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THE PARABOLICITY OF BRELOT'S HARMONIC SPACES

HIDEO IMAI

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Abstract

The parabolicity of Brelot's harmonic spaces is characterized by the fact that every positive harmonic function is of minimal growth at the ideal boundary.

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Take a locally compact Hausdorff space R which is connected and locally connected, and a sheaf H of functions on R. We assume that R is not compact and has a countable base. Consider a Brelot harmonic space (R, H). Let $H^+(R)$ be the set of all non-negative harmonic functions on R. According to the classification of elliptic equations with respect to the existence of positive solutions ([8, Definition 2.1], cf. also [7, 9, 10]) we provide the following classification of Brelot's harmonic spaces: A Brelot harmonic space (R, H) is called *positively degenerate* if $H^+(R) = \{0\}$. In case $H^+(R) \neq \{0\}$, a Brelot harmonic space (R, H) is called *parabolic* if there exist no potentials on R and *hyperbolic* if there exists a potential on R. In the theory of harmonic spaces the term 'parabolic' is sometimes used in an entirely different meaning (cf. [2]). However we wish to retain the term 'parabolic' in the classification of Brelot's harmonic spaces since it is traditional terminology of classification theory (cf. [11]) in classical potential theory. Hereafter we merely refer to Brelot's harmonic spaces as harmonic spaces. We will also sometimes loosely call R itself or a subregion of R the harmonic space if H is well understood. Denote by δR the boundary of R relative to the one point compactification of R. On the complement $R \setminus F$ of a compact region F in R we will define the Dirichlet operator (cf. [11, p. 403]) which assigns the harmonic function $D_{R\setminus F}\phi$ on $R \setminus F$ with 'boundary values' ϕ on ∂F and zero at the ideal boundary δR for each function ϕ in $C(\partial F)$. Our purpose is to prove the following result:

A harmonic space (R, H) is parabolic if and only if $H^+(R) \neq \{0\}$ and $D_{R\setminus F}h = h$ holds on $R \setminus F$ for each h in $H^+(R) \setminus \{0\}$.

The condition $D_{R\setminus F}h = h$ on $R \setminus F$ is equivalent to the following condition which was used to characterize the parabolicity of elliptic differential operators in [7, Theorem 1.1] and [10, Corollary 4.3]: h is a function with minimal growth at infinity.

The *harmonic measure* $\omega_{R\setminus F}1$ *of the ideal boundary* δR with respect to $R \setminus F$ is the harmonic function on $R \setminus F$ with 'boundary values' 1 at δR and 0 on ∂F. In the case where the constant 1 is harmonic the parabolicity of (R, H) was characterized by $\omega_{R\setminus F}1 = 0$ ([5, Theorem 5.8]). But we still see that this characterization is not true in general and in fact there are cases in which the harmonic measure $\omega_{R\setminus F}1$ is equal to infinity.

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Section 1

Let V be a relatively compact resolutive region in R. A point x_0 in ∂V is *regular* for V (with respect to H) if $\lim_{V\ni x\to x_0} D_V^\phi(x) = \phi(x_0)$ for any ϕ in $C(\partial V)$, where $D_V^\phi(x)$ is the Dirichlet solution for ϕ in V. A region V is called *regular* if V is a relatively compact resolutive region and each boundary point x_0 of V is regular for V. Denote by μ_x^V the harmonic measure of V at $x \in V$. The a lower semi-continuous, lower finite function s on R is *superharmonic* if $s \geq \int_{\partial V} s\, d\mu_x^V$ on V for any regular subregion V of R. For any subregion V of R a *potential* p on V is a positive superharmonic function p on V such that, if $p \geq h$ holds on V for a harmonic function h on V, then $h \leq 0$ on V. A compact region is the closure of a relatively compact region. A non-empty compact region K in R is called *outer regular* ([3, p. 439; 5, p. 184]) if every component of $V \setminus K$ is regular for each regular region V of R with $V \supset K$. We also denote by $\mathring{K}$ the interior of K. We will first prove:

LEMMA 1. *Suppose that a harmonic space (R, H) is not positively degenerate. Let x_0 be any point in R and V be any region in R which contains x_0. Then there exists an outer regular compact region K_V with $x_0 \in \mathring{K}_V \subset K_V \subset V$ such that $R \setminus K_V$ is hyperbolic.*

PROOF. By taking V smaller, if necessary, we may assume (cf. [6, Corollary 2.3]) that there exists a potential on V. Hence by [3, Lemma 7.1] there exists an outer regular compact region K_V in V with $x_0 \in \mathring{K}_V \subset K_V \subset V$. Choose a function h in $H^+(R) \setminus \{0\}$. Let u be the harmonic function on $V \setminus K_V$ with boundary values 0 on ∂K_V and h on ∂V. We set $s = u$ on $V \setminus K_V$ and $s = h$ on $R \setminus V$. Since $h > 0$ on R by the minimum principle, the function s is a positive superharmonic on $R \setminus K_V$ which is not harmonic there. By the Riesz decomposition theorem (cf. [2, 6]) there exists a potential on $R \setminus K_V$.

LEMMA 2. [3] *Suppose that a harmonic space (R, H) is not positively degenerate. Then for any compact region K and any relatively compact region D in R with $K \subset D$ there exists a regular region G with $K \subset G \subset \bar{G} \subset D$.*

PROOF. This lemma was shown by Hervé [3, Proposition 7.1] in the case where R is hyperbolic. Take a region V in $R \setminus \bar{D}$ and a point x_0 in V. Then in view of Lemma 1 we can take an outer regular compact region K_V with $x_0 \in \mathring{K}_V \subset K_V \subset V$ such that $R \setminus K_V$ is hyperbolic. Therefore by [3, Proposition 7.1] we obtain a desired region G in $R \setminus K_V$ and hence in R.

As a corollary of Lemma 2 we have:

COROLLARY. [5] *If (R, H) is not positively degenerate, then there exists a regular exhaustion $\{\Omega\}$ of R, that is, an upper directed family $\{\Omega\}$ of regular regions Ω of R such that $R = \bigcup \Omega$.*

Section 2

We assume that (R, H) is not positively degenerate. Let F be a given outer regular compact region in R. Hereafter we only consider regular exhaustions $\{\Omega\}$ of R in which each Ω contains F. For a function h in $H^+(R) \setminus \{0\}$ we denote by $D_{\Omega \setminus F} h$ the harmonic function on $\Omega \setminus F$ with continuous boundary values h on ∂F and 0 on $\partial \Omega$. Then by the minimum principle we have $D_{\Omega \setminus F} h \leq D_{\Omega' \setminus F} h \leq h$ on $\Omega \setminus F$ for every Ω and Ω' in $\{\Omega\}$ with $\bar{\Omega} \subset \Omega'$. Thus $\lim_{\Omega \to R} D_{\Omega \setminus F} h$ exists on $R \setminus F$ which we denote by $D_{R \setminus F} h$. Again by the minimum principle we get either

$$h = D_{R \setminus F} h \qquad \text{or} \qquad h > D_{R \setminus F} h \tag{1}$$

on each connected component of $R \setminus F$ for any h in $H^+(R)$.

Let ϕ be a positive continuous function on R. We also consider the harmonic function $\omega_{\Omega \setminus F} \phi$ on $\Omega \setminus F$ with continuous boundary values ϕ on $\partial \Omega$ and 0 on ∂F. If $\lim_{\Omega \to R} \omega_{\Omega \setminus F} \phi$ exists, then we set $\omega_{R \setminus F} \phi = \lim_{\Omega \to R} \omega_{\Omega \setminus F} \phi$ and we call it the ϕ-*harmonic measure* of the ideal boundary δR with respect to $R \setminus F$ (cf. [4, p. 5]). In particular if $\omega_{R\setminus F}1$ exists, it is referred to as the *harmonic measure* of the ideal boundary δR with respect to $R \setminus F$ (cf. [5, p. 186; 11, p. 157]). Take any non-zero function h in $H^+(R)$. Then by the minimum principle we have $0 \le \omega_{\Omega'\setminus F}h \le \omega_{\Omega\setminus F}h \le h$ on $\Omega \setminus F$ for every Ω and Ω' in $\{\Omega\}$ with $\bar{\Omega} \subset \Omega'$. Consequently $\lim_{\Omega\to R} \omega_{\Omega\setminus F}h$ exists on $R \setminus F$. Therefore $\omega_{R\setminus F}h$ exists if $h \in H^+(R)$. Observe that $\omega_{\Omega\setminus F}h + D_{\Omega\setminus F}h = h$ is valid on $\Omega \setminus F$ for every Ω. As $\Omega \to R$ we have

$$\omega_{R\setminus F}h + D_{R\setminus F}h = h \tag{2}$$

on $R \setminus F$ for each h in $H^+(R)$ and any outer regular compact region F in R.

Section 3

We next suppose that (R, H) is hyperbolic and take a point y in $\mathring{F}$. Then there exists a potential $p_{R,y}$ on R which is harmonic on $R \setminus \{y\}$ by [3, Théorème 16.1]. In addition, for each Ω in $\{\Omega\}$ there exists a unique potential $p_{\Omega,y}$ on Ω which is harmonic on $\Omega \setminus \{y\}$ and the identity $p_{\Omega,y} = p_{R,y} - R^{C\Omega}_{p_{R,y}}$ holds on Ω by [3, Théorème 16.4] where $R^{C\Omega}_{p_{R,y}}$ is the reduced function of $p_{R,y}$ relative to $C\Omega$ on R. Therefore the inequality $D_{\Omega'\setminus F}p_{R,y} \ge p_{\Omega,y}$ is valid on $\Omega \setminus F$ for any Ω and Ω' in $\{\Omega\}$ with $\bar{\Omega} \subset \Omega'$. As $\Omega' \to R$ and then as $\Omega \to R$ we have $D_{R\setminus F}p_{R,y} \ge p_{R,y}$ on $R \setminus F$. Thus we obtain that

$$p_{R,y} = D_{R\setminus F}p_{R,y} \tag{3}$$

on $R \setminus F$ since $p_{R,y} \ge D_{R\setminus F}p_{R,y}$ on $R \setminus F$. We will show the following:

THEOREM. *Assume that (R, H) is not positively degenerate. Then the following statements are equivalent:*

(a) *(R, H) is a parabolic harmonic space;*
(b) *$D_{R\setminus F}h = h$ holds on $R \setminus F$ for some and hence for any pair (F, h) of an outer regular compact region F in R and a function h in $H^+(R) \setminus \{0\}$;*
(c) *$\omega_{R\setminus F}h = 0$ holds on $R \setminus F$ for some and hence for any pair (F, h) of an outer regular compact region F in R and a function h in $H^+(R) \setminus \{0\}$.*

PROOF. We assume (b). Suppose that there exists a potential $p_{R,y}$ on R which is harmonic on $R \setminus \{y\}$ with y in $\mathring{F}$. We set $c = \max_{\partial F}\{h/p_{R,y}\}$. Then $cD_{R\setminus F}p_{R,y} \ge D_{\Omega\setminus F}h$ on $\Omega \setminus F$ for any Ω in $\{\Omega\}$. As $\Omega \to R$ we have $cp_{R,y} \ge h$ on $R \setminus F$ by (3). Also it follows from the minimum principle that $cp_{R,y} \ge h$ is valid on F. Thus the inequality $cp_{R,y} \ge h$ holds on R. Therefore we have $h \le 0$ on R. But this contradicts the assumption. Hence there exist no potentials on R. Thus (b) implies (a).

We next suppose that $h \ne D_{R\setminus F}h$ on $R \setminus F$ for some outer regular compact region F and some h in $H^+(R) \setminus \{0\}$. We denote by W the union of all connected components of $R \setminus F$ on which $h - D_{R\setminus F}h > 0$ holds. Then $h = D_{R\setminus F}h$ is valid on $(R \setminus F) \setminus W$ by (1). We set $u = D_{R\setminus F}h$ on W and $u = h$ on $R \setminus W$. Clearly u is a positive superharmonic function on R. Let v be a harmonic function on R which is dominated by u on R. Setting $m = \max_{\partial F}(v/h)$, we have $mh \geq v$ and $D_{R\setminus F}h - v \geq (1-m)h$ on ∂F. Hence we have $mh \geq v$ on F and $D_{R\setminus F}h - v \geq (1-m)D_{R\setminus F}h$ on $R \setminus F$. These inequalities imply $mD_{R\setminus F}h \geq v$ on W and $mh \geq v$ on $R \setminus W$. Thus we obtain $mu \geq v$ on R. Suppose that $m > 0$. Then we have $mh \geq mu \geq v$ on R. This with the fact $h > u$ on W implies that $mh - v > 0$ on W. Thus the minimum principle yields $mh - v > 0$ on R. In particular we have $m > (v/h)$ on ∂F. But this is a contradiction. Thus we have $m \leq 0$ and a fortiori $v \leq 0$ on R. Hence u is a potential on R. Thus the assertion (b) follows from (a).

In view of (2), (b) and (c) are equivalent.

REMARK. If 1 is harmonic, then $\omega_{R\setminus F}1$ exists. In this sense we may regard the h-harmonic measure $\omega_{R\setminus F}h$ with $h \in H^+(R) \setminus \{0\}$ as a generalization of the harmonic measure of the ideal boundary of Riemann surfaces (cf. [11, p. 157]).

Section 4

We will state some remarks concerning conditions in the theorem. Let u be a positive harmonic function on $R \setminus F$. Following [1] (cf. [10, p. 956]) u is said to be a function with *minimal growth at infinity* if u satisfies the following condition: For any outer regular compact region F' in R and any positive harmonic function v on $R \setminus F'$ there exists a constant $c > 0$ and an outer regular compact region F'' in R with $F'' \supset F' \cup F$ such that $u \leq cv$ on $R \setminus F''$. Then we have:

PROPOSITION 1. *Let h be a function in $H^+(R) \setminus \{0\}$. Then h is a function with minimal growth at infinity if and only if $D_{R\setminus F}h = h$ on $R \setminus F$ for some and hence for any outer regular compact region F in R.*

PROOF. If $D_{R\setminus F}h = h$ on $R \setminus F$, then h is clearly a function with minimal growth at infinity. Conversely suppose that h is a function with minimal growth at infinity. Since $D_{R\setminus F}h$ is a positive harmonic function on $R \setminus F$, there exists a constant $c > 1$ such that $cD_{R\setminus F}h \geq h$ on $R \setminus F$. For any regular region Ω in R we have $cD_{R\setminus F}h - h \geq (c-1)D_{\Omega\setminus F}h$ on $\Omega \setminus F$. As $\Omega \to R$ the inequality $D_{R\setminus F}h \geq h$ holds on $R \setminus F$. Hence we have $D_{R\setminus F}h = h$ on $R \setminus F$.

PROPOSITION 2. [8, 10] *Suppose that (R, H) is parabolic and let x_0 be any fixed point in R. Then there exists a unique positive harmonic function h on R with $h(x_0) = 1$.*

PROOF. Suppose that there exist two different functions $h_j (j = 1, 2)$ in $H^+(R)\setminus\{0\}$ with $h_j(x_0) = 1$. We set $h = \min(h_1, h_2)$ on R. Then h is a positive superharmonic function on R which is not harmonic. Therefore by the Riesz decomposition theorem there exists a potential on R. But this contradicts the assumption.

Let h be a function in $H^+(R) \setminus \{0\}$. We denote by $h^{-1}H$ such a sheaf of functions on R that for each region V in R, $h^{-1}H(V)$ is a real linear space given by $\{u/h : u \in H(V)\}$. Evidently $(R, h^{-1}H)$ is a harmonic space in which the constant 1 is harmonic on R (cf. [5, p. 193], [6]). We also denote by $h^{-1}H^c(R \setminus F)$ the family of harmonic functions on $R \setminus F$ with respect to $h^{-1}H$ which is continuous on ∂F. We will show:

PROPOSITION 3. *Let h be a function in $H^+(R) \setminus \{0\}$ and F be any outer regular compact region in R. Then $\omega_{R\setminus F}h = 0$ on $R \setminus F$ if and only if* $\max_{\partial F} v = \sup_{R\setminus F} v$ *for any v in $h^{-1}H^c(R \setminus F)$ which is bounded from above.*

PROOF. Observe that 1 is in $h^{-1}H(R)$ and therefore the harmonic measure $\tilde{\omega}_{R\setminus F}1$ of δR with respect to the sheaf $h^{-1}H$ can be defined. Suppose $\omega_{R\setminus F}h = 0$ on $R \setminus F$. Then $\tilde{\omega}_{R\setminus F}1$ vanishes identically on $R \setminus F$ since $\tilde{\omega}_{R\setminus F}1 = h^{-1}\omega_{R\setminus F}h$ on $R \setminus F$. Therefore $(R, h^{-1}H)$ is parabolic by (c) in the Theorem. Let v be any function in $h^{-1}H^c(R \setminus F)$ which is bounded above. Then for each Ω we have $(\sup_{R\setminus F} v)\tilde{\omega}_{\Omega\setminus F}1 + (\max_{\partial F} v)1 \geq v$ on $\Omega \setminus F$. As $\Omega \to R$ $\max_{\partial F} v \geq v$ holds on $R \setminus F$. Therefore we have $\max_{\partial F} v = \sup_{R\setminus F} v$.

The converse assertion is almost trivial. Since $h \geq \omega_{R\setminus F}h$ on $R \setminus F$, the function $(\omega_{R\setminus F}h)/h$ is in $h^{-1}H^c(R \setminus F)$ which is bounded from above. Thus from the assumption we have $(\omega_{R\setminus F}h)/h = 0$ and a fortiori $\omega_{R\setminus F}h = 0$ on $R \setminus F$.

Section 5

We will provide examples which indicate that we can not distinguish parabolic harmonic spaces from hyperbolic harmonic spaces in terms of the harmonic measures of the ideal boundary. In this section we denote by R the punctured unit open ball in $R^n : R = \{0 < |x| < 1\}$, $(n \geq 2)$, so that $\{0\}$ and $|x| = 1$ give rise to the Alexandroff point δR. Consider the elliptic differential equation

$$Lu(x) \equiv \Delta u(x) + P(|x|)u(x) = 0 \tag{4}$$

on R, where Δ is the Laplacian and

$$P(|x|) = \frac{(n-1)(n-3)}{4|x|^2} + \frac{1}{4(1-|x|)^2}\left\{1 + \frac{1}{(\log(1-|x|)^{-1})^2}\right\} \tag{5}$$

on R. Let H be the sheaf of solutions of (4) on R. Then by [3, Théorème 34.1] (R, H) is a harmonic space.

Consider the differential equation

$$\ell u(r) \equiv \frac{d^2}{dr^2}u(r) + \frac{n-1}{r}\frac{d}{dr}u(r) + P(r)u(r) = 0$$

in (0, 1), where P(r) is the function given by (5) with $|x| = r$. We set $p(r) = r^{-(n-1)/2}(1-r)^{1/2}(\log(1-r)^{-1})^{1/2}$ and $q(r) = p(r)\log_2(1-r)^{-1}$, where $\log_2(1-r)^{-1}$ is the iterated logarithm $\log(\log(1-r)^{-1})$. Then we can easily see that $p(r)$ and $q(r)$ are linearly independent solutions of $\ell u(r) = 0$ in (0, 1). Evidently $p(|x|)$ and $q(|x|)$ with $|x| = r$ are solutions of (4) on R.

Take constants η, s_0, t_0, t, s with $0 < \eta < s_0 < t_0 < 1/2 < t < s < 1$ and we set $R_0 = \{x : \eta < |x| < 1\}$. Let H_0 be the sheaf of solutions of (4) on R_0. The set $F_0 \equiv \{t_0 \le |x| \le t\}$ is an outer regular compact region in R and R_0 respectively. We also denote by F an outer regular compact region in F_0 such that $R_0 \setminus F$ is connected. We will see:

EXAMPLE. The harmonic space (R, H) is parabolic and (R_0, H_0) is hyperbolic. But both $\omega_{R\setminus F}1$ and $\omega_{R_0\setminus F}1$ are identically infinity on $R \setminus F$ and on $R_0 \setminus F$ respectively.

The harmonic space (R, H) is not positively degenerate because $p(|x|)$ is a positive solution of (4) on R. We denote by $D_{s_0,t_0}p(|x|)$ the solution of (4) on $s_0 < |x| < t_0$ with boundary values $p(|x|)$ on $|x| = t_0$ and 0 on $|x| = s_0$, and also by $D_{t,s}p(|x|)$ the solution of (4) on $t < |x| < s$ with boundary values $p(|x|)$ on $|x| = t$ and 0 on $|x| = s$. Then we have $D_{s_0,t_0}p(|x|) = \{(\log_2(1-s_0)^{-1} - \log_2(1-|x|)^{-1})/(\log_2(1-s_0)^{-1} - \log_2(1-t_0)^{-1})\}p(|x|)$ on $s_0 \le |x| \le t_0$ and $D_{t,s}p(|x|) = \{(\log_2(1-|x|)^{-1} - \log_2(1-s)^{-1})/(\log_2(1-t)^{-1} - \log_2(1-s)^{-1})\}p(|x|)$ on $t \le |x| \le s$. Thus the following identities hold:

$$\lim_{s_0 \to 0} D_{s_0,t_0}p(|x|) = p(|x|) \qquad \text{on } 0 < |x| \le t_0, \qquad \text{and}$$
$$\lim_{s \to 1} D_{t,s}p(|x|) = p(|x|) \qquad \text{on } t \le |x| < 1.$$

Therefore $D_{R\setminus F_0}p(|x|) = p(|x|)$ holds on $R \setminus F_0$ and hence (R, H) is a parabolic harmonic space by the theorem. Then, in view of Lemma 1, (R_0, H_0) is a hyperbolic harmonic space. Let $\omega_{t,s}(x)$ be the solution of (4) on $t < |x| < s$ with boundary values 0 on $|x| = t$ and 1 on $|x| = s$. Then $\omega_{t,s}(x) = \{(\log_2(1-t)^{-1} - \log_2(1-|x|)^{-1})/(\log_2(1-t)^{-1} - \log_2(1-s)^{-1})\}\{p(|x|)/p(s)\}$ holds on $t \le |x| \le s$. Observe that $\omega_{R\setminus F}1 \ge \omega_{R\setminus F_0}1$ and $\omega_{R_0\setminus F}1 \ge \omega_{R_0\setminus F_0}1$ are valid on $\{t \le |x| < 1\}$ by the minimum principle. Hence we have $\omega_{R\setminus F}1 = \omega_{R_0\setminus F}1 \ge \lim_{s\to 1}\omega_{t,s}(x) = \infty$ on $\{t \le |x| < 1\}$. Thus we conclude $\omega_{R\setminus F}1 = \infty$ on $R \setminus F$ and $\omega_{R_0\setminus F}1 = \infty$ on $R_0 \setminus F$.

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Department of Mathematics
Daido Institute of Technology
Takiharu, Minami, Nagoya 457
Japan