theorem 5.11 *Let $(X,\Psi), (Y,\Phi)$, and (Z,Ω) be an IFTS's. If $f\colon X \to Y$ is IFγ-continuous and $g\colon Y \to Z$ is IF-continuous, then $g \circ f$ is IFγ-continuous.*

Proof Obvious. ■

Remark 5.12 The composition of two IFγ-continuous functions need not be IFγ-continuous as shown by the following example.

Example 5.13 Referring to Example 5.4, let $X = \{a, b, c\}$ and $Y = Z = \{1, 2, 3\}$. Then the families $\Psi = \{\underset{\sim}{0}, \underset{\sim}{1}, G_1, G_2\}$ and $\Phi = \Omega = \{\underset{\sim}{0}, \underset{\sim}{1}, H\}$ are IFT's in X and Y=Z respectively. If we define $f\colon X \to Y$ by $f(a) = 3$, $f(b) = 1$, $f(c) = 2$ and define the identity function $g\colon Y \to Z$, then it is clear that each of f and g is IFγ-continuous, but $g \circ f$ is not IFγ-continuous.

References

[1] K. Atanassov, *Intuitionistic fuzzy sets VII.* ITKR's Session, Sofia, Bulgaria, 1983.
[2] ———, *Intuitionistic fuzzy sets.* Fuzzy Sets and Systems **20**(1986), no. 1, 87–96.
[3] D. Coker , *An introduction to intuitionistic fuzzy topological spaces.* Fuzzy Sets and Systems **88**(1997), no. 1, 81–89.
[4] D. Coker and M. Demirci, *On intuitionistic fuzzy points.* Notes IFS **1**(1995), no. 2, 79–84.
[5] H. Gurcay, D. Coker, and A. H. Es, *On fuzzy continuity in intuitionistic fuzzy topological spaces.* J. Fuzzy Math. **5**(1997), no. 2, 365–378.
[6] I. M. Hanafy, *Fuzzy γ-open sets and fuzzy γ-continuity.* J. Fuzzy Math.**7**(1999), no. 2, 419–430.
[7] L. A. Zadeh, *Fuzzy sets.* Information and Control **8**(1965), 338–353.

Department of Mathematics, Faculty of Education, Suez Canal University, El-Arish, Egypt
e-mail: ihanafy@hotmail.com

https://doi.org/10.4153/CMB-2009-055-0 Published online by Cambridge University Press