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HOMOMORPHISMS FROM $C(X)$ INTO C^*-ALGEBRAS

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ABSTRACT. Let A be a simple C^*-algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$ of countable rank. We show that a monomorphism ϕ: $C(S^2) \longrightarrow A$ can be approximated pointwise by homomorphisms from $C(S^2)$ into A with finite dimensional range if and only if certain index vanishes. In particular, we show that every homomorphism ϕ from $C(S^2)$ into a UHF-algebra can be approximated pointwise by homomorphisms from $C(S^2)$ into the UHF-algebra with finite dimensional range. As an application, we show that if A is a simple C^*-algebra of real rank zero and is an inductive limit of matrices over $C(S^2)$ then A is an AF-algebra. Similar results for tori are also obtained. Classification of $Hom\Big(C(X), A\Big)$ for lower dimensional spaces is also studied.

0. **Introduction.** The original purpose of the paper is to show that every homomorphism ϕ from $C(S^2)$ into a UHF-algebra can be approximated pointwise by homomorphisms from $C(S^2)$ into the UHF-algebra with finite dimensional range, *i.e.*, for any $\varepsilon > 0$ and any finitely many $f_1, f_2, \ldots, f_n \in C(S^2)$, there are mutually orthogonal projections $p_1, p_2, \ldots, p_m$ (in the algebra) and points $\xi_1, \xi_2, \ldots, \xi_m \in S^2$ such that

$$\Big\|\phi(f_i) - \sum_{j=1}^{m} f_i(\xi_j)p_j\Big\| < \varepsilon, \quad i = 1, 2, \ldots, n.$$

Since AF-algebras are regarded as non-commutative zero dimensional spaces, the above result is expected. This approximation problem and some closely related problems have been considered for some time (see [ExL3] for example). However, it is much more difficult than one might have first thought given the fact that until very recently it was not even known that UHF-algebras have the property (FN). (Recall that a C^*-algebra A has the property (FN), if every normal element $x \in A$ can be approximated by normal elements (in A) with finite spectrum ([Bl1] and [Ln10]).)

If we replace UHF-algebras by general simple AF-algebras A, monomorphisms ϕ: $C(S^2) \longrightarrow A$ may not be approximated by homomorphisms from $C(S^2)$ into A with finite dimensional range. In fact, it is shown in [EL] that a map $\phi\colon C(S^2) \longrightarrow A$ may induce an injective map on K_0, if $K_0(A)$ has nonzero infinitesimal elements. These maps are certainly not approximated by homomorphisms from $C(S^2)$ into A with finite dimensional range. Therefore we have to understand why the absence of infinitesimal elements in $K_0(A)$ should make the approximation possible.

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There are several related problems. For example, what can one conclude if we replace 2-sphere by 2-torus, or if we replace UHF-algebras by more general simple C^*-algebras with real rank zero? It turns out that the latter question related to C^*-algebra extension theory. We also found that these problems have interesting connection with the classification theory of C^*-algebras of real rank zero. (The results in this paper are also used to study almost commuting unitaries.) In this paper we will study homomorphisms from $C(X)$ into C^*-algebras, where X is a lower dimensional compact metric space. When A has nonzero real rank, a homomorphism $\phi: C(X) \longrightarrow A$ is certainly not expected to be approximated by homomorphisms from $C(X)$ into A with finite dimensional range even if ϕ induces a trivial map on K_*. However, some weaker approximation may be possible.

DEFINITION 0.1. Let X be a compact metric space and A be a unital C^*-algebra. Suppose that $\phi_1, \phi_2: C(X) \longrightarrow A \otimes K$ are two homomorphisms, where K is the C^*-algebra of compact operators on l^2. We will write $\phi_1 \sim \phi_2$ if for any $\varepsilon > 0$ and finitely many elements $f_1, f_2, \ldots, f_m \in C(X)$ there are homomorphisms $\psi_1, \psi_2: C(X) \longrightarrow A \otimes K$ with finite dimensional range and a unitary $U \in (A \otimes K)\tilde{}$ such that

$$\left\|\phi_1(f_i) \oplus \psi_1(f_i) - U^*\left[\phi_2(f_i) \oplus \psi_2(f_i)\right]U\right\| < \varepsilon,$$

$i = 1, 2, \ldots, m$.

Note that since X is compact, $C(X)$ is unital. Therefore there is a projection $p \in A \otimes K$ such that im $\phi \subset p(A \otimes K)p$. So there is a unitary $U \in (A \otimes K)\tilde{}$ such that $U^*\phi U: C(X) \longrightarrow M_n(A)$ for some integer n.

Note also that if a homomorphism $\psi: C(X) \longrightarrow B$ has finite dimensional range then there are mutually orthogonal projections $p_1, p_2, \ldots, p_k \in B$ and points $\xi_1, \xi_2, \ldots, \xi_k \in X$ such that

$$\psi(f) = \sum_{i=1}^{k} f(\xi_i)p_i, \quad f \in C(X).$$

It is easy to see that "$\sim$" is an equivalence relation. If $\phi \sim \psi$, we say ϕ and ψ are sau-equivalent (stably approximately unitarily equivalent).

DEFINITION 0.2. Let A be a unital C^*-algebra and X be a compact metric space. Let $\mathbf{E}$ be the set of those homomorphisms $\phi: C(X) \longrightarrow A \otimes K$ which satisfy the following: for any $\varepsilon > 0$ and finitely many functions $f_1, f_2, \ldots, f_n \in C(X)$, there are homomorphisms $\psi_1, \psi_2: C(X) \longrightarrow A \otimes K$ with finite dimensional range such that

$$\|\phi(f_i) \oplus \psi_1(f_i) - \psi_2(f_i)\| < \varepsilon$$

$i = 1, 2, \ldots, n$. So $\phi \in \mathbf{E}$ if and only if ϕ is sau-equivalent to a homomorphism with finite dimensional range.

Denote by $\mathbf{Hom}\big(C(X), A\big)$ the equivalence classes of homomorphisms from $C(X)$ into $A \otimes K$ modulo $\mathbf{E}$.

Let $\phi_1, \phi_2: C(X) \longrightarrow A \otimes K$ be two homomorphisms. We define $\phi_1 + \phi_2: C(X) \longrightarrow A \otimes K$ by

$$(\phi_1 + \phi_2)(f) = \phi_1(f) \oplus \phi_2(f), \quad f \in C(X).$$

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It is easy to check that $\mathbf{Hom}\big(C(X),A\big)$ is a semigroup.

We show that $\mathbf{Hom}\big(C(X),A\big)$ is a group whenever X is a finite CW complex in the plane, X is homeomorphic to S^2, or to $S^1\times S^1$. We also discuss the relation of sau-equivalence with other equivalences. We show that two homomorphisms being sau-equivalent implies that two maps are stably homotopic, if the space X is a finite CW complex in the plane. For higher dimensional space X, it is possible that two sau-equivalent unital homomorphisms may not be stably homotopic but they are homotopic as asymptotic homomorphisms.

DEFINITION 0.3. Let $\phi\colon C(X)\longrightarrow A$ be a homomorphism. Then ϕ induces two maps:

$$\phi_*^{(0)}\colon K_0\big(C(X)\big)\longrightarrow K_0(A)\quad\text{and}\quad \phi_*^{(1)}\colon K_1\big(C(X)\big)\longrightarrow K_1(A).$$

If x is a normal element in A, then x gives a monomorphism $\phi\colon C\big(\mathrm{sp}(x)\big)\longrightarrow A$. We denote by $\gamma(x)$ the map $\phi_*^{(1)}$.

When X is a finite CW complex in the plane, we will show that

$$\mathbf{Hom}\big(C(X),A\big)\cong \hom\Big(K_1\big(C(X)\big),K_1(A)\Big).$$

In the case that $X=S^2$, or $X=S^1\times S^1$, classification of $\mathbf{Hom}\big(C(X),A\big)$ will also be given for some special C^*-algebras A. For example, we show that

$$\mathbf{Hom}\big(C(S^2),A\big)\cong \hom\Big(\ker d,\inf\big(K_0(A)\big)\Big)$$

and

$$\mathbf{Hom}\big(C(S^1\times S^1)\big)\cong\bigg(\hom\Big(K_1\big(C(S^1\times S^1)\big),K_1(A)\Big),\hom\Big(\ker d,\inf\big(K_0(A)\big)\Big)\bigg)$$

for some special C^*-algebras, where $\ker\, d$ is a subgroup of $K_0\big(C(X)\big)$ defined in 2.1.

The paper is organized as follows. Section 1 studies $\mathbf{Hom}\big(C(X),A\big)$ for the case that X is a compact subset of the plane. Most results are more or less known and are taken from [Ln10, 3]. Section 2 contains the main result. We show that if A is a simple C^*-algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$ of countable rank, then a monomorphism $\phi\colon C(S^2)\longrightarrow A$ can be approximated pointwise by homomorphisms from $C(S^2)$ into A with finite dimensional range if and only if

$$\phi_*^{(0)}(\ker\, d)=0.$$

Section 3 shows that results in Section 2 are also valid for the case that $X=S^1\times S^1$. In Section 4, we study $\mathbf{Hom}\big(C(S^2),A\big)$ and $\mathbf{Hom}\big(C(S^1\times S^1),A\big)$. Finally, in Section 5, we give applications of these results to the study of classification of C^*-algebras of real rank zero. For example, we show that if A is a simple C^*-algebra of real rank zero which is an inductive limit of matrix algebras over $C(S^2)$ then A is in fact an AF-algebra.

Before we end this introduction, we would like to review several terminologies:

(1) A C^*-algebra has *real rank zero* if the set of selfadjoint elements with finite spectrum is dense in the $A_{a.s}$, the set of selfadjoint elements of A (see [BP]).
(2) A C^*-algebra has *stable rank one* if the set of invertible elements is dense in A (see [Rf2]). If A has real rank zero, then A has stable rank one, if and only if it has cancellation (see [BH]).
(3) An ordered group (G, G_+) is *unperforated* if $nx \geq 0$ for some integer n implies that $x \geq 0$. (G, G_+) is weakly unperforated if $nx > 0$ implies that $x > 0$ (*cf.* [EHS]).
(4) An ordered group with order unit has *countable rank* if it has only countably many extreme states (see [Ln10, 4.7]). (All groups that used in this paper are countable groups.)

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1. **Compact subsets of the plane.**

LEMMA 1.1 (CF. [EGLP, 2.1], [LR, 8(I)]). *For any $\varepsilon > 0$, there is an integer k such that for any selfadjoint element x in a unital C^*-algebra A with $\|x\| \leq 1$ there are selfadjoint elements $y \in M_k(A)$ and $z \in M_{k+1}(A)$ with finite spectrum* $\mathrm{sp}(y)$, $\mathrm{sp}(z) \subset \mathrm{sp}(x)$ *such that*

$$\|x \oplus y - z\| < \varepsilon.$$

PROOF. The case that $\mathrm{sp}(x)$ is an interval follows directly from [LR, 8(i)].

There are m mutually disjoint open intervals $I_1, I_2, \ldots, I_m \subset [0,1]$ such that $\mathrm{sp}(x) \subset \cup_{i=1}^m \overline{I_i}$ and for any $\xi \in I_i$ there exists $\lambda \in \mathrm{sp}(x)$ with $\mathrm{dist}(\xi, \lambda) < \varepsilon/2$. Therefore there are mutually orthogonal projections $p_1, p_2, \ldots, p_m \in A$ such that

$$p_i x = x p_i \text{ and } \mathrm{sp}(x p_i) \subset \overline{I_i} \text{ (as an element in } p_i A p_i), \quad i = 1, 2, \ldots, m.$$

By applying [Ph3, 2.4] or [LR, 8(i)], we obtain an integer k such that there are selfadjoint elements $y_i \in M_k(p_i A p_i)$ and $z_i \in M_k(p_i A p_i)$ with finite spectrum in $\overline{I_i}$ (as an element in $p_i A p_i$) such that

$$\|x p_i \oplus y_i - z_i\| < \varepsilon/2.$$

So

$$\left\| x \oplus \sum_{i=1}^m y_i - \sum_{i=1}^m z_i \right\| < \varepsilon/2.$$

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There are selfadjoint elements $y \in M_k(A)$ and $z \in M_{k+1}(A)$ such that

$$\Big\|y - \sum_{i=1}^{m} y_i\Big\| < \varepsilon/2 \quad \Big\|z - \sum_{i=1}^{m} z_i\Big\| < \varepsilon/2$$

and $\mathrm{sp}(y), \mathrm{sp}(z) \subset \mathrm{sp}(x)$. We have

$$\|x \oplus y - z\| < \varepsilon. \qquad \blacksquare$$

LEMMA 1.2. *Let A be a unital C^*-algebra. Suppose that*

$$z + (\lambda p - x) \in \mathrm{Inv}_0(A)$$

and $z \in \mathrm{Inv}_0\big((1-p)A(1-p)\big)$, where $z \in (1-p)A(1-p)$ and $x \in pAp$. Then, for any $\xi \neq \lambda$,

$$\lambda(p \oplus 1) - x \oplus \xi \in \mathrm{Inv}_0\big((p \oplus 1)M_2(A)(p \oplus 1)\big).$$

PROOF. Let z^{-1} be the inverse of z in $(1-p)A(1-p)$. Then $z^{-1} + p \in \mathrm{Inv}_0(A)$. So $(1-p) + (\lambda p - x) \in \mathrm{Inv}_0(A)$. Therefore

$$\alpha(1-p) + (\lambda p - x) \in \mathrm{Inv}_0(A)$$

for any $\alpha \neq 0$. This implies that

$$(\lambda p - x) \oplus (\lambda - \xi) \in \mathrm{Inv}_0\big((p \oplus 1)M_2(A)(p \oplus 1)\big)$$

for any $\xi \neq \lambda$. $\blacksquare$

THEOREM 1.3 (CF. [LN10, 3.13]). *Let A be a (unital) C^*-algebra and $x \in A$ be a normal element with $\gamma(x) = 0$. For any $\varepsilon > 0$, there exist an integer k, normal elements $y \in M_k(A)$ and $z \in M_{k+1}(A)$ with finite spectra $\mathrm{sp}(y), \mathrm{sp}(z) \subset \mathrm{sp}(x)$ such that*

$$\|x \oplus y - z\| < \varepsilon.$$

Theorem 1.3 is a consequence of the following which will be used in later sections.

THEOREM 1.4 (CF. [LN10, 3.12]). *Let Ω be a compact subset of the plane. For any $\varepsilon > 0$ there exists $\delta > 0$ satisfying the following: Suppose that B is a unital C^*-algebra, A is a unital C^*-subalgebra of A, $x \in B$ is a normal element, if $p \in A$ is a projection and if $pf(x), f(x)p \in A$ for any $f \in C\big(\mathrm{sp}(x)\big)$,*

$$\|px - xp\| < \delta,\ \lambda p - pxp \in \mathrm{Inv}(pAp) \quad \text{and} \quad [\lambda p - pxp] = 0 \ \big(\textit{in } K_1(pAp)\big)$$

for $\lambda \notin \Omega$. Then there is an integer L, normal elements $y \in M_L(pAp)$ and $z \in M_{L+1}(pAp)$ with finite spectrum $\mathrm{sp}(y), \mathrm{sp}(z) \subset \Omega$ such that

$$\|pxp \oplus y - z\| < \varepsilon.$$

PROOF. The proof is essentially contained in [Ln10, 3]. We sketch the proof here.

LEMMA 1.5 (CF. [LN10, 3.4]). *Let*

$$X = \{\lambda : |\operatorname{Re}\lambda| \le 1/2, |\operatorname{Im}\lambda| \le 1/2\}.$$

For any $\epsilon > 0$, there exist $\delta > 0$ and an integer k, for any unital C^*-algebra A and an element $x \in A$ with

$$\|\operatorname{Re} x\| \le 1/2 \text{ and } \|\operatorname{Im} x\| \le 1/2,$$

if

$$\|x^*x - xx^*\| < \delta,$$

there are normal element $y \in M_k(A)$ and normal element $z \in M_{k+1}(A)$ with finite spectra $\mathrm{sp}(y), \mathrm{sp}(z) \subset X$ *and*

$$\|x \oplus y - z\| < \epsilon.$$

The proof of 1.5 is almost identical to that of [Ln10, 3.4]. Instead of using the condition that the algebra A has real rank zero, we apply 1.1.

LEMMA 1.6 (CF. [LN10, 3.5]). *Let X be a compact subset of the plane which is homeomorphic to the unit disk. For any $\epsilon > 0$, there exist $\delta > 0$ and an integer k, if x is a normal element with* $\mathrm{sp}(x) \subset X$ *in a C^*-algebra B and A is a C^*-subalgebra of B, and if $p \in A$ is a nonzero projection such that $pf(x), f(x)p \in A$ for all $f \in C\big(\mathrm{sp}(x)\big)$,*

$$\|px - xp\| < \delta, \quad \mathrm{sp}(pxp) \subset X,$$

then there are normal elements $y \in M_k(pAp)$ and $z \in M_{k+1}(pAp)$ with finite spectra $\mathrm{sp}(y), \mathrm{sp}(z) \subset X$ *such that*

$$\|x \oplus y - z\| < \epsilon.$$

The proof of this lemma is the same as that of [Ln10, 3.5]. The place where the proof of [Ln10, 3.5] uses [Ln10, 3.4], we use 1.5 here.

We also notice that [Ln10, 3.8] does not need to assume that the algebra A has real rank zero.

LEMMA 1.7 (CF. [LN10, 3.9]). *Let $a > 0$. For any $\varepsilon > 0$, there exists $\delta > 0$ satisfying the following: Suppose that B is a C^*-algebra, A is a unital C^*-subalgebra of B and $x \in B$ is a normal element with polar decomposition $x = uh$, where $0 < a \le h \le 1$ (so u is a unitary in B). If p is a projection in A such that*

$$\|px - xp\| < \delta, \quad pxp \in \mathrm{Inv}_0(pAp) \quad \text{and} \quad pf(x), f(x)p \in A \text{ for all } f \in C\big(\mathrm{sp}(x)\big),$$

and

$$\mathrm{sp}(x) \subset X = \{re^{i\theta} : a \le r \le 1, -\pi \le \theta \le \pi\},$$

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then there are an integer k, normal elements $y \in M_k(pAp)$ *and* $z \in M_{k+1}(pAp)$ *with finite spectrum contained in X such that*

$$\|pxp \oplus y - z\| < \varepsilon.$$

The proof is essentially the same as that of [Ln10, 3.9]. We notice that here k possibly depends on A and x. This is because that A is no longer assumed to have real rank zero. So the length of the path $\{v(t)\}$ (in the proof of [Ln10, 3.9]) may depend on the element x and the algebra A.

The following notation is used in 1.8.

Let

$$X' = \{z : 0 \le \mathrm{Re}(z) \le 1, \, |\,\mathrm{Im}(z)| \le b\} \setminus \bigcup_{j=1}^{k} \{z : |(2j-1)/2k - z| < r\},$$

$0 < r < \min(b, 1/4k)$, and $X = \{z - 1/2 : z \in X'\}$. So X is a square with k holes.

LEMMA 1.8 ([LN10, 3.11]). *Let* Ω *be a compact subset of the plane which is homeomorphic to the subset X described above. For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *satisfying the following: Suppose that B is a unital* C^**-algebra and* $x \in B$ *is a normal element with* $\mathrm{sp}(x) \subset X$ *and A is a unital* C^**-subalgebra of B. If p is a projection in A such that* $pf(x), f(x)p \in A$ *for all* $f \in C\big(\mathrm{sp}(x)\big)$,

$$\|px - xp\| < \delta,$$
$$\lambda p - pxp \in \mathrm{Inv}(pAp) \quad \text{and} \quad [\lambda p - pxp] = 0 \ \big(\text{in } K_1(pAp)\big)$$

for $\lambda \notin X$, *then there are an integer L, normal elements* $y \in M_L(pAp)$ *and* $z \in M_{L+1}(pAp)$ *with finite spectrum* $\mathrm{sp}(y), \mathrm{sp}(z) \subset Y$ *such that*

$$\|pxp \oplus y - z\| < \epsilon.$$

Again, the proof is essentially the same as that of [Ln10, 3.11]. But 2.3 of [Ln10] can not be directly applied since A may not have real rank zero. So $\lambda p_j - p_j x_j p_j$ may not be in $\mathrm{Inv}_0(p_jAp_j)$. However, we can apply 1.2. So

$$\lambda(p_j \oplus 1) - p_j x_j p_j \oplus \xi_j \cdot 1 \in \mathrm{Inv}_0\big((p \oplus 1)M_2(A)(p \oplus 1)\big)$$

for some $\xi_j \in Y_j$. Since we are allowed to have a large integer L, we can choose y_j such that they are mutually orthogonal. Notice that integer L in 1.8 may depend on A and x.

We are now ready to prove Theorem 1.4 as in [Ln10, 3.12]. ∎

REMARK 1.9. We notice that in 1.3 the integer k may depend not only on ε and $\mathrm{sp}(x)$, but also on A and x, while in [Ln10, 3.13] the integer k depends only on ε and the space $\mathrm{sp}(x)$. This happens because if A is not of real rank zero, there is no control of the exponential length of hereditary C^*-subalgebras and the map $U(A)/U_0(A)$ may not be injective in general. If we assume that the exponential length of hereditary C^*-subalgebras of A is bounded, then the integer k in 1.7 depends only on ε. If we further assume that $U(pAp)/U_0(pAp) \longrightarrow K_1(pAp)$ is injective for each projection $p \in A$ (for example A has stable rank one), then the integer k depends only on ε in 1.8.

PROPOSITION 1.10. *Let X be a compact metric space and A be a unital C^*-algebra. Suppose that two homomorphisms $\phi_1, \phi_2\colon C(X) \longrightarrow A \otimes \mathcal{K}$ are in* **E**. *Then $\phi_1 \sim \phi_2$.*

PROOF. By 0.2, for any $\varepsilon > 0$ and $f_1, f_2, \ldots, f_N \in C(X)$, there are homomorphisms $\psi_1, \psi_2\colon C(X) \longrightarrow M_{nL}(A)$ with finite dimensional range and $\psi_3, \psi_4\colon C(X) \longrightarrow M_{n(L+1)}(A)$ with finite dimensional range such that

$$\|\phi_1(f_i) \oplus \psi_1(f_i) - \psi_3(f_i)\| < \varepsilon/2$$

and

$$\|\phi_2(f_i) \oplus \psi_2(f_i) - \psi_4(f_i)\| < \varepsilon/2,$$

$i = 1, 2, \ldots, N$. There is a unitary $U \in M_{2n(L+1)}(A)$ such that

$$\phi_1(f_i) \oplus \psi_1(f_i) \oplus \phi_2(f_i) \oplus \psi_2(f_i) = U^*\big[\phi_2(f_i) \oplus \psi_2(f_i) \oplus \phi_1(f_i) \oplus \psi_1(f_i)\big]U,$$

$i = 1, 2, \ldots, N$. So

$$\Big\|\big[\phi_1(f_i) \oplus \psi_1(f_i) \oplus \psi_4(f_i)\big] - U^*\big[\phi_2(f_i) \oplus \psi_2(f_i) \oplus \psi_3(f_i)\big]U\Big\| < \varepsilon,$$

$i = 1, 2, \ldots, N$. ∎

COROLLARY 1.11. *Let X be a compact subset of the plane and A be unital C^*-algebra. Then $\phi\colon C(X) \longrightarrow A \otimes K$ is in* **E** *if and only if $\phi_*^{(1)} = 0$. Furthermore, two homomorphism $\phi_1,\ \phi_2\colon C(X) \longrightarrow A \otimes \mathcal{K}$ are sau-equivalent if*

$$(\phi_1)_*^{(1)} = (\phi_2)_*^{(1)} = 0.$$

PROOF. This follows from 1.3 and 1.10. ∎

LEMMA 1.12. *Let A be a unital C^*-algebra which is not scattered (see [J]) and X be a finite* CW *complex in the plane. Then, for any homomorphism*

$$\gamma\colon K_1\big(C(X)\big) \longrightarrow K_1(A),$$

there is a monomorphism

$$\phi\colon C(X) \longrightarrow A \otimes \mathcal{K}$$

such that $\phi_*^{(1)} = \gamma$.

PROOF. Let $f_1, f_2, \ldots, f_n\colon X \longrightarrow S^1$ be continuous functions such that $([f_1], [f_2], \ldots, [f_n])$ form a system of free generators for the (free) abelian group $K_1\big(C(X)\big)$. Suppose that $\gamma([f_i]) = a_i$ (in $K_1(A)$). We will produce monomorphisms $\phi_1, \phi_2, \ldots, \phi_n$ such that

$$\big[\phi_i(f_i)\big] = a_i \text{ and } \big[\phi_i(f_j)\big] = 0 \quad \text{for} \quad i \neq j.$$

Choose $g_1, g_2, \ldots, g_n\colon S^1 \longrightarrow X$ such that $f_i \circ g_j$ is homotopic to the identity map for $i = j$ and to the constant map for $i \neq j$. Let $v_1, v_2, \ldots, v_n \in A \otimes \mathcal{K}$ be normal partial isometries

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such that $[v_i] = a_i$ in $K_1(A)$. Since A is not scattered, by [AS, p. 61], there is a selfadjoint element $h \in A$ such that $\mathrm{sp}(h) = [0,1]$. Suppose that $X = \cup_{k=1}^m X_i$, where each X_i is a path connected component of X. It is well known (see [N, p. 89]) that, for each k, there is a function $F_k \in C([0,1])$ such that $\mathrm{sp}\big(F_k(h)\big) = X_k$, $k = 1, 2, \ldots, m$. Set

$$x = \bigoplus_{k=1}^m F_k(h).$$

It is easy to see that $\mathrm{sp}(x) = X$ and $\gamma(x) = 0$. Define

$$\phi_i(f) = f\big(g_i(v_i) \oplus x\big), \quad f \in C(X).$$

So

$$\big[\phi_i(f_i)\big] = a_i \text{ and } \big[\phi_i(f_j)\big] = 0 \text{ for } i \neq j.$$

Define

$$\phi(f) = \bigoplus_{i=1}^n \phi_i(f), \quad f \in C(X).$$

Then ϕ is as required. ■

LEMMA 1.13. *Let X be a finite* CW *complex in the plane and let $\phi: C(X) \longrightarrow A$ be a homomorphism. Then there is $\tilde{\phi}: C(X) \longrightarrow M_n(A)$ for some integer n such that $\phi_*^{(1)} = -(\tilde{\phi})_*^{(1)}$.*

PROOF. If A is scattered, then A is an AF-algebra (see [Ln2, 5.1] but it is known long before that). Therefore $K_1(A) = 0$. Consequently, $\phi_*^{(1)} = -\phi_*^{(1)}$.

In the case that A is not scattered, 1.13 follows immediately from 1.12. ■

THEOREM 1.14. *Let A be a unital C^*-algebra and X be a finite* CW *complex in the plane. Then $\mathbf{Hom}\big(C(X), A\big)$ is a group.*

PROOF. If A is scattered, $K_1(A) = 0$. By 1.11, $\mathbf{Hom}\big(C(X), A\big) = \{0\}$. Now we assume that A is not scattered. Let $\phi: C(X) \longrightarrow A \otimes \mathcal{K}$ be a monomorphism. By 1.13, there is $\tilde{\phi}: C(X) \longrightarrow A \otimes \mathcal{K}$ such that

$$(\tilde{\phi})_*^{(1)} = -\phi_*^{(1)}.$$

It follows from 1.3 that $[\phi \oplus \tilde{\phi}] = 0$ in $\mathbf{Hom}\big(C(X), A\big)$. ■

The proof of the following is much shorter if we assume that X is a finite CW complex in the plane.

THEOREM 1.15. *Let X be a compact subset of the plane and A be a unital C^*-algebra. Suppose that $\phi_1, \phi_2: C(X) \longrightarrow M_n(A)$ for some integer n. Then ϕ_1 is sau-equivalent to ϕ_2 if and only if $(\phi_1)_*^{(1)} = (\phi_2)_*^{(1)}$.*

PROOF. The "only if" part is trivial.

Now we assume that $(\phi_1)_*^{(1)} = (\phi_2)_*^{(1)}$. By 1.3 and 1.11, we may assume that A is not scattered. Let $\varepsilon > 0$ and $f_1, f_2, \ldots, f_m \in C(X)$ be fixed. There exists $\delta > 0$ such that

$$|f_i(\xi) - f_i(\zeta)| < \varepsilon/2$$

whenever $\operatorname{dist}(\xi,\zeta) < \delta$, $i = 1, 2, \ldots, m$. There is a finite CW complex Y in the plane such that $X \subset Y$ and for any $\zeta \in Y$ there is $\xi \in X$ with

$$\operatorname{dist}(\xi,\zeta) < \delta.$$

By the proof of 1.12, there is a monomorphism $\phi_0 : C(Y) \longrightarrow A \otimes \mathcal{K}$ such that $(\phi_0)_*^{(1)} = 0$. Denote by $i: X \longrightarrow Y$ the embedding. We define

$$\Phi_j(f) = \phi_j(f \circ i) \oplus \phi_0(f)$$

$f \in C(Y)$ and $j = 1, 2$. So Φ_j is a monomorphism from $C(Y)$ into $A \otimes \mathcal{K}$ and $(\Phi_1)_*^{(1)} = (\Phi_2)_*^{(1)}$. By 1.12, there is a monomorphism $\tilde{\Phi}: C(Y) \longrightarrow A \otimes \mathcal{K}$ with $\tilde{\Phi}_*^{(1)} = -(\Phi_1)_*^{(1)} = -(\Phi_2)_*^{(1)}$. It follows from 1.11 that

$$[\Phi_1 \oplus \tilde{\Phi}] = [\Phi_2 \oplus \tilde{\Phi}] \text{ in } \mathbf{Hom}\big(C(Y), A\big).$$

Hence $[\Phi_1 \oplus \tilde{\Phi} \oplus \Phi_2] = [\Phi_2 \oplus \tilde{\Phi} \oplus \Phi_2]$. Since $(\tilde{\Phi} \oplus \Phi_2)_*^{(1)} = 0$, by 1.3, $[\tilde{\Phi} \oplus \Phi_2] = 0$. Therefore $\Phi_1 \sim \Phi_2$. Let $\overline{f_1}, \overline{f_2}, \ldots, \overline{f_n} \in C(Y)$ such that $\overline{f_k}|_X = f_k$, $k = 1, 2, \ldots, n$. There are homomorphisms Ψ_1, $\Psi_2 : C(Y) \longrightarrow A \otimes \mathcal{K}$ with finite dimensional range and a unitary $U \in (A \otimes \mathcal{K})\tilde{}$ such that

$$\left\| \Phi_1(\overline{f_k}) \oplus \Psi_1(\overline{f_k}) - U^*\big(\Phi_2(\overline{f_k}) \oplus \Psi_2(\overline{f_k})\big)U \right\| < \varepsilon/4,$$

$k = 1, 2, \ldots, n$. By 1.3, there are homomorphisms $\psi_1, \psi_2 : C(Y) \longrightarrow A \otimes \mathcal{K}$ with finite dimensional range such that

$$\|\phi_0(\overline{f_k}) \oplus \psi_1(\overline{f_k}) - \psi_2(\overline{f_k})\| < \varepsilon/4.$$

This implies that, for a unitary $W \in (A \otimes \mathcal{K})\tilde{}$,

$$\left\| \phi_1(f_k) \oplus \psi_2(\overline{f_k}) \oplus \Psi_1(\overline{f_k}) - W^*\big(\phi_2(f_k) \oplus \psi_2(\overline{f_k}) \oplus \Psi_2(\overline{f_k})\big)W \right\| < \varepsilon/2.$$

Notice that

$$\Psi_j(f) = \sum_{l=1}^{n} f\big(\xi_l^{(j)}\big) p_l^{(j)}, \quad j = 1, 2,$$

$$\psi_2(f) = \sum_{l=1}^{N} f(\zeta_l) q_l$$

for $f \in C(Y)$, where $\xi_l^{(j)}, \zeta_l \in Y$, $j = 1, 2$, and $q_l's$ are mutually orthogonal and $p_l^{(j)}$ are mutually orthogonal for each j. Since for each $\zeta \in Y$ there is $\xi \in X$ such that $\operatorname{dist}(\xi,\zeta) < \delta$, we have homomorphisms $\Psi_1', \Psi_2' : C(X) \longrightarrow A \otimes \mathcal{K}$ with finite dimensional range and a unitary $V \in (A \otimes \mathcal{K})\tilde{}$ such that

$$\left\| \phi_1(f_k) \oplus \Psi_1'(f_k) - V^*\big(\phi_2 \oplus \Psi_2'(f_k)\big)V \right\| < \varepsilon,$$

$k = 1, 2, \ldots, n$. ∎

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THEOREM 1.16. *Let A be a unital C^*-algebra and X be a finite* CW *complex in the plane. Then there is a bijection*

$$\gamma\colon \mathbf{Hom}\big(C(X),A\big) \longrightarrow \mathrm{hom}\Big(K_1\big(C(X)\big),K_1(A)\Big).$$

PROOF. If A is scattered, as in 1.12, we know that $K_1(A) = 0$. So, by 1.3, $\mathbf{Hom}\big(C(X),A\big) = \{0\} = \mathrm{hom}\Big(K_1\big(C(X)\big),K_1(A)\Big)$. Therefore, we may assume that A is not scattered. The theorem follows from 1.3, 1.12 and 1.15. ∎

REMARK 1.17. One may compare 1.16 with [BDF1, 10]. When A is the Calkin algebra, Theorem 1.16 says the map

$$\gamma\colon \mathbf{Hom}\big(C(X),A\big) \longrightarrow \mathrm{hom}\Big(K_1\big(C(X)\big),\mathbf{Z}\Big)$$

is bijective. So by certain absorption lemma and a standard quasidiagonal argument, one could obtain [BDF1, 10] by applying 1.16 (*cf.* [Ln7]).

REMARK 1.18. In [Ln10, 4.4], we show that purely infinite simple C^*-algebras and separable simple C^*-algebras with real rank zero, stable rank one, weakly unperforated $K_0(A)$ with countable rank have property weak (FN), *i.e.*, a normal element x with $\gamma(x) = 0$ can be approximated by normal elements with finite spectrum. The stable approximation discussed in this section is much weaker but is the right one for general C^*-algebras.

Now we show that if two homomorphisms ϕ_1, $\phi_2\colon C(X) \longrightarrow A\otimes K$ are sau-equivalent if and only if they are stably homotopic. The following is a result of Terry Loring [Lor2].

LEMMA 1.19 ([LOR2, THEOREM B]). *Let X be a finite* CW *complex in the plane. There exist $\varepsilon > 0$ and $f_1, f_2, \ldots, f_k \in C(X)$ such that whenever ϕ_1, $\phi_2\colon C(X) \longrightarrow A$ are two homomorphisms, where A is a unital C^*-algebra, such that*

$$\|\phi_1(f_i) - \phi_2(f_i)\| < \varepsilon, \quad i = 1, 2, \ldots, k,$$

then ϕ_1 and ϕ_2 are homotopic.

PROOF. This is an immediate consequence of Theorem B and Proposition 3.1 of [Lor2]. ∎

THEOREM 1.20. *Let X be a finite* CW *complex of the plane and A be a unital C^*-algebra. Suppose that ϕ_1, $\phi_2\colon C(X) \longrightarrow A\otimes K$ are two homomorphisms. Then the following are equivalent:*

(1) $\phi_1 \sim \phi_2$,

(2) There are two homotopically trivial homomorphisms ψ_1, $\psi_2\colon C(X) \longrightarrow A \otimes K$ such that

$$\phi_1 \oplus \psi_2 \text{ and } \phi_2 \oplus \psi_2$$

are homotopic,

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(3) $(\phi_1)_*^{(1)} = (\phi_2)_*^{(1)}$.

PROOF. The equivalence of (1) and (2) follows from 1.15. That (2) implies (3) is trivial. We now show that (1) implies (2).

Given $\varepsilon > 0$ and $f_1, f_2, \ldots, f_k \in C(X)$. By 1.10, there are homomorphisms ψ_1, ψ_2: $C(X) \longrightarrow A \otimes K$ with finite dimensional range and a unitary $U \in (A \otimes K)\tilde{}$ such that

$$\left\| \phi_1(f_i) \oplus \psi_1(f_i) - U^*\left[\phi_2(f_i) \oplus \psi_2(f_i)\right]U \right\| < \varepsilon, \quad i = 1, 2, \ldots, k.$$

It follows from 1.19 that $\phi_1 \oplus \psi_1$ and $U^*(\phi_2 \oplus \psi_2)U$ are homotopic. It follows from [M] that there is a continuous path of unitaries $U_t : t \in [0, 1]$ in $M(A \otimes K)$ such that $U_0 = U$ and $U_1 = 1$. Since $U_t \in M(A \otimes K)$,

$$U_t^*\big(\phi_2(f) \oplus \psi_2(f)\big)U_t \in A \otimes K, \quad f \in C(X).$$

This implies that $U^*(\phi_2 \oplus \psi_2)U$ and $\phi_2 \oplus \psi_2$ are homotopic. Therefore $\phi_1 \oplus \psi_1$ and $\phi_2 \oplus \psi_2$ are homotopic. Notice that a homomorphism with finite dimensional range is homotopically trivial. ∎

2. **The 2-sphere S^2.**

2.1. Let X be a compact metric space. For any projection $p \in C(X, M_k)$, where M_k is the $k \times k$ matrices over $\mathbf{C}$, let $\dim\big(p(x)\big)$ be the dimension of $p(x)$. So $\dim\big(p(x)\big)$ is a function in $C(X)$. It is easy to see that the map $d: K_0\big(C(X)\big) \longrightarrow C(X, \mathbf{Z})$ defined by dim is a surjective homomorphism. So we have the short exact sequence

$$0 \longrightarrow \ker(d) \longrightarrow K_0\big(C(X)\big) \longrightarrow C(X, \mathbf{Z}) \longrightarrow 0$$

Let $\phi: C(X) \longrightarrow A$ be a homomorphism, where A is a unital C^*-algebra.

2.2. Regard S^2 as consisting of two copies of D_1 and D_2 of the unit disk D with the boundaries identified. So a function in $C(S^2)$ is a pair (f, g) of functions f, $g \in C(D)$ such that $f(z) = g(z)$, if $|z| = 1$. Let

$$P(z) = \left(\begin{pmatrix} |z|^2 & \bar{z}(1 - |z|^2)^{1/2} \\ z(1 - |z|^2)^{1/2} & 1 - |z|^2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right).$$

It is easy to see that $P(z)$ is a projection in $C(S^2) \otimes M_2$. Let L be the Hopf line bundle on S^2. Then L can be defined by the projection P. Let $\phi: C(S^2) \longrightarrow A$ be a homomorphism. Suppose that $\phi_*^{(0)}|_{\ker d} = 0$. If A has stable rank one, there is a partial isometry $V \in M_2(A)$ such that

$$VV^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } V^*V = P,$$

where $P = \phi^{(2)}\big(P(z)\big)$ and $\psi^{(2)}: C(S^2) \otimes M_2 \longrightarrow M_2(A)$ is the map induced by ϕ. Let $[L] \cdot \phi$ be the map from $C(S^2)$ into $PM_2(A)P$ defined by

$$[L] \cdot \phi(f) = \begin{pmatrix} \phi(f) & 0 \\ 0 & \phi(f) \end{pmatrix} \cdot P, \quad f \in C(S^2).$$

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(Note that P commutes with $\begin{pmatrix} \phi(f) & 0 \\ 0 & \phi(f) \end{pmatrix}$. Notice that $V[L]\cdot\phi V^*$ defines a homomorphism ψ from $C(S^2)$ into A. It is easy to check that

$$V = \begin{pmatrix} t_1 & s_1 \\ 0 & 0 \end{pmatrix} \in M_2(A).$$

Set

$$t_2 = t_1\phi\big((\bar{z},\bar{z})\big) + s_1\phi\Big(\big((1-|z|^2)^{1/2},0\big)\Big).$$

LEMMA 2.3.

(1) $t_1^*t_1 = \phi\big((|z|^2,1)\big)$,
(2) $t_1^*s_1 = \phi\Big(\big(\bar{z}(1-|z|^2)^{1/2},0\big)\Big)$,
(3) $s_1^*s_1 = \phi\big((1-|z|^2,0)\big)$,
(4) $s_1^*t_1 = \phi\Big(\big(z(1-|z|^2)^{1/2},0\big)\Big)$,
(5) $t_1t_1^* = \psi\big((|z|^2,1)\big)$,
(6) $t_2^*t_2 = \phi\big((1,|z|^2)\big)$
(7) $t_2^*t_1 = \phi\big((z,z)\big)$,
(8) $t_1t_2^* = \psi\big((z,z)\big)$ *and*
(9) $t_2t_2^* = \psi\big((1,|z|^2)\big)$,
(10) $s_1s_1^* = \psi\big((1-|z|^2,0)\big)$,
(11) $t_1\phi\big((f,g)\big) = \psi\big((f,g)\big)t_1$ *for all* $(f,g)\in C(S^2)$,
(12) $s_1\phi\big((f,g)\big) = \psi\big((f,g)\big)s_1$ *for all* $(f,g)\in C(S^2)$.

PROOF. The first four equations follow immediately from the fact that $V^*V = P$. We compute

$$\begin{aligned} t_2^*t_1 &= \Big[\phi\big((z,z)\big)t_1^* + \phi\big((1-|z|^2)^{1/2},0\big)s_1^*\Big]t_1 \\ &= \phi\big((z,z)(|z|^2,1)\big) + \phi\Big(\big((1-|z|^2)^{1/2}z(1-|z|^2)^{1/2},0\big)\Big) \\ &= \phi\big((z,z)\big) \end{aligned}$$

and

$$\begin{aligned} t_2^*t_2 &= \Big[\phi\big((z,z)\big)t_1^* + \phi\Big(\big((1-|z|^2)^{1/2},0\big)\Big)s_1^*\Big]\Big[t_1\phi\big((\bar{z},\bar{z})\big) \\ &\qquad + s_1\phi\Big(\big((1-|z|^2)^{1/2},0\big)\Big)\Big] \\ &= \phi(|z|^2,|z|^2) + \phi\Big(\big(|z|^2(1-|z|^2),0\big)\Big) + \phi\Big(\big(|z|^2(1-|z|^2),0\big)\Big) \\ &\qquad + \phi\Big(\big((1-|z|^2)^2,0\big)\Big) \\ &= \phi\big((1,|z|^2)\big). \end{aligned}$$

These computations show (7) and (6). For (11), we notice that

$$t_1t_1^* + s_1s_1^* = 1$$

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and

$$V\left(\begin{pmatrix} \phi\big((|z|,1)\big) & 0 \\ 0 & \phi\big((|z|,1)\big) \end{pmatrix}\right)V^* = \left(\begin{pmatrix} t_1\phi(|z|^2,1)t_1^* + s_1\phi\big((|z|,1)\big)s_1^* & 0 \\ 0 & 0 \end{pmatrix}\right).$$

So

$$\psi\big((|z|^2,1)\big) = t_1\phi\big((|z|^2,1)\big)t_1^* + s_1\phi\big((|z|^2,1)\big)s_1^*$$

Therefore

$$\begin{aligned}\psi\big((f,g)\big)t_1 &= \Big(t_1\phi\big((|z|^2,1)\big)t_1^* + s_1\phi\big((|z|^2,1)\big)s_1^*\Big)t_1 \\ &= t_1\phi\big((|z|^2,1)\big)\phi\big((|z|^2,1)\big) + s_1\phi\big((|z|^2,1)\big)\phi\Big(\big(z(1-|z|^2)^{1/2},0\big)\Big) \\ &= (t_1t_1^*t_1 + s_1s_1^*t_1)\phi\big((f,g)\big) \\ &= (t_1t_1^* + s_1s_1^*)t_1\phi\big((f,g)\big) = t_1\phi\big((f,g)\big).\end{aligned}$$

Similarly, we can show (12).

Let $p_{t_1t_1^*}$ be the range projection of $t_1t_1^*$ in A^{**}. From (11), we have

$$t_1t_1^*t_1 = \psi\big((|z|^2,1)\big)t_1.$$

This implies that

$$t_1t_1^* = t_1t_1^*p_{t_1t_1^*} = \psi\big((|z|^2,1)\big)p_{t_1t_1^*}.$$

It follows that $p_{t_1t_1^*}$ commutes with $\psi\big((|z|^2,1)\big)$. Similarly, from (12), we have

$$s_1s_1^* = s_1s_1^*p_{s_1s_1^*} = \psi\big((1-|z|^2,0)\big)p_{s_1s_1^*}$$

and $\psi\big((1-|z|^2,0)\big)$ commutes with $p_{s_1s_1^*}$, where $p_{s_1s_1^*}$ is the range projection of $s_1s_1^*$ in A^{**}. Therefore we have

$$\psi\big((|z|^2,1)\big)p_{t_1t_1^*} + \psi\big((1-|z|^2,0)\big)p_{s_1s_1^*} = t_1t_1^* + s_1s_1^* = 1.$$

Since

$$\psi\big((|z|^2,1)\big)(1-p_{t_1t_1^*}) + \psi\big((1-|z|^2,0)\big)(1-p_{s_1s_1^*}) \geq 0,$$

and

$$\psi\big((|z|^2,1)\big) + \psi\big((1-|z|^2,0)\big) = 1,$$

we conclude that

$$\psi\big((|z|^2,1)\big)p_{t_1t_1^*} = \psi(|z|^2,1) \text{ and } \psi\big((1-|z|^2,0)p_{s_1s_1^*} = \psi(1-|z|^2,0)\big).$$

So we have proved (5) and (10). Others follow similar computation. ∎

The proof of Theorem 2.12 is an approximative version of [Ln12, 2.4] (see also [BDF2, 6.6]). If t_1 has a polar decomposition $t_1 = u|t_1|$ in A, then the condition $\phi_*^{(0)}(\ker d) = 0$ implies that the unitary u commutes with $\phi\big(C(S^2)\big)$. So the C^*-algebra generated by u and $\phi\big(C(S^2)\big)$ is isomorphic to $C(X)$ for some compact Hausdorff space X and in $C(X)$ the sphere breaks down. Here we have to use an approximative form of this idea carefully. We also need an absorption method used in [Ln10].

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LEMMA 2.4. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if there is a unitary* $u \in A$ *such that*

$$\|u|t_1| - t_1\| < \delta,$$

then

$$\|u^*\psi(f_i, g_i)u - \phi(f_i, g_i)\| < 4\|(f, g)\|\varepsilon.$$

for all $(f, g) \in C(S^2)$.

PROOF. If $\delta(< \varepsilon)$ is small enough,

$$\|u|t_1|u^* - (t_1t_1^*)^{1/2}\| < \varepsilon.$$

Then, by 2.3 (5),

$$\Big\|u\phi\big((|z|, 1)\big) - \psi\big((|z|, 1)\big)u\Big\| < 2\varepsilon.$$

By 2.3 (11),

$$\begin{aligned}\Big\|u\phi\big((|z|f, g)\big) - \psi\big((|z|f, g)\big)u\Big\| &\le \Big\|u\phi\big((|z|f, g)\big) - t_1\phi\big((f, g)\big)\Big\| \\ &\quad + \Big\|\psi\big((f, g)\big)t_1 - \psi\big((f, g)\big)u|t_1|\Big\| \\ &\quad + \Big\|\psi\big((f, g)\big)u|t_1| - \psi((|z|f, g))u\Big\| \\ &< \|(f, g)\|\varepsilon + \|(f, g)\|\varepsilon + 2\|(f, g)\|\varepsilon \\ &= 4\|(f, g)\|\varepsilon. \qquad \blacksquare\end{aligned}$$

2.5. Let $Z = (z, z)$ and $H = (1 - |z|, |z| - 1)$. Then $C(S^2)$ is generated by Z and H.

LEMMA 2.6. *For any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if there is a unitary* $u \in A$ *with*

$$\|u|t_1| - t_1\| < \delta,$$

then

$$\|Yf(|Y|) - \phi(Z)\| < \varepsilon \text{ and } \||Y| - 1 - \phi(H)\| < \varepsilon,$$

where $Y = t_2^* u(2 - |t_1|)$ *and*

$$f = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ t^{-1}(2 - t) & \text{if } 1 \le t \le 2 \end{cases}$$

PROOF. It follows from 2.4 that, if δ is small enough,

$$\begin{aligned}\Big\|(Y^*Y) - \phi\big((2 - |z|, |z|)^2\big)\Big\|&\|(2 - |t_1|)u^*t_2t_2^*u(2 - |t_1|) - (2 - |t_1|)t_2^*t_2(2 - |t_1|)\| \\ &< 8\|u|t_1|u^* - (t_1t_1^*)^{1/2}\|.\end{aligned}$$

We also have $f\Big(\phi\big((2 - |z|, |z|)\big)\Big) = \phi\Big(\big((2 - |z|)^{-1}|z|, 1\big)\Big)$ and

$$\Big\|Yf\Big(\phi\big((2 - |z|, |z|)\big)\Big) - Z\Big\| = \|t_2^*u|t_1| - t_2^*t_1\| < \delta.$$

Therefore if δ is small enough,

$$\|Yf(|Y|) - Z\| < \varepsilon \text{ and } \|\,|Y| - 1 - H\| < \varepsilon. \qquad \blacksquare$$

The following lemma is certainly known even though the exact form may not be found in literature.

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LEMMA 2.7. *For any element x in a C^*-algebra A with $\|x\| < r$, and an analytic function f on D_r, where*

$$D_r = \{\xi : |\xi| < r\},$$

we have

$$\|\operatorname{Re} f(x)\| \le \|\operatorname{Re} f\|_{D_r} \text{ and } \|\operatorname{Im} f(x)\| \le \|\operatorname{Im} f\|_{D_r}.$$

PROOF. We may assume that $\|x\| \le 1$. It is known (see [S]) that there is a C^*-algebra B containing A such that there is a projection $e \in B$ which satisfies the condition that $ea = ae = a$ for all $a \in A$, and there is a unitary $u \in B$ such that

$$x^n = eu^n e \text{ and } (x^*)^n = e(u^*)^n e$$

for all positive integer n. Then

$$\|\operatorname{Re} f(x)\| \le \left\|e\big(\operatorname{Re} f(u)\big)e\right\| \le \|\operatorname{Re} f(u)\| \le \|\operatorname{Re} f\|_{D_r}.$$

Similarly,

$$\|\operatorname{Im} f(x)\| \le \|\operatorname{Im} f\|_{D_r}.$$

∎

LEMMA 2.8. *For any $\epsilon > 0$ and $r > 0$, there exist $\delta > 0$ and an integer k satisfying the following: If x is an element in a C^*-algebra A with $\|x\| \le r$ such that*

$$\|x^*x - xx^*\| < \delta$$

then there are normal elements $y \in M_k(A)$ and $z \in M_{k+1}(A)$ with finite spectrum and $\|y\|, \|z\| \le r$ such that

$$\|x \oplus y - z\| < \epsilon.$$

PROOF. Set

$$S = \{\lambda : |\operatorname{Re}\lambda| < 1/2, |\operatorname{Im}\lambda| < 1/2\}.$$

For any $\varepsilon > 0$, by the Riemann mapping theorem, there is a conformal mapping f from $D_{r+\varepsilon/2}$ onto S. There is $\eta > 0$ such that

$$f(D_r) \subset \{\xi : |\operatorname{Re}(\xi)| < 1/2 - \eta, \quad |\operatorname{Im}(\xi)| < 1/2 - \eta\}.$$

It follows from 2.7 that

$$\left\|\operatorname{Re}\big(f(x)\big)\right\| \le \|\operatorname{Re}(f)\|_{D_{r+\varepsilon/2}}$$

and

$$\left\|\operatorname{Im}\big(f(x)\big)\right\| \le \|\operatorname{Im}(f)\|_{D_{r+\varepsilon/2}}.$$

By Lemma 1.4, for any $\sigma > 0$, if δ is small enough, there exist an integer k (which does not depend on A or x but does depend on r and ε), and normal elements $y_1 \in M_k(A)$ and $z_1 \in M_{k+1}(A)$ with finite spectra $\operatorname{sp}(y_1)$, $\operatorname{sp}(z_1) \subset S$ such that

$$\|f(x) \oplus y_1 - z_1\| < \sigma.$$

If σ is small enough,

$$\|x \oplus f^{-1}(y_1) - f^{-1}(z_1)\| < \varepsilon/2.$$

Since $\operatorname{sp}\big(f^{-1}(y_1)\big)$ $\operatorname{sp}\big(f^{-1}(z_1)\big) \in D_{r+\varepsilon/2}$, by changing the spectrum of $f^{-1}(y_1)$ and $f^{-1}(z_1)$ slightly (within $\varepsilon/2$), we obtain normal elements y and z as required. ∎

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LEMMA 2.9. *In 2.6, there is $\eta > 0$ such that if there is a projection $q \in A$ such that*

$$\|qY - Yq\| < \eta,$$

then there are an integer L, mutually orthogonal projections $q_1, q_2, \ldots, q_n$ in $M_L(qAq)$ and mutually orthogonal projections $d_1, d_2, \ldots, d_m$ in $M_{L+1}(qAq)$ and points $\xi_1, \xi_2, \ldots, \xi_n$ and $\zeta_1, \zeta_2, \ldots, \zeta_m \in S^2$ such that

$$\Big\| q\phi(Z)q \oplus \sum_{i=1}^{n} Z(\xi_i)q_i - \sum_{j=1}^{m} Z(\zeta_j)d_j \Big\| < 2\varepsilon$$

and

$$\Big\| q\phi(H)q \oplus \sum_{i=1}^{n} H(\xi_i)q_i - \sum_{j=1}^{m} H(\zeta_j)d_j \Big\| < 2\varepsilon.$$

PROOF. By 2.3 (7), we have

$$Y = 2t_2^* u - t_2^* t_1 = 2t_2^* u - \phi\big((z,z)\big).$$

Therefore,

$$\begin{aligned}\|Y^*Y - YY^*\| &= 4\|t_2^* uu^* t_2 - u^* t_2 t_2^* u\| \\ &= 4\Big\| t_2^* t_2 - u^* \psi\big((1, |z|)\big) u \Big\|.\end{aligned}$$

It follows from 2.3 and 2.4 that if δ is small enough,

$$\|Y^*Y - YY^*\| \text{ is small enough.}$$

Thus if η is small enough,

$$\|(qYq)^* qYq - qYq(qYq)^*\|$$

and

$$\|qf(|Y|) - f(|Y|)q\|$$

are small enough. It follows from 2.8 that for any $\sigma > 0$, there are complex numbers α_i and β_j with $|\alpha_i|$, $|\beta_j| \le 2$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ and projections q_i and d_j as described such that

$$\Big\| qYq \oplus \sum_{i=1}^{n} \alpha_i q_i - \sum_{j=1}^{m} \beta_j d_j \Big\| < \sigma.$$

If η is small enough, then

$$\|qYf(|Y|)q - qYqf(|qYq|)\|$$

is small. So if η is small enough, we have

$$\Big\| qYf(|Y|)q \oplus \sum_{i=1}^{n} \alpha_i f(|\alpha|_i) q_i - \sum_{j=1}^{m} \beta_j f(|\beta_j|) d_j \Big\| < \sigma.$$

Therefore

$$\Big\| q\phi(Z)q \oplus \sum_{i=1}^{n} \alpha_i f(|\alpha_i|) q_i - \sum_{j=1}^{m} \beta_j f(|\beta_j|) d_j \Big\| < \varepsilon + \sigma$$

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and

$$\Big\| q\phi(H)q \oplus \sum_{j=1}^{m}(|\alpha_i|-1)q_i - \sum_{j=1}^{m}(|\beta_j|-1)d_j \Big\| < \varepsilon + \sigma.$$

We will again identify S^2 with two copies of D_1 and D_2 of the unit disk with the boundaries identified. We now map

$$\Omega = \{\lambda : |\lambda| \le 2\}$$

onto S^2. First, we map $\{\lambda : |\lambda| \le 1\}$ to D_1. Then we map $\{\lambda : 1 \le |\lambda| \le 2\}$ onto D_2 by the formular

$$\lambda f(|\lambda|) = \lambda(2-|\lambda|)/|\lambda|.$$

It then follows easily that there are $\xi_i, \zeta_j \in S^2$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ such that

$$\Big\| q\phi(Z)q \oplus \sum_{i=1}^{n} Z(\xi_i)q_i - \sum_{j=1}^{m} Z(\zeta_i)d_j \Big\| < 2\varepsilon$$

and

$$\Big\| q\phi(H)q \oplus \sum_{j=1}^{m} H(\xi_i)q_i - \sum_{j=1}^{m} H(\zeta_j)d_j \Big\| < 2\varepsilon$$

if σ is small enough. ■

LEMMA 2.10 ([LN10, 4.8] AND [EGLP, 4.4]). *Let A be a separable simple unital C^*-algebra of real rank zero, stable rank one and with weakly unperforated $K_0(A)$ of countable rank. Suppose that $\phi: C(X) \longrightarrow A$ is a monomorphism, where X is a compact metric space. For any $\epsilon > 0$, any finitely many $f_1, f_2, \ldots, f_m \in C(X)$ and any integer $K > 0$ there are mutually orthogonal projections $p_1, p_2, \ldots, p_n$ in A and $\lambda_1, \lambda_2, \ldots, \lambda_n \in X$ such that*

$$\Big\| \phi(f_s) - \Big(y_s + \sum_{i=1}^{n} f_s(\lambda_i)p_i\Big)\Big\| < \epsilon, \quad s = 1, 2, \ldots, m$$

where $y_s = (1-\sum_{i=1}^{n} p_i)\phi(f_s)(1-\sum_{i=1}^{n} p_i)$,

$$\Big\| \Big(1-\sum_{i=1}^{n} p_i\Big)\phi(f_s) - \phi(f_s)\Big(1-\sum_{i=1}^{n} p_i\Big)\Big\| < \epsilon, \text{ and } [p_k] > K\Big[1-\sum_{i=1}^{n} p_i\Big],$$

$s = 1, 2, \ldots, m$, $k = 1, 2, \ldots, n$, and for any $\lambda \in X$, there is λ_i such that

$$\operatorname{dist}(\lambda, \lambda_i) < \epsilon.$$

2.11. In Lemma 2.10, let $X = S^2$ and $q = 1 - \sum_{i=1}^{n} p_i$. Suppose that

$$f_1(z) = (|z|^2, 1), \quad f_2(z) = \big(\bar{z}(1-|z|^2)^{1/2}, 0\big)$$
$$f_3(z) = \big(z(1-|z|^2)^{1/2}, 0\big), \quad f_4(z) = (1-|z|^2, 0)$$

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and $m > 4$. We also assume that $n = 2L$ and $\lambda_i \in D_1$, $i = 1, 2, \ldots, L$ and $\lambda_i \in D_2$, $i = L+1, L+2, \ldots, 2L$. Set

$$P_0 = \begin{pmatrix} \sum_{i=1}^{2L} f_1(\lambda_i)p_i & \sum_{i=1}^{2L} f_2(\lambda_i)p_i \\ \sum_{i=1}^{2L} f_3(\lambda_i)p_i & \sum_{i=1}^{2L} f_4(\lambda_i)p_i \end{pmatrix}.$$

For any $\sigma > 0$, if ε is small enough, then

$$\|P_0P - P_0\| < \sigma \text{ and } \left\|(P - P_0)\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} - (P - P_0)\right\| < \sigma.$$

Set

$$V_0 = \begin{pmatrix} \sum_{i=1}^{L} \lambda_i p_i + \sum_{i=L+1}^{2L} p_i & \sum_{i=1}^{2L}(1 - |\lambda_i|^2)^{1/2} p_i \\ 0 & 0 \end{pmatrix}.$$

Then

$$V_0V_0^* = \begin{pmatrix} 1-q & 0 \\ 0 & 0 \end{pmatrix} \text{ and } V_0^*V_0 = P_0.$$

There is a unitary $W \in M_2(A)$ such that

$$\|1 - W\| < 2\sigma \text{ and } W^*P_0W \leq P.$$

Let $\bar{V}_0 = V_0W$ and $W^*P_0W = \bar{P}_0$. Then

$$\|\bar{V}_0 - V_0\| < 2\sigma.$$

We may assume that

$$\bar{V}_0 = \begin{pmatrix} t_1^0 & s_1^0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\|(1-q)t_1^0 - t^0(1-q)\| < 2\sigma \text{ and } \|(1-q)s_1^0 - s_1^0(1-q)\| < 2\sigma.$$

Since A has stable rank one, there is partial isometry $V_1 \in M_2(A)$ such that

$$V_1^*V_1 = P - \bar{P}_0 \text{ and } V_1V_1^* = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}.$$

We may assume that

$$V_1 = \begin{pmatrix} t_1' & s_1' \\ 0 & 0 \end{pmatrix}.$$

Since $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} V_1 = V_1$, we may also assume that

$$qt_1' = t_1' \text{ and } qs_1' = s_1'.$$

Since

$$\left\|(P - P_0)\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} - (P - P_0)\right\| < \sigma,$$
$$\|(P - P_0) - (P - \bar{P}_0)\| < 2\sigma$$

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and

$$V_1(P - \overline{P}_0) = V_1,$$

we see that

$$\left\| V_1 \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} - V_1 \right\| < 4\sigma.$$

Therefore,

$$\|t_1'(1-q)\| < 4\sigma \text{ and } \|s_1'(1-q)\| < 4\sigma.$$

Set $V' = \overline{V}_0 + V_1$. Then

$$V'V'^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } V'^*V' = P.$$

We also have

$$V' = \begin{pmatrix} \overline{t_1} & \overline{s_1} \\ 0 & 0 \end{pmatrix},$$

where $\overline{t_1} = t_1^0 + t_1'$ and $\overline{s_1} = s_1^0 + s_1'$. From above, we have

$$\|q\overline{t_1} - \overline{t_1}q\| < 6\sigma.$$

THEOREM 2.12. *Let A be a simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(A)$ of countable rank. Suppose that $\phi\colon C(S^2) \longrightarrow A$ is a monomorphism. If $\phi_*\big(\ker(d)\big) = 0$, then for any $\varepsilon > 0$ and $f_1, f_2, \ldots, f_n \in C(S^2)$ there is a homomorphism $\Phi\colon C(S^2) \longrightarrow A$ with finite dimensional range such that*

$$\|\phi(f_i) - \Phi(f_i)\| < \varepsilon,$$

$i = 1, 2, \ldots, n$.

PROOF. Note since $C(S^2)$ is unital, by considering $\phi\big(1_{C(S^2)}\big)A\phi\big(1_{C(S^2)}\big)$, we may assume that A is unital. Let Z and H be as in 2.5. Since $C(S^2)$ is generated by Z and H, it is enough to show the case that $n = 2, f_1 = Z$ and $f_2 = H$. By applying 2.10, we have for any $\epsilon_1 > 0$, $g_1, g_2, \ldots, g_m \in C(S^2)$ and an integer $K > 0$ there are mutually orthogonal projections $p_1, p_2, \ldots, p_n$ in A and $\lambda_1, \lambda_2, \ldots, \lambda_{n'} \in S^2$ such that

$$\left\| \phi(g_s) - \Big(y_s + \sum_{i=1}^{n'} g_s(\lambda_i) p_i \Big) \right\| < \epsilon_1, \quad s = 1, 2, \ldots, m$$

where $y_s = (1 - \sum_{i=1}^{n'} p_i)\phi(g_s)(1 - \sum_{i=1}^{n'} p_i)$,

$$\left\| \Big(1 - \sum_{i=1}^{n'} p_i\Big)\phi(g_s) - \phi(g_s)\Big(1 - \sum_{i=1}^{n'} p_i\Big) \right\| < \epsilon_1,$$

$$[p_k] > K\Big[1 - \sum_{i=1}^{n'} p_i\Big],$$

$s = 1, 2, \ldots, m, j = 1, 2, \ldots, l$ and for any $\lambda \in S^2$, there is λ_i such that

$$\operatorname{dist}(\lambda, \lambda_i) < \epsilon_1.$$

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Here we assume that

$$g_1 = Z,\ g_2 = H, g_3 = (|z|, 1),\ g_4 = (1, |z|),$$
$$g_5 = (\bar{z}, \bar{z}),\ g_6 = \big((1-|z|^2)^{1/2}, 0\big),\ g_7 = (|z|^2, 1),$$
$$g_8 = \big(\bar{z}(1-|z|^2)^{1/2}, 0\big),\ g_9 = \big(z(1-|z|^2)^{1/2}, 0\big),\ g_{10} = (1-|z|^2, 0)$$

and $m \geq 10$. Let $q = 1 - \sum_{i=1}^{n} p_i$. By 2.11, we may assume that $t_1 = \overline{t_1}$ and $s_1 = \overline{s_1}$ ($\overline{s_1}$, $\overline{t_1}$ are as in 2.11) so that t_1 and s_1 almost commute with q. Since A has stable rank one, there is an invertible element $a \in A$ such that $\|a - t_1\|$ is small. Suppose that $a = u|a|$, where u is a unitary in A. We may assume that

$$\|u|t_1| - t_1\| < \delta \quad (\delta \text{ as in } 2.6).$$

Furthermore, since $\|qt_1 - t_1 q\|$ is small, we may also assume that

$$\|qu - uq\|$$

is small. Let $Y = t_2^* u(2 - |t_1|)$ be as in 2.6. Since $\|qt_1 - t_1 q\|$, $\|qu - uq\|$ and $\|qt_2 - t_2 q\|$ are small, we may assume that they are small enough (this requires ε_1 is small enough) such that

$$\|qY - Yq\| < \eta \quad (\eta \text{ as in } 2.9).$$

Now we can apply 2.9. So there are mutually orthogonal projections $q_1, q_2, \ldots, q_N \in M_L(qAq)$, $d_1, d_2, \ldots, d_M \in M_{L+1}(qAq)$ and points $\xi_1, \xi_2, \ldots, \xi_N, \zeta_1, \zeta_2, \ldots, \zeta_M \in S^2$ such that

$$\Big\| q\phi(Z)q \oplus \sum_{i=1}^{N} Z(\xi_i)q_i - \sum_{j=1}^{M} Z(\zeta_j)d_j \Big\| < 2\varepsilon$$
$$\Big\| q\phi(H)q \oplus \sum_{i=1}^{N} H(\xi_i)q_i - \sum_{j=1}^{M} Z(\zeta_j)d_j \Big\| < 2\varepsilon.$$

Notice that the number L depends on ε alone and K is arbitrary (K does not depend on ε_1), we may assume that $L \leq K$. Without loss of generality, one may also assume that $N = n'$ and $\lambda_i = \xi_i$. Since $L \leq K$, there is a partial isometry

$$v \in \big((1-q) \oplus q \oplus \cdots \oplus q\big) M_{L+1}(A) \big((1-q) \oplus q \oplus \cdots \oplus q\big)$$

(there are L copies of q) such that

$$v^* q_i v \leq p_i, \quad i = 1, 2, \ldots, N,$$
$$v^* v = \sum_{i=1}^{N} p_i', \text{ and } vv^* = q \oplus \cdots \oplus q$$

(there are L copies of p), where $p_i' = v^* q_i v$. Set

$$u = q \oplus v.$$

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Notice that

$$u^*\Big(y_s + \sum_{i=1}^{N} g_s(\xi)q_i\Big)u = \sum_{i=1}^{n'} g_s(\lambda_i)p_l' \oplus y_s, \quad (n' = N, \lambda_i = \xi_i),$$

where $y_s = q\phi(g_s)$, $s = 1, 2$ and uu^* is the identity of $M_{L+1}(qAq)$. Now we have

$$\Big\| y_s + \sum_{i=1}^{M} g_s(\lambda_i)p_i - \sum_{i=1}^{L} g_s(\lambda_i)(p_i - p_i') - u^*\Big(\sum_{j=1}^{M} g_s(\zeta_j)d_j\Big)u\Big\| < \varepsilon_1 + 2\varepsilon,$$

$s = 1, 2$. Therefore

$$\Big\|\phi(g_s) - \sum_{i=1}^{N} g_s(\lambda_i)(p_i - p_i') \oplus \sum_{j=1}^{M} g_s(\zeta_j)u^*d_ju\Big\| < 2\varepsilon_1 + 2\varepsilon$$

$s = 1, 2$. Notice that $g_1 = Z$ and $g_2 = H$. ∎

REMARK 2.13. The condition $\phi_*(\ker d) = 0$ is also necessary in 2.12. In fact, if ϕ can be approximated by homomorphisms with finite dimensional range, then

$$\|\psi^{(2)}(P) - \psi^{(2)}(P)\|$$

is small, where $\psi\colon C(S^2) \longrightarrow A$ has finite dimensional range. Since $\psi^{(2)}(P)$ is equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\phi^{(2)}(P)$ is also equivalent to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore

$$\phi_*^{(0)}(\ker d) = 0.$$

DEFINITION 2.14. An element s in an ordered group (G, G_+, u) with order unit is said be an *infinitesimal* element if $\tau(s) = 0$ for all states τ on (G, G_+, u). We denote by $\inf(G)$ the set of infinitesimal elements of G. Clearly, $\inf(G)$ is a subgroup. If X is a compact metric space, then

$$\inf\Big(K_0\big(C(X)\big)\Big) = \ker d.$$

Let $\phi\colon C(X) \longrightarrow A$ be a homomorphism. Suppose that $K_0(A)$ is an ordered group with unit (this is always the case if A is unital and stably finite). Then it is easy to see that

$$\phi_*^{(0)}(\ker d) \subset \inf\big(K_0(A)\big).$$

COROLLARY 2.15. *Let A be a simple AF-algebra with countably many extreme traces. Then a homomorphism $\phi\colon C(S^2) \longrightarrow A$ can be approximated pointwise by homomorphisms from $C(S^2)$ into A with finite dimensional range if and only if*

$$\phi_*^{(0)}(\ker d) = 0.$$

PROOF. If ϕ is not injective, then $\operatorname{im}\phi \cong C(F)$, where F is a compact subset of the plane. So there is a normal element $x \in \operatorname{im}\phi$ generates $\operatorname{im}\phi$. So $\phi_*^{(0)}(\ker d) = 0$. It is

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enough to show that x can be approximated by normal elements (in A) with finite spectrum. But this follows from [Ln9, Theorem A]. When ϕ is a monomorphism, 2.15 follows from 2.12 and 2.13.

REMARK 2.16. If A is a matroid algebra, or A is a UHF-algebra, then $\inf\big(K_0(A)\big) = \{0\}$. Thus every homomorphism from $C(S^2)$ into A is approximated by homomorphisms with finite dimensional range.

REMARK 2.17. Notice that if F is a proper compact subset of S^2, then F is homeomorphic to a compact subset of the plane. It follows from [Ln10, 4.9] that 2.12 holds for $X = F$, if we replace the condition $\phi_*^{(0)}(\ker d) = 0$ by $\phi_*^{(1)} = 0$.

3. **The 2-torus $S^1 \times S^1$.**

LEMMA 3.1. *For any $\varepsilon > 0$ and a compact subset of the plane X, there are $0 < \sigma < \varepsilon$, $\delta > 0$ and $d > 0$ satisfying the following. Let A be a simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(A)$ of countable rank, and x be a normal element in A. If $p \in A$ is a projection,*

$$\|px - xp\| < \delta,$$
$$\lambda p - pxp \in \mathrm{Inv}_0(pAp) \text{ for } \lambda \notin X$$

and $p_\sigma \le p \le p_d$, where p_σ is the spectral projection in A^{**} corresponding to

$$X_{\sigma/2} = \big\{\xi : \mathrm{dist}(\xi, X) < \sigma/2\big\}$$

and p_d is the spectral projection in A^{**} corresponding to

$$X_{\sigma/2+d} = \{\xi : \mathrm{dist}(\xi, X) < \sigma/2 + d\},$$

then there is a normal element $y \in pAp$ with finite spectrum $\mathrm{sp}(y) \subset X$ such that

$$\|pxp - y\| < \varepsilon.$$

PROOF. The proof is a combination of 3.12, 4.8 and 4.9 of [Ln10]. Let $\Omega = X \cup \mathrm{sp}(x)$. By functional calculus, we have a homomorphism $\phi: C(\Omega) \longrightarrow C\big(\mathrm{sp}(x)\big) \longrightarrow A$. We will use the similar notation used in the proof of 4.8 in [Ln10]. Using the argument of 4.8 of [Ln10], we obtain $\sigma > 0$ and an one-dimensional CW complex $S \subset X_{\sigma/2}$ such that

$$\big\{\xi : \mathrm{dist}(\xi, X) = \sigma/4\big\} \subset S,$$

$\mu_\tau(S) = 0$ for every $\tau \in \Delta$ and $X_{\sigma/2} \setminus S$ is a finite union of open subsets $O_1, O_2, \ldots, O_N$, where

$$\mathrm{diam}(O_i) < \varepsilon/4.$$

Let B_{O_i} be the hereditary C^*-subalgebra corresponding to the open subset O_i (see the notation in 4.8 of [Ln10]), $i = 1, 2, \ldots, N$, and let $B_{O_{N+1}}$ be the hereditary C^*-subalgebra

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corresponding to the open subset $\mathrm{sp}(x)\setminus \bar{X}_{\sigma/2}$. As in 4.8 of [Ln10], for any given integer L, there are projections $p_i \in B_{O_i}$ such that

$$\tau(p_i) > (L+1)\tau\Big(1-\sum_{n=1}^{N} p_i\Big), \quad i=1,2,\ldots,N+1, \tau\in\Delta.$$

There is $d>0$ such that

$$\mu_\tau(Y_d) < (1/L+1)\tau(p_i), \quad i=1,2,\ldots,N,$$

where

$$Y_d = \big\{\xi : \mathrm{dist}(\xi, X) < \sigma/2 + d\big\} \setminus X_{\sigma/2}.$$

If $p_\sigma \le p \le p_d$, $p \ge p_\sigma \ge p_i$. So, by 4.3 of [Ln10],

$$\Big\| pxp - \Big(\sum_{i=1}^{N} \lambda_i p_i + \Big(p - \sum_{i=1}^{N} p_i\Big)x\Big(p - \sum_{n=1}^{N} p_i\Big)\Big)\Big\| < \varepsilon/2,$$

where $\lambda_i \in O_i$, $i=1,2,\ldots,N$. By 3.12 of [Ln10], there are normal element $y' \in M_K\big((p-\sum_{i=1}^N p_i)A(p-\sum_{i=1}^N p_i)\big)$ and normal element $z' \in M_k\big((p-\sum_{i=1}^N p_i)A(p-\sum_{i=1}^n p_i)\big)$ with finite spectrum $\mathrm{sp}(y')$, $\mathrm{sp}(z') \subset X_{\sigma/2+d}$ such that

$$\Big\|\Big(p-\sum_{i=1}^{N} p_i\Big)x\Big(p-\sum_{i=1}^{N} p_i\Big) \oplus y' - z'\Big\| < \varepsilon/4,$$

provided that δ is small enough (Note δ does not depend on A). Notice that K depends only $X_{\sigma/2+d}$ and ε. Since

$$(L+1)\tau\Big(p-\sum_{n=1}^{N} p_i\Big) < \tau(p_i), \quad i=1,2,\ldots,N, \tau\in\Delta$$

and L is any given number, as in 4.8 and 4.9 of [Ln10], an absorption argument shows that there is a normal element $y\in pAp$ with finite spectrum $\mathrm{sp}(y)\subset X$ such that

$$\Big\|\Big[\sum_{n=1}^{N} \lambda_i p_i + \Big(p-\sum_{n=1}^{N} p_i\Big)x\Big(p-\sum_{n=1}^{N} p_i\Big)\Big] - y\Big\| < \varepsilon/2.$$

Therefore

$$\|pxp - y\| < \varepsilon. \qquad \blacksquare$$

LEMMA 3.2. *Let X be a locally compact metric space, $G\subset X$ be an open subset,*

$$I = \big\{f\in C(X) : f(x) = 0 \text{ if } x\notin G\big\},$$

and A be any (unital) C^*-algebra. Suppose that $\phi\colon C_0(X)\longrightarrow A$ is a monomorphism and $\phi|_I$ is approximated by homomorphisms from $I\longrightarrow A$ with finite dimensional range. Then, for any $\varepsilon>0$, $\sigma>0$ and $f_1, f_2,\ldots,f_n\in C_0(X)$, there exists a projection $p\in A$ such that

$$q_\sigma \le p \le q_{\sigma/4}, \quad \|p\phi(f_i) - \phi(f_i)p\| < \varepsilon$$

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and

$$\left\| p\phi(f_i)p - \sum_{k=1}^{m} f_i(\xi_k)p_k \right\| < \varepsilon,$$

where q_σ and $q_{\sigma/4}$ are spectral projections in A^{**} corresponding to (the homomorphism ϕ and) the subset

$$\Omega_\sigma = \left\{ \xi \in G : \operatorname{dist}(\xi, X \setminus G) > \sigma \right\}$$

and

$$\Omega_{\sigma/4} = \left\{ \xi \in G : \operatorname{dist}(\xi, X \setminus G) > \sigma/4 \right\},$$

respectively, $\xi_k \in \Omega_\sigma$ and $\{p_k\}$ are mutually orthogonal projections in pAp.

PROOF. Let $F = X \setminus G$. For any positive number $d > 0$, denote by Ω_d the set

$$\left\{ \xi \in G : \operatorname{dist}(\xi, F) > d \right\}$$

and denote by q_d be the spectral projection of ϕ in A^{**} corresponding to the subset Ω_d.

We claim that there exists a projection $e \in A$ such that

$$q_{6\sigma/16} \leq e \leq q_{\sigma/4}.$$

Let $g_i \in C_0(X)$, $i = 0, 1, 2, 3, 4$ such that

$$0 \leq g_i \leq 1, \quad g_0(\xi) = 0 \text{ if } \xi \notin \Omega_{\sigma/4},$$
$$g_1(\xi) = 0 \text{ if } \xi \notin \Omega_{5\sigma/16}, \quad g_2(\xi) = 0 \text{ if } \xi \notin \Omega_{6\sigma/16},$$
$$g_3(\xi) = 0 \text{ if } \xi \notin \Omega_{7\sigma/16}, \quad g_4(\xi) = 0 \text{ if } \xi \notin \Omega_{\sigma/2},$$

and $g_0(\xi) = 1$ if $\xi \in \Omega_{5\sigma/16}$, $g_1(\xi) = 1$ if $\xi \in \Omega_{6\sigma/16}$, $g_2(\xi) = 1$ if $\xi \in \Omega_{7\sigma/16}$, $g_3(\xi) = 1$ if $\xi \in \Omega_{\sigma/2}$, $g_4(\xi) = 1$ if $\xi \in \Omega_{3\sigma/4}$. Note $g_i \in I$, $i = 0, 1, 2, 3$. For any $\eta > 0$, by our assumption, there are $\xi_1, \xi_2, \ldots, \xi_m \in G$ and mutually orthogonal projections $p_1, p_2, \ldots, p_m \in A$ such that

$$\left\| \phi(g_i) - \sum_{j=1}^{m} g_i(\xi_j)p_j \right\| < \eta, \quad i = 0, 1, 2, 3$$

and

$$\left\| \phi(g_k f_i) - \sum_{j=1}^{m} g_k f_i(\xi_j)p_j \right\| < \eta, \quad i = 1, 2, \ldots, n, \; k = 0, 1, 2.$$

Let $p_{5\sigma/16} = \sum_{\xi_j \in \Omega_{5\sigma/16}} p_j$. We obtain

$$\|\phi(g_0)p_{5\sigma/16} - p_{5\sigma/16}\| < \eta.$$

Thus

$$\|\phi(g_0)p_{5\sigma/16}\phi(g_0) - p_{5\sigma/16}\| < 2\eta.$$

Let B_0 be the hereditary C^*-subalgebra generated by $\phi(g_0)$. Then by [Eff, A8], there is a projection $e_0 \in B_0$ such that

$$\|e_0 - p_{5\sigma/16}\| < 4\eta.$$

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By [Eff, A8], there exists a unitary $v_0 \in A$ such that

$$\|v_0 - 1\| < 4\eta \text{ and } v_0^* p_{5\sigma/16} v_0 = e_0.$$

To save the notation, without loss of generality, by replacing p_j by $v_0^* p_j v_0$, we may assume that $p_{5\sigma/16} \le q_{\sigma/4}$.

Now we will use an argument of L. G. Brown (*cf.* [Bn]). We have

$$\begin{aligned}\left\|\phi(g_1)p_{5\sigma/16} - \phi(g_1)\right\| \le& \left\|\phi(g_1)p_{5\sigma/16} - \sum_{k=1}^{m} g_1(\xi_k)p_k p_{5\sigma/16}\right\| \\ &+ \left\|\sum_{k=1}^{m} g_1(\xi_k)p_k p_{5\sigma/16} - \sum_{k=1}^{m} g_1(\xi_k)p_k\right\| \\ &+ \left\|\sum_{k=1}^{m} g_1(\xi_k)p_k - \phi(g_1)\right\| < 2\eta.\end{aligned}$$

Working now in the unitized algebra $\tilde{B}_0$ (with 1 denoting the identity in $\tilde{B}_0$), we set $x = \big(1 - \phi(g_1)\big)^{1/2}(1 - p_{5\sigma/16})$. Then

$$\|(1 - p_{5\sigma/16}) - x^*x\| = \|(1 - p_{5\sigma/16})\phi(g_1)(1 - p_{5\sigma/16})\| < 2\eta.$$

So x^*x is invertible in $(1 - p_{5\sigma/16})\tilde{B}_0(1 - p_{5\sigma/16})$. Let $w = x|x|^{-1}$, where the inverse is taken in $(1 - p_{5\sigma/16})\tilde{B}_0(1 - p_{5\sigma/16})$. Then

$$w^*w = 1 - p_{5\sigma/16} \text{ and } ww^* = d$$

are projections in $\tilde{B}_0$. Moreover, $d \in \overline{(x\tilde{B}_0 x^*)}$. Since

$$\big(1 - \phi(g_1)\big)q_{6\sigma/16} = q_{6\sigma/16}\big(1 - \phi(g_1)\big) = 0,$$

we conclude that $d \le 1 - q_{6\sigma/16}$. From a direct computation, one sees that there is $b \in B_0$ such that $w = 1 + b$. It follows that $e = 1 - ww^* = 1 - d$ is a projection in B_0. Furthermore,

$$q_{6\sigma/16} \le e \le q_{\sigma/4}.$$

This proves the claim.

Let $p_{7\sigma/16} = \sum_{\xi_j \in \Omega_{7\sigma/16}} p_j$. As in the proof of the claim, we have

$$\|e p_{7\sigma/16} - p_{7\sigma/16}\| < \eta.$$

As in the proof of the claim, again, we may assume (by considering $v_1^* p_{7\sigma/16} v_1$ for a suitable unitary $v_1 \in A$) that $p_{7\sigma/16} \le e \le q_{\sigma/4}$.

Now we consider inequalities

$$\left\|\phi(g_i) - \sum_{k=1}^{m} g_i(\xi_k)p_k\right\| < \eta \text{ and } \left\|\phi(g_i f_j) - \sum_{k=1}^{m} (g_i f_j)(\xi_k)p_k\right\| < \eta,$$

$i = 3, 4, j = 1, 2, \ldots, n$. In these inequalities, we may assume that $\xi_k \in \Omega_{7\sigma/16}$ and $p_k \le p_{7\sigma/16} \le e$. Set $p_{\sigma/2} = \sum_{xi_j \in \Omega_{\sigma/2}} p_j$. We obtain

$$\|q_{3\sigma/4} - q_{3\sigma/4}p_{\sigma/2}\| < 2\eta \text{ and } \|q_{7\sigma/16}p_{\sigma/2} - p_{\sigma/2}\| < 2\eta.$$

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Note that the claim is true for any σ. By repeating the proof of the claim, we obtain a projection $p' \in A$ such that

$$q_{3\sigma/4} \le p' \le q_{7\sigma/16}.$$

We have

$$\|p' p_{\sigma/2} - p'\| < 2\eta.$$

This implies that

$$\|p_{\sigma/2} p' p_{\sigma/2} - p'\| < 4\eta.$$

By [Eff, A8], there is a projection $e' \in p_{\sigma/2} A p_{\sigma/2}$ such that

$$\|e' - p'\| < 8\eta.$$

By [Eff, A8] again, there is a unitary $v \in eAe$ such that

$$\|v - e\| < 8\eta \text{ and } e \ge v^* p_{\sigma/2} v \ge p' \ge q_\sigma.$$

We also have

$$\|p_{\sigma/2}\phi(g_3 f_i) - \phi(g_3 f_i) p_{\sigma/2}\| < 2\eta \text{ and } \Big\| p_{\sigma/2}\phi(f_i) p_{\sigma/2} - \sum_{\xi_j \in \Omega_{\sigma/2}} (g_3 f_i)(\xi_j) p_j \Big\| < \eta.$$

Since

$$\|p_{\sigma/2} - p_{\sigma/2} q_{7\sigma/16}\| < 2\eta \text{ and } \|p_{\sigma/2}\phi(f_i) - p_{\sigma/2} q_{7\sigma/16}\phi(f_i)\| < 2\eta.$$

Notice that $q_{7\sigma/4}\phi(f_i) = q_{7\sigma/16}\phi(g_3 f_i)$. Then

$$\|p_{\sigma/2}\phi(f_i) - p_{\sigma/2}\phi(g_3 f_i)\| < 4\eta.$$

Similarly,

$$\|\phi(f_i) p_{\sigma/2} - \phi(g_3 f_i) p_{\sigma/2}\| < 4\eta.$$

Therefore

$$\|p_{\sigma/2}\phi(f_i) - \phi(f_i) p_{\sigma/2}\| < 10\eta, \quad i = 1, 2, \ldots, N$$

and

$$\Big\| p_{\sigma/2}\phi(f_i) p_{\sigma/2} - \sum_{\xi_j \in \Omega_{\sigma/2}} f_i(\xi_j) p_j \Big\| < 10\eta.$$

Notice that

$$\|v^* p_{\sigma/2} v - p_{\sigma/2}\| < 16\eta.$$

We take $p = v^* p_{\sigma/2} v$ and $\eta < \varepsilon/64$. ■

We in fact have proved the following which will be used in Section 4.

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COROLLARY 3.3. *Let X be a locally compact metric space, $G \subset X$ be an open subset,*

$$I = \{ f \in C(X) : f(x) = 0 \text{ if } x \notin G \},$$

A be any (unital) C^*-algebra, $\varepsilon > 0, \sigma$ be positive numbers and $f_1, f_2, \dots, f_n \in C_0(X)$. Suppose that $\phi\colon C_0(X) \longrightarrow A$ is a monomorphism satisfying the following: there are $\delta > 0$, $\xi_1, \xi_2, \dots, \xi_m \in G$ and mutually orthogonal projections $d_1, d_2, \dots, d_m \in A$ such that

$$\Big\| \phi(g_i) - \sum_{k=1}^{m} g_i(\xi_k) d_k \Big\| < \delta \quad \text{and} \quad \Big\| \phi(g_i f_j) - \sum_{k=1}^{m} (g_i f_j)(\xi_k) d_k \Big\| < \delta,$$

where g_i is the same as in the proof of 3.2, $i = 0, 1, \dots, 4$. Then there exists a projection $p \in A$ such that

$$q_\sigma \le p \le q_{\sigma/4}, \quad \| p\phi(f_i) - \phi(f_i)p \| < \varepsilon$$

and

$$\Big\| p\phi(f_i)p - \sum_{k=1}^{m} f_i(\xi_k) p_k \Big\| < \varepsilon,$$

where q_σ and $q_{\sigma/4}$ are spectral projections in A^{**} corresponding to (the homomorphism ϕ and) the subset

$$\Omega_\sigma = \{ \xi \in G : \operatorname{dist}(\xi, X \setminus G) > \sigma \}$$

and

$$\Omega_{\sigma/4} = \{ \xi \in G : \operatorname{dist}(\xi, X \setminus G) > \sigma/4 \},$$

respectively, $\xi_k \in \Omega_\sigma$ and $\{p_k\}$ are mutually orthogonal projections in pAp.

We also need the following computation. Note X is a figure eight curve.

LEMMA 3.4. *Let B be a unital C^*-algebra,* $\operatorname{sp}(x) = X = \{\xi : |\xi - 1| = 1$ *or* $|\xi + 1| = 1\}$, *$f_1, f_2 \in C(X)$ be nonnegative such that*

$$f_1(\xi) = \begin{cases} 1 + \xi, & \text{if } \operatorname{Re}(\xi) < 0; \\ 1, & \text{if } \operatorname{Re}(\xi) \ge 0 \end{cases}$$

and

$$f_2(\xi) = \begin{cases} \xi - 1, & \text{if } \operatorname{Re}(\xi) > 0; \\ -1, & \text{if } \operatorname{Re}(\xi) \le 0. \end{cases}$$

For any $\sigma > 0$, there exists $\delta > 0$ satisfying the following: if there exists a projection p in a unital C^*-subalgebra $A \subset B$ such that

$$\begin{gathered} \|(1-p)x - x(1-p)\| < \delta, \\ \|(1-p)f_i(x) - f_i(x)(1-p)\| < \delta, \quad i = 1, 2, \\ (1-p)f(x)(1-p) \in A \end{gathered}$$

for all $f \in C(X)$ and there exist unitaries $w_1, w_2 \in (1-p)A(1-p)$ with $[w_i] = 0$ in $K_1\big((1-p)A(1-p)\big)$ and

$$\|(1-p)f_i(x)(1-p) - w_i\| < \delta,$$

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then for any $\lambda \notin X_\sigma$,

$$\big[\lambda(1-p)-(1-p)x(1-p)\big]=0$$

in $\mathrm{GL}_\infty\big((1-p)A(1-p)\big)/\,\mathrm{GL}_\infty\big((1-p)A(1-p)\big)_0$, *where*

$$X_\sigma=\big\{\xi : \mathrm{dist}(\xi,X)\le\sigma\big\}.$$

PROOF. Define a continuous function in $C([-\pi,\pi])$ as follows

$$h_0(t)=\begin{cases}0, & \text{if } -\pi/6<t<\pi/6;\\ \pi, & \text{if } t=\pi;\\ -\pi, & \text{if } t=-\pi;\\ \textit{linear}, & \text{if } -\pi\le t\le-\pi/6;\\ \textit{linear}, & \text{if } \pi/6\le t\le\pi\end{cases}$$

Then define $h_1\in C(S^1)$ by $h_1(e^{it})=e^{ih_0(t)}$, $-\pi\le t\le\pi$. Let $h_2(\xi)=(\xi+1)|\xi+1|^{-1}$ for $\xi\ne-1$. Then

$$h_1\circ f_1=h_1\circ h_2 \quad \text{on } X.$$

It is enough to show that

$$\big[\pm(1-p)-(1-p)x(1-p)\big]=0$$

in $\mathrm{GL}_\infty\big((1-p)A(1-p)\big)/\,\mathrm{GL}_\infty\big((1-p)A(1-p)\big)_0$. Let $y=(1-p)x(1-p)$. To show that $[(1-p)-(1-p)x(1-p)]=0$, it is enough to show that $[h_2(y)]=0$ in $K_1\big((1-p)A(1-p)\big)$. Let P be a polynomial of z and z^* such that

$$\|P-h_1\|<\delta.$$

Again, we will use the notation $P(z,z^*)$ for the corresponding linear combination of $z^i(z^J)^*$. As in the proof of [Ln10, 2.11], for any $\varepsilon>0$, if δ is small enough, we obtain and z^* such that

$$\Big\|P\big(h_2(y),h_2(y)^*\big)-h_1\big(h_2(y)\big)\Big\|<\varepsilon,\quad \Big\|P(w_1,w_1^*)-h_1(w_1)\Big\|<\varepsilon,$$
$$\Big\|P\big(h_2(y),h_2(y)^*\big)-(1-p)h_1\big(h_2(x)\big)(1-p)\Big\|<\varepsilon$$

and

$$\Big\|P\Big((1-p)f_1(x)(1-p),\big((1-p)f_1(1-p)\big)^*\Big)-(1-p)h_1\big(f_1(x)\big)(1-p)\Big\|<\varepsilon.$$

Now we have

$$\begin{aligned}\Big\|h_1(h_2(y)-h_1(w_1)\Big\| &\le \Big\|h_1\big(h_2(y)\big)-P\big(h_2(y),h_2(y)^*\big)\Big\|\\ &+\Big\|P\big(h_2(y),h_2(y)^*\big)-(1-p)h_1\big(h_2(x)\big)(1-p)\Big\|\\ &+\Big\|(1-p)h_1\big(h_2(x)\big)(1-p)-(1-p)h_1\big(f_1(x)\big)(1-p)\Big\|\\ &+\Big\|(1-p)h_1\big(f_1(x)\big)(1-p)\\ &\qquad -P\big((1-p)f_1(x)(1-p),(1-p)f_1(x)^*(1-p)\big)\Big\|\\ &+\Big\|P\big((1-p)f_1(x)(1-p),(1-p)f_1(x)^*(1-p)\big)-P(w_1,w_1^*)\Big\|\\ &+\Big\|P(w_1,w_1^*)-h_1(w_1)\Big\|<5\varepsilon.\end{aligned}$$

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So if $\varepsilon < 1/6$, $\big[h_1\big(h_2(y)\big)\big] = [h_1(w_1)]$ in $K_1\big((1-p)A(1-p)\big)$. But it is easy to see that

$$\big[h_1\big(h_2(y)\big)\big] = \big[h_2(y)\big] \text{ and } \big[h_1(w_1)\big] = [w_1] = 0.$$

Therefore $[h_2(y)] = 0$ in $K_1\big((1-p)A(1-p)\big)$. Thus $[(1-p)-(1-p)x(1-p)] = 0$ in $\mathrm{GL}_\infty\big((1-p)A(1-p)\big)/\,\mathrm{GL}_\infty\big((1-p)A(1-p)\big)_0$. Similarly, we can show that $[-(1-p)-(1-p)x(1-p)] = 0$. ∎

THEOREM 3.5. *Let A be a simple C^*-algebra with real rank zero, stable rank one, weakly unperforated $K_0(A)$ of countable rank, and let F be a compact subset of $S^1 \times S^1$. Suppose that $\phi: C(F) \longrightarrow A$ is a monomorphism, $\phi_*^{(1)} = 0$ and $\phi_*^{(0)}|_{\ker d} = 0$. Then ϕ can be approximated pointwise by maps from $C(F)$ into A with finite dimensional range.*

PROOF. We will identity $S^1 \times S^1$ with

$$\big\{(e^{it}, e^{i\tau}) : 0 \le t \le 2\pi, 0 \le \tau \le 2\pi\big\}.$$

Set $u_1, u_2 \in C(F)$ such that $u_1\big((e^{it}, e^{i\tau})\big) = e^{it}$ for $0 \le t \le 2\pi$ and $u_2\big((e^{it}, e^{i\tau})\big) = e^{i\tau}$ for $0 \le \tau \le 2\pi$, if $(e^{it}, e^{i\tau}) \in F$. Then $C(F)$ is generated by u_1 and u_2. It is enough to show that for any $\varepsilon > 0$ there is a homomorphism $\psi: C(F) \longrightarrow A$ with finite dimensional range such that

$$\big\|\phi(u_i) - \psi(u_i)\big\| < \varepsilon, \quad i = 1, 2.$$

Let $X = (\{1\} \times S^1 \cup S^1 \times \{1\}) \cap F$ and

$$I = \{f \in C(F) : f(\xi) = 0 \text{ for } \xi \in X\}.$$

Note that $\tilde{I} \cong C(G)$, where G is a compact subset of S^2. Since $\phi_*^{(0)}(\ker d) = 0$ and $\phi_*^{(1)} = 0$,

$$K_0\big(\phi(I)\big) \cap \ker\theta = 0 \text{ and } K_1\big(\tilde{\phi}(I)\big) = 0.$$

It follows from 2.12 and Remark 2.20 that the map $\phi|_I$ can be approximated by homomorphisms from I into A with finite dimensional range. By 3.2, for any $\delta > 0$ and $\sigma > 0$, there exists a projection $p \in A$ such that

$$q_\sigma \le p \le q_{3\sigma/4}, \quad \|p\phi(u_j) - \phi(u_j)p\| < \delta/8$$

and

$$\Big\|p\phi(u_j)p - \sum_{k=1}^{m} u_j(\xi_k)p_k\Big\| < \delta/8,$$

where q_σ is the spectral projection in A^{**} corresponding to the subset

$$\Omega_\sigma = \big\{\xi \in F \setminus X : \mathrm{dist}(\xi, X) \ge \sigma\big\},$$
$$\xi_k \in \big\{\xi \in F \setminus X : \mathrm{dist}(\xi, X) > \sigma\big\},$$

and $\{p_k\}$ are mutually orthogonal projections in pAp.

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There is a continuous map $r\colon F \longrightarrow F$ such that $r|_{F\setminus\Omega_\sigma}$ is a retraction on X. Set $v_i = u_i \circ r$, $i = 1, 2$. By choosing a small σ and an appropriate r, we may assume that

$$\|\phi(u_i) - \phi(v_i)\| < \delta/16.$$

We have that

$$\|\phi(v_i)(1-p) - (1-p)\phi(v_i)\| < 3\delta/16.$$

Let f be a homeomorphism from X onto a compact subset Y of a figure eight curve on the plane and $\bar{f} \in C(F)$ such that

$$\bar{f}|_{F\setminus\Omega_\sigma} = f \circ r.$$

There are $f_1, f_2 \in C(Y)$ such that

$$f_i\big(\phi(\bar{f})(1-q_\sigma)\big) = \phi(v_i)(1-q_\sigma), \quad i = 1, 2.$$

Therefore $a_i = (1-p)f_i\big(\phi(\bar{f})(1-q_\sigma)\big)(1-p) = (1-p)\phi(v_i)(1-p) \in A$,

$$\|(1-p)\phi(\bar{f}) - \phi(\bar{f})(1-p)\| < 3\delta/16$$

and

$$\|a_i - (1-p)\phi(u_i)(1-p)\| < \delta/16.$$

For each i, there is a polynomial P_i of two variables such that

$$\|P_i(z, \bar{z}) - f_i(z)\|_Y < \delta/32.$$

Set $y = (1-p)\phi(\bar{f})(1-p)$. Let ε be a positive number. If we define $P_i(y, y^*)$ to be the corresponding linear combination of terms $y^s(y^*)^t$ (y^* appears after y, see [Ln10, 2.10]), one computes (see [Ln10, 2.11]) that, if δ is small enough,

$$\Big\|P_i(y, y^*) - (1-p)P_i\big(\phi(\bar{f}), \phi(\bar{f})^*\big)(1-p)\Big\| < \varepsilon/32$$

and

$$\Big\|(1-p)P_i\big(\phi(\bar{f}), \phi(\bar{f})^*\big)(1-p) - a_i\Big\| < \varepsilon/32.$$

Therefore

$$\|P_i(y, y^*) - a_i\| < \varepsilon/16.$$

Moreover, we have

$$\Big\|a_i + \sum_{\xi_i \in \Omega_{\sigma/2}} u_i(\xi_i)p_i - \phi(u_i)\Big\| < \delta/4.$$

Therefore, since $\phi_*^{(1)} = 0$, there are unitaries $w_i \in (1-p)A(1-p)$ with $[w_i] = 0$ in $K_1\big((1-p)A(1-p)\big)$, $i = 1, 2$ such that w_i are close to a_i. By 3.4, we conclude that, for any $\sigma' > 0$, if δ is small enough,

$$\lambda(1-p) - (1-p)\phi(\bar{f})(1-p) \in \mathrm{Inv}_0\big((1-p)A(1-p)\big)$$

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for all $\lambda \notin Y_{\sigma'}$, where

$$Y_{\sigma'} = \{\lambda : \mathrm{dist}(\lambda, Y) < \sigma'\}.$$

Notice also

$$1 - q_{\sigma/4} \le 1 - p \le 1 - q_\sigma$$

It follows from 3.1 that, for any $\varepsilon' > 0$, if σ' and δ are small enough, there are $\lambda_1, \lambda_2, \ldots, \lambda_n \in X$, mutually orthogonal projections $d_1, d_2, \ldots, d_n \in (1-p)A(1-p)$ such that

$$\Big\| y - \sum_{j=1}^{n} f(\lambda_j) d_j \Big\| < \varepsilon'.$$

If ε' is small enough,

$$\Big\| P_i(y, y^*) - \sum_{j=1}^{n} f_i\big(f(\lambda_j)\big) d_j \Big\| < \varepsilon/16.$$

Therefore

$$\Big\| a_i - \sum_{j=1}^{n} f_i\big(f(\lambda_j)\big) d_j \Big\| < \varepsilon/16 + \varepsilon/16, \quad i = 1, 2.$$

Thus

$$\Big\| (1-p)\phi(u_i)(1-p) - \sum_{j=1}^{n} u_i(\lambda_j) d_j \Big\| < \delta/16 + 2\varepsilon/16, \quad i = 1, 2.$$

This implies that, if δ is small enough,

$$\Big\| \phi(u_i) - \sum_{\xi_j \in \Omega_{\sigma/2}} u_i(\xi_j) p_j - \sum_{j=1}^{n} u_i(\lambda_j) d_j \Big\| < \delta/16 + \delta/16 + \varepsilon/8 < \varepsilon. \qquad ■$$

COROLLARY 3.6. *Let A be a simple AF-algebra with countably many extreme traces. Then a homomorphism $\phi: C(S^1 \times S^1) \longrightarrow A$ can be approximated pointwise by homomorphisms from $C(S^1 \times S^1)$ into A if and only if*

$$\phi_*^{(0)}(\ker d) = 0.$$

4. Hom$\big(C(S^2), A\big)$ and Hom$\big(C(S^1 \times S^1), A\big)$.

LEMMA 4.1 (CF. [G, LEMMA 2]). *For any $\varepsilon > 0$, $\varepsilon/2 > \eta > 0$, any unital C^*-algebra A and any nonzero element $a \in A$, there exists an invertible element $b \in M_2(A)$ such that*

$$\| a \oplus \eta - b \| < \varepsilon.$$

PROOF. Set

$$b = \begin{pmatrix} a & \varepsilon/2 \\ \varepsilon/2 & 0 \end{pmatrix}$$

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Then b is invertible and

$$\|a \oplus 0 - b\| < \varepsilon/2.$$

Therefore

$$\|a \oplus \eta - b\| < \varepsilon.$$

THEOREM 4.2. *Let A be a unital C^*-algebra and $\phi\colon C(S^2) \longrightarrow A \otimes K$ be a monomorphism. If*

$$\phi_*^{(0)}(\ker d) = 0,$$

then, for any $\varepsilon > 0$ and $f_1, f_2, \ldots, f_m \in C(S^2)$, there exist homomorphisms ψ_1, ψ_2: $C(S^2) \longrightarrow A \otimes K$ with finite dimensional range such that

$$\|\phi(f_i) \oplus \psi_1(f_i) - \psi_2(f_i)\| < \varepsilon,$$

$i = 1, 2, \ldots, m$. In other words, $\phi \in \mathbf{E}$.

PROOF. The proof is essentially contained in Section 2. In fact, the proof is easier, since we are allowed to add any homomorphism from $C(S^2) \longrightarrow A \otimes K$ with finite dimensional range to ϕ and we do not need to absorb that summand ψ_1. We will sketch the proof as follows.

There are two places in Section 2 where we used the condition of stable rank one. One place is in 2.2. If A does not have stable rank one, one may not have a partial isometry $V \in M_2(A)$ such that

$$VV^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } V^*V = P.$$

However, since these two projections have the same image in $K_0(A)$ $\big(\phi_*^{(0)}(\ker d) = 0\big)$, there exist an integer k, a projection $e \in M_k(A)$ and a partial isometry $V \in M_{k+1}(A)$ such that

$$VV^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus e \text{ and } V^*V = P \oplus e.$$

By adding $(1_k - e)$, we may assume that $e = 1_k$, where 1_k is the identity of $M_k(A)$.

Pick a point ξ in the boundary on D_1. We then consider the map

$$\phi_1(f) = \phi(f) \oplus f(\xi) \cdot 1_k, \quad f \in C(S^2).$$

Hence, $\phi_1^{(2)}\big(P(z)\big) = P \oplus 1_k$. Therefore there is a partial isometry $U \in M_2\big(M_{k+1}(A)\big)$ such that

$$UU^* = \begin{pmatrix} 1_{k+1} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } U^*U = \phi_1^{(2)}\big(P(z)\big).$$

The second place is Lemma 2.6. One may not be able to find a unitary $u \in A$ such that $\|u|t_1| - t_1\|$ is small if we do not assume that A has stable rank one. However, we can apply 4.2. For $\varepsilon > 0$,

$$\|t_1 \oplus \eta - b\| < \varepsilon,$$

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for some invertible element $b \in M_2(A)$. Let $u = b|b|^{-1}$. If η is small enough, $\|u(|t_1| \oplus |\zeta|) - t_1 \oplus \zeta\|$ is small, where $\zeta \in D_1$ and $|\zeta| < \eta$. Denote by $g = (|z|, 1)$. Then $g(\zeta) = |\zeta|$. We then consider the map

$$\phi_2(f) = \phi_1(f) \oplus f(\zeta), \quad f \in C(S^2).$$

Since we are freely allowed to add a homomorphism (from $C(S^2)$ into $A \otimes \mathcal{K}$) with finite dimensional range to ϕ, at these two places, the condition of having stable rank one can be removed.

The condition of real rank zero can be removed as the same way as in Section 1. ■

THEOREM 4.3. *Let A be a unital C^*-algebra and F be a compact subset of $S^1 \times S^1$. Suppose that $\phi: C(F) \longrightarrow A \otimes \mathcal{K}$ is a homomorphism. If*

$$\phi_*^{(0)}(\ker d) = 0 \text{ and } \phi_*^{(1)} = 0,$$

then $\phi \in \mathbf{E}$.

PROOF. The proof is a modification of that of 3.5. In fact, it is easier, since we are allowed to add a homomorphism with finite dimensional range. We will keep the notation in the proof of 3.5.

It is enough to show that for any $\varepsilon > 0$ there exist homomorphisms $\psi_1: C(F) \longrightarrow M_I(A)$ and $\psi_2: C(F) \longrightarrow M_{I+1}(A)$ with finite dimensional range such that

$$\|\phi(u_i) \oplus \psi_1(u_i) - \psi_2(u_i)\| < \varepsilon, \quad i = 1, 2.$$

It follows from 4.2 that for any $\delta > 0$ and $\sigma > 0$, there are a homomorphism $\psi: C(F \setminus X) \longrightarrow M_k(A)$ (for some integer) with finite dimensional range and

$$\xi_1, \xi_2, \ldots, \xi_m \in F \setminus X$$

and mutually orthogonal projections $p_1, p_2, \ldots, p_m \in A$ such that

$$\Big\|\Phi(g) - \sum_{j=1}^m g(\xi_j)p_j\Big\| < \eta, \quad \Big\|\Phi(g_i) - \sum_{j=1}^m g_1(\xi_j)p_j\Big\| < \eta$$

and

$$\Big\|\Phi(gu_i) - \sum_{j=1}^m gu_i(\xi_j)p_j\Big\| < \eta, \quad i = 1, 2,$$

where $\Phi(f) = \phi(f) \oplus \psi(f)$ for $f \in C(G)$. We also define $\Phi(f) = \phi(f) \oplus \psi(f|_F)$ for $f \in C(F)$. From now on, we replace ϕ by Φ in the proof of 3.5.

As in the proof of 3.5, by applying 3.4, we conclude that, for any $\varepsilon' > 0$, if δ is small enough,

$$\Big[\Big(\lambda(1-p) - (1-p)\Phi(\bar{f})(1-p)\Big) \oplus (1-p)\Big] = 0$$

in $K_1\big((1-p)A(1-p)\big)$ for all $\lambda \notin Y_{\sigma'}$, where

$$Y_\sigma = \big\{\lambda : \operatorname{dist}(\lambda, X) < \sigma'\big\}.$$

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Notice also

$$\mathrm{sp}\big[\Phi(\bar{f})(1-q_\sigma)\big] = Y.$$

It follows from 1.3 that, for any $\varepsilon' > 0$, if δ is small enough, there are $\lambda_1, \lambda_2, \ldots, \lambda_n$, $\lambda'_1, \lambda'_2, \ldots, \lambda'_l \in X$, mutually orthogonal projections $d_1, d_2, \ldots, d_n \in M_L\big(M_{k+1}(A)\big)$ and

$$d'_1, d'_2, \ldots, d'_l \in \big((1-p)\oplus 1_L\big)M_{L+1}\big(M_{k+1}(A)\big)\big((1-p)\oplus 1_L\big)$$

such that

$$\Big\| y \oplus \sum_{j=1}^{n} f(\lambda_j)d_j - \sum_{j'=1}^{l} f(\lambda'_{j'})d'_{j'} \Big\| < \varepsilon'.$$

If ε' is small enough,

$$\Big\| P_i(y, y^*) \oplus \sum_{j=1}^{n} f_i\big(f(\lambda_j)\big)d_j - \sum_{j'=1}^{l} f_i\big(f(\lambda'_{j'})\big)d'_{j'} \Big\| < \varepsilon/16.$$

Therefore

$$\Big\| a_i \oplus \sum_{j=1}^{n} f_i\big(f(\lambda_j)\big)d_j - \sum_{j'=1}^{l} f_i\big(f(\lambda'_{j'})\big)d'_{j'} \Big\| < \varepsilon/8, \quad i = 1, 2.$$

Thus there are $\zeta_1, \zeta_2, \ldots, \zeta_n, \zeta'_1, \zeta'_2, \ldots, \zeta'_l \in X$ such that

$$\Big\| (1-p)\Phi(u_i)(1-p) \oplus \sum_{j=1}^{n} u_i(\zeta_j)d_j - \sum_{j'=1}^{l} u_i(\zeta'_{j'})d'_{j'} \Big\| < 3\varepsilon/16 + \delta/4 + \delta/16, \quad i = 1, 2.$$

This implies that, if δ is small enough,

$$\Big\| \Phi(u_i) \oplus \sum_{j=1}^{n} u_i(\zeta_j)d_j - \sum_{\xi_j \in \Omega_{\sigma/2}} u_i(\xi_j)p_j - \sum_{j'=1}^{l} u_i(\zeta'_{j'})d'_{j'} \Big\| < \varepsilon. \qquad \blacksquare$$

THEOREM 4.4. *Both*

$$\mathbf{Hom}\big(C(S^2), A\big) \text{ and } \mathbf{Hom}\big(C(S^1 \times S^1), A\big)$$

are groups.

PROOF. Let ϕ^* be the map defined in [EGLP, 3.1], then, by [EGLP, 3.4],

$$(\phi \oplus \phi^*)_*^{(0)}(\ker d) = 0$$

and (in the case of $S^1 \times S^1$)

$$(\phi \oplus \phi^*)_*^{(1)} = 0. \qquad \blacksquare$$

THEOREM 4.5. *Let $X = S^2$, or $X = S^1 \times S^1$. Then two homomorphisms ϕ_1, ϕ_2: $C(X) \longrightarrow A \otimes K$ are sau-equivalent if and only if*

$$(\phi_1)_*^{(0)}|_{\ker d} = (\phi_2)_*^{(0)}|_{\ker d} \text{ and } (\phi_1)_*^{(1)} = (\phi_2)_*^{(1)}.$$

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PROOF. The proof is the same as that of 1.14, instead of applying 1.13, here we apply 4.3. ∎

DEFINITION 4.6. Let A be a unital C^*-algebra and X be a compact metric space. Given a homomorphism $\phi\colon C(X) \longrightarrow A$, define

$$\Gamma(\phi) = \left(\phi_*^{(0)}|_{\ker\, d}, \phi_*^{(1)}\right).$$

The map Γ gives a homomorphism from $\mathbf{Hom}\big(C(X), A\big)$ into

$$\Big(\hom\Big(\ker\, d, \inf\big(K_0(A)\big)\Big), \hom\Big(K_1\big(C(X)\big), K_1(A)\Big)\Big).$$

If X is a compact subset of the plane, X is homeomorphic to S^2, or X is homeomorphic to $S^1 \times S^1$, from what we have established, Γ is injective.

THEOREM 4.7. *Let A be a unital AF-algebra which has no finite dimensional quotient. Then*

$$\mathbf{Hom}\big(C(S^1 \times S^1), A\big) \cong \hom\Big(\ker\, d, \inf\big(K_0(A)\big)\Big).$$

PROOF. It is enough to show that the map

$$\Gamma\colon \mathbf{Hom}\big(C(S^1 \times S^1), A\big) \longrightarrow \hom\Big(\ker\, d, \inf\big(K_0(A)\big)\Big)$$

is surjective. Let b be the Bott element in $K_0\big(C(S^1 \times S^1)\big)$ and $x \in \inf\big(K_0(A)\big)$. Then, by [EL, 7.3], there is a monomorphism $\phi\colon C(S^1 \times S^1) \longrightarrow A$ such that

$$\phi_*^{(0)}(b) = x. \qquad ∎$$

PROPOSITION 4.8. *Let A be a unital C^*-algebra with real rank zero and stable rank one. Suppose that H is a hereditary subgroup of $K_0(A)$. Then the closure of*

$$\Big\{a \in A : a^*a \leq \sum_{i=1}^{k} \lambda_i p_i, \lambda_i \geq 0, [p_i] \in H\Big\}$$

is a (closed) ideal of A.

PROOF. Let I_+ be the closure of

$$\Big\{a \in A_+ : a \leq \sum_{i=1}^{k} \lambda_i p_i, \lambda_i \geq 0, [p_i] \in H\Big\}.$$

Then I_+ is a closed cone of A_+. It follows from [Pd, 1.52] that I is a (closed) left ideal. To show I is an ideal, it suffices to show that if $a \in I$ then $a^* \in I$. Now suppose that $a \in I$

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and $a = u(a^*a)^{1/2}$ is the polar decomposition of a in A^{**}. Let D_1 be the hereditary C^*-subalgebra generated by $|a|$. Then, for any $d \in D_1$, $ud \in A$ (see [Ln4, 1.4] for example. This is certainly known before [Ln4]). Furthermore,

$$\Phi(d) = udu^*, \quad d \in D$$

gives an isomorphism from D onto the hereditary C^*-subalgebra D_2 generated by $|a^*|$ (see [Cu1, 1.7]). For any projection $p \in D$, $upu^* \in A$ and $[upu^*] = [p]$ in $K_0(A)$. Therefore, $upu^* \in I_+$. Since D_2 has real rank zero (see [BP, 2.6]), D_2 is generated by its projections. From the above, we conclude that $D_2 \subset I$. In particular, $aa^* \in I_+$. Therefore, $a^* \in I$. ∎

THEOREM 4.9. *Let A be a unital C^*-algebra with no finite dimensional quotient, with real rank zero, stable rank one and unperforated $K_0(A)$. Then*

$$\mathbf{Hom}\Big(C(S^1 \times S^1), A\Big) \cong \bigg(\mathrm{hom}\Big(\ker d, \inf\big(K_0(A)\big)\Big), \mathrm{hom}\Big(K_1\big(C(S^1 \times S^1)\big), K_1(A)\Big)\bigg).$$

PROOF. It follows from [Ln4, 2.9] that there is a unital AF-algebra B and a unital monomorphism $\Phi: B \longrightarrow A$ such that $\Phi_*^{(0)}$ is an isomorphism. Furthermore, (by identifying $\Phi(B)$ with B) for any projection $p \in A$ there is a projection $q \in B$ and partial isometry $v \in A$ such that $v^*v = p$ and $vv^* = q$. We claim that B has no finite dimensional quotient.

Assume that I_0 is an ideal of B such that B/I_0 is of finite dimensional. Let H be the hereditary subgroup of $K_0(B) = K_0(A)$ corresponding to the ideal I_0. Set I be the ideal generated by H as in 4.7. The definition of B, it is clear that $B \cap I = I_0$. So $B/I \cong B/B \cap I = B/I_0$ is finite dimensional. Write $B/I = M_{k_1} \oplus M_{k_2} \oplus \cdots \oplus M_{k_n}$. If $\bar{p} \in A/I$ be a projection, since A has real rank zero, by [Zh2, 3.2], there is a projection $p \in A$ such that $\bar{p}$ is the image of p in the quotient. Therefore, there is a projection $q \in B$ and a partial isometry $v \in A$ such that $vv^* = p$ and $v^*v = q$. Denote by $\bar{q}$, $\bar{v}$ the image of q and image of v in the quotient, respectively. Then $\bar{v}^*\bar{v} = \bar{q}$ and $\overline{vv^*} = \bar{p}$. Thus, every projection in A is in the ideal generated by $e_1, e_2, \ldots, e_n$, where each e_i is a (nonzero) minimal projection in M_{k_i}. Since A has stable rank one, the above also implies that a (nonzero) minimal projection in B must be a (nonzero) minimal projection in A. In particular, e_i is a minimal projection in A. Since A has real rank zero, we must have $e_iAe_i \cong \mathbf{C}$. Therefore $(\sum_{i=1}^n e_i)A(\sum_{i=1}^n e_i)$ has to be finite dimensional. Also, the ideal generated by $e_1, e_2, \ldots, e_n$ is A. Thus by [Bn1], $(\sum_{i=1}^n e_i)A(\sum_{i=1}^n e_i) \otimes \mathcal{K} \cong A \otimes \mathcal{K}$. But $(\sum_{i=1}^n e_i)A(\sum_{i=1}^n e_i) \otimes \mathcal{K}$ is isomorphic to a finite direct sum of $\mathcal{K}$. Thus A which is isomorphic to a unital hereditary C^*-subalgebra of it must be finite dimensional. This contradicts our assumption that A has no finite dimensional quotient. This proves the claim.

Now let b be the Bott element, u_1, $u_2 \in C(S^1 \times S^1)$ such that $u_1(e^{i\theta}, e^{i\zeta}) = e^{i\theta}$ and $u_2(e^{i\theta}, e^{i\zeta}) = e^{i\zeta}$, and $x \in \inf\big(K_0(A)\big)$, y, $z \in K_1(A)$. As in 4.7, there is a monomorphism $\phi_0: C(S^1 \times S^1) \longrightarrow B$ such that $(\phi_0)_*^{(0)}(b) = x$. Set $\phi_1 = \Phi \circ \phi_0$. Then

$$(\phi_1)_*^{(0)}(b) = x \text{ and } (\phi_1)_*^{(1)} = 0,$$

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since $K_1(B) = 0$. Let $v_1 \in M_{m_1}(A)$ be a unitary such that $[v_1] = y$ (in $K_1(A)$) and $v_2 \in M_{m_2}(A)$ be a unitary such that $[v_2] = z$. Fix $\xi \in S^1$. Define maps

$$j_1, j_2 \colon S^1 \longrightarrow S^1 \times S^1$$

by $j_1(e^{i\theta}) = (e^{i\theta}, \xi)$ and $j_2(e^{i\zeta}) = (\xi, e^{i\zeta})$. Let $h_i \colon C(S^1 \times S^1) \longrightarrow C(S^1)$ be the surjective homomorphism induced by j_i, $i = 1, 2$, and let $\psi_i \colon C(S^1) \longrightarrow M_{m_i}(A)$ be the homomorphism induced by the unitary v_i, $i = 1, 2$. Now we define

$$\phi_2 \colon C(S^1 \times S^1) \longrightarrow M_{1+m_1+m_2}(A)$$

by

$$\phi_2(f) = \phi_1(f) \oplus \psi_1\big(h_1(f)\big) \oplus \psi_2\big(h_2(f)\big)$$

for $f \in C(S^1 \times S^1)$. It is easy to check that

$$(\phi_2)_*^{(0)}(b) = x, \quad (\phi_2)_*^{(1)}\big([u_1]\big) = y \text{ and } (\phi_2)_*^{(1)}\big([u_2]\big) = z. \qquad \blacksquare$$

COROLLARY 4.10. *Let A be a unital AF-algebra with no finite dimensional quotient. Then*

$$\mathbf{Hom}\big(C(S^2), A\big) \cong \mathrm{hom}(\ker d, \inf\big(K_0(A)\big).$$

PROOF. Let

$$I = \big\{f \in C(S^1 \times S^1) : f\big(S^1 \times \{1\}\big) = f\big(\{1\} \times S^1\big) = 0\big\}.$$

Then $\tilde{I} \cong C(S^2)$. From the six term exact sequence in K-theory, we obtain

$$0 \longrightarrow K_0(I) \longrightarrow K_0\big(C(S^1 \times S^1)\big) \longrightarrow K_0\big(C(S^1 \times S^1)/I\big) \longrightarrow 0.$$

So the map from $\ker\, d$ of $C(S^2)$ onto $\ker\, d$ of $C(S^1 \times S^1)$ is an isomorphism. Therefore 4.9 follows immediately from 4.8. $\blacksquare$

COROLLARY 4.11. *Let A be a unital C^*-algebra with no finite dimensional quotient, with real rank zero, stable rank one and unperforated $K_0(A)$. Then*

$$\mathbf{Hom}\big(C(S^2), A\big) \cong \mathrm{hom}\Big(\ker d, \inf\big(K_0(A)\big)\Big).$$

COROLLARY 4.12. *Let A be a unital C^*-algebra which has no infinitesimal elements in $K_0(A)$. Then*

$$\mathbf{Hom}\big(C(S^2), A\big) = \{0\}$$

and

$$\mathbf{Hom}\big(C(S^1 \times S^1), A\big) = \mathrm{hom}\big(K_1(S^1 \times S^1), K_1(A)\big).$$

PROOF. The first part follows from 4.2. Note that in 4.9, the assumption that A has no finite dimensional quotient is used only to get an element in $\mathbf{Hom}\big(C(S^1 \times S^1), A\big)$ to induce a given map from $\ker\, d$ to $\inf\big(K_0(A)\big)$. Since now $\inf\big(K_0(A)\big) = 0$, this assumption is no longer needed. $\blacksquare$

THEOREM 4.13. *Let A be a purely infinite simple C^*-algebra and X is a compact subset of S^2 or $S^1 \times S^1$. Suppose that $\phi\colon C(X) \longrightarrow A$. If*

$$\phi_*^{(0)}(\ker d) = 0 \text{ and } \phi_*^{(1)} = 0,$$

then, for any $\varepsilon > 0$ and $f_1, f_2, \ldots, f_s \in C(X)$, there exists a homomorphism $\psi\colon C(X) \longrightarrow A$ with finite dimensional range such that

$$\|\phi(f_i) - \psi(f_i)\| < \varepsilon,$$

$i = 1, 2, \ldots, s$.

PROOF. Since X is compact, without loss of generality, we may assume that A is unital. By 4.2 or 4.3, for any $\varepsilon > 0$, there exist an integer k, homomorphisms $\Phi_1\colon C(X) \longrightarrow M_k(A)$ and $\Phi_2\colon C(X) \longrightarrow M_{k+1}(A)$ with finite dimensional range such that

$$\|\phi(f_i) \oplus \Phi_1(f_i) - \Phi_2(f_i)\| < \varepsilon/3,$$

$i = 1, 2, \ldots, s$. We may also assume that Φ_1 and Φ_2 are unital and

$$\Phi_1(f) = \sum_{l=1}^{L} f(\lambda_l)d_l \quad \text{and} \quad \Phi_2(f) = \sum_{m=1}^{M} f(\alpha_m)e_m,$$

for $f \in C(X)$, where $\{d_l\}$ and $\{e_m\}$ are mutually orthogonal projections, and $\lambda_l, \alpha_m \in X$. If follows from [EGLP, 4.1] (see [Ln10, 4.3] also) that there are mutually orthogonal projections $p_1, p_2, \ldots, p_n \in A$ and points $\xi_1, \xi_2, \ldots, \xi_n \in X$ such that

$$\Big\|\phi(f_i) - \Big[\sum_{j=1}^{n} f_i(\xi_j)p_j + (1-p)\phi(f_i)(1-p)\Big]\Big\| < \varepsilon/3,$$

$i = 1, 2, \ldots, s$, and for any $\zeta \in X$ there is j such that

$$\operatorname{dist}(\zeta, \xi_j) < \varepsilon/3,$$

where $p = \sum_{j=1}^{n} p_j$. Without loss of generality (with an error within $\varepsilon/3$), we may assume that $L = n$ and $\lambda_l = \xi_l$. Since A is purely infinite and simple, there is a partial isometry $v\,\big(\in M_{k+1}(A)\big)$ such that

$$v^*(p_j \oplus d_j)v \le p_j, \quad j = 1, 2, \ldots, n \text{ and } vv^* = \sum_{j=1}^{n} p_j \oplus \sum_{j=1}^{n} d_j.$$

Set $q_j = p_j - v^*(p_j \oplus d_j)v$ and $u = v \oplus (1-p)$. Then

$$\Big\|\sum_{j=1}^{n} f_i(\xi_j)p_j \oplus (1-p)\phi(f_i)(1-p) - u^*\Psi_2(f_i)u\Big\| < 2\varepsilon/3,$$

where

$$\Psi_2(f) = \Phi(f) \oplus \sum_{j=1}^{n} f(\xi_j)q_j \quad \text{for } f \in C(X).$$

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This implies that

$$\|\phi(f_i) - u^*\Phi_2(f_i)u\| < \varepsilon.$$

Notice that $u^*\Phi_2 u$ has finite dimensional range. ■

REMARK 4.14. It is shown in [EGLP, 4.2] that if A is a purely infinite simple C^*-algebra and X is homeomorphic to S^2 or to $S^1 \times S^1$, $\phi\colon C(X) \longrightarrow A$ is a homotopically trivial monomorphism, then ϕ can be approximated pointwise by homomorphisms from $C(X)$ into A with finite dimensional range. Theorem 4.13 is certainly a stronger result.

COROLLARY 4.15. *Let X be homeomorphic to S^2 or $S^1 \times S^1$. Then every monomorphism $\phi\colon C(X) \longrightarrow O_2$ can be approximated pointwise by homomorphisms from $C(X)$ into O_2 with finite dimensional range.*

PROOF. $K_0(O_2) = K_1(O_2) = \{0\}$. ■

REMARK 4.16. To study (essential) extensions of $C(X)$ by A, one studies the monomorphisms $\tau\colon C(X) \longrightarrow M(A)/A$. In many cases, $M(A)/A$ is purely infinite and simple (see [Ln1, Ln3]). Results like 4.14 are certainly related to C^*-algebra extensions. We will not give a detailed discussion here.

5. **Applications to classification theory.**

5.1. Let A be the C^*-algebraic inductive limit of a sequence:

$$A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} \cdots A_n \xrightarrow{\phi_n} A_{n+1} \xrightarrow{\phi_{n+1}} \cdots.$$

We will write $A = \lim_{n\to\infty}(A_n, \phi_n)$. We will also use the notations $\phi_{n,m}$ for the composition $\phi_m \circ \phi_{m-1} \circ \cdots \circ \phi_n$ and ϕ_∞ for the map $\phi_{n,\infty}\colon A_n \longrightarrow A$ for each n.

Let $A = \lim_{n\to\infty}(A_n, \phi_{n,})$ be a C^*-algebra of real rank zero, where

$$A_n = \bigoplus C(S^1) \otimes M_{[n,i]},$$

with each $[n,i]$ a positive integer. It is shown in [Ell2] that such C^*-algebras can be completely classified by their graded ordered K_*-groups. Conversely, for any (graded) (weakly) unperforated (see [Ell3] for the generalized definition) ordered torsion free group G with the Riesz decomposition property, there is a C^*-algebra A of the above-mentioned inductive limit form, with real rank zero such that $K_*(A) = G$. It has been shown [EE] that the irrational rotation C^*-algebras are in this class of inductive limits. In this section we will show that in fact many other inductive limits belong to this class.

We will consider those inductive limits such that

$$A_n = \bigoplus C(X_{n,i}) \otimes M_{[n,i]}$$

for some lower dimensional spaces $X_{n,i}$. The problem when such inductive limits have real rank zero has been studied (*cf.* [BBEK] and [BDR]). For example, it is shown that in the case that A is a simple unital C^*-algebra with slow dimension growth, then A has real rank zero if and only if the projections separate the tracial states.

In what follows, we will identify $C(X)\otimes M_n$ with $C(X, M_n)$, the continuous maps from X into M_n.

LEMMA 5.2. *Let A be a separable unital simple C^*-algebra with real rank zero, stable rank zero and weakly unperforated $K_0(A)$ of countable rank, and let X be a compact subset of the plane. Suppose that $x \in A$ is a normal element with* $\mathrm{sp}(x) \subset X$ *and*

$$\lambda - x \in \mathrm{Inv}_0(A) \text{ for } \lambda \notin X.$$

Then, for any $\varepsilon > 0$ and integer k, there are mutually orthogonal projections $p_1, p_2, \ldots, p_s \in A$ and $\lambda_1, \lambda_2, \ldots, \lambda_s \in X$ such that

$$\Big\| x - \Big(x' + \sum_{i=1}^{s} \lambda_i p_i \Big) \Big\| < \varepsilon/8,$$

where $x' = qxq$ and $q = 1 - \sum_{i=1}^{s} p_i$,

$$\|qx - xq\| < \varepsilon,$$
$$\|x' \oplus y - z\| < \varepsilon/8,$$

where $y \in M_L(qAq)$ and $z \in M_{L+1}(qAq)$ are normal elements with finite spectra $\mathrm{sp}(y)$ $\mathrm{sp}(z) \in X$, *and*

$$k(2L+2)[q] < [p_i], \quad i = 1, 2, \ldots, s.$$

PROOF. This is an easy consequence of [Ln10, Sections 3 and 4]. This is certainly contained in the proof of 3.1. But the easiest way to obtain this is to directly apply 3.12 in [Ln10] and 2.10. ∎

LEMMA 5.3. *Let A be a non-scattered, simple unital C^*-algebra with real rank zero, and X be a compact subset of the plane. For any $\varepsilon > 0$, there exists an integer k, such that, for any nonzero projection $p \in A$ and any normal element $x \in A$ with* $\mathrm{sp}(x) \subset X$ *there are normal elements $y_i \in pAp$ with*

$$\mathrm{sp}(y_i) \subset X_\varepsilon = \big\{ \xi : \mathrm{dist}(\xi, X) < \varepsilon \big\}$$

satisfying the following: $\mathrm{sp}(y_i)$ *is homeomorphic to the unit circle and*

$$\lambda - x \oplus \bigoplus_{i=1}^{k} y_i \in \mathrm{Inv}_0\big(M_{k+1}(A)\big)$$

for any $\lambda \notin X_\varepsilon$.

PROOF. Since X is compact, there exists a finite CW complex Z in the plane such that $X \subset Z$ and for any $\zeta \in Z$ there is $\lambda \in X$ such that

$$\mathrm{dist}(\zeta, \lambda) < \varepsilon/2.$$

It follows from the proof of 1.12 that there is a normal element $z \in A$ such that $\gamma(z) = 0$ and $\mathrm{sp}(z) = Z$. Notice that since A is simple, the embedding $e\colon pAp \longrightarrow A$ induces an

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isomorphism from $K_1(pAp) \longrightarrow K_1(A)$. It is easy to check (see the proof of 1.12) that there are normal elements $y_i \in pAp$ such that $\mathrm{sp}(y_i) \subset Z$, $\mathrm{sp}(y_i)$ is homeomorphic to the unit circle and

$$\gamma\Big(x \oplus z \oplus \bigoplus_{i=1}^{m} y_i\Big) = 0.$$

Therefore, since A has real rank zero, by [Ln10, 2.4],

$$\gamma\Big(x \oplus \bigoplus_{i=1}^{m} y_i\Big) = 0. \qquad \blacksquare$$

THEOREM 5.4. *Let $A = \lim_{\rightarrow}(A_n, \phi_n)$ be a simple C^*-algebra of real rank zero, where each A_n has the form*

$$A_n = \sum \bigoplus_{i=1}^{k} C(X_n^i) \otimes M_{m_i},$$

where X_n^i is a compact subset of the plane. Suppose that $K_0(A)$ has countable rank. Then A is an inductive limit of finite direct sum of matrix algebras over $C(S^1)$. Consequently, those algebras are classified (up to isomorphisms) by their graded ordered group $\big(K_0(A), K_1(A)\big)$. In particular A is an AF-algebra if and only if $K_1(A) = 0$.

PROOF. It follows from [DNNP] that A has stable rank one. By the proof of [GL, 3.3], $K_0(A)$ is weakly unperforated. For any $\epsilon > 0$ and $x_1, x_2, \ldots, x_m \in A$, there are an integer N and $y_1, y_2, \ldots, y_m \in \phi_\infty(A_N)$ such that

$$\|x_i - y_i\| < \epsilon/2.$$

It follows from [Ell1] (see also [LR]) that it is sufficient to show that there are a C^*-subalgebra $B \subset A$ which is isomorphic to a finite direct sum of matrix algebras over $C(S^1)$ and $z_1, z_2, \ldots, z_m \in B$ such that

$$\|y_i - z_i\| < \epsilon/2.$$

To save notation, (without loss of generality), we may assume that

$$A_N = C(X) \otimes M_d\big(\cong C(X, M_d)\big),$$

where X is a compact subset of the plane. Since $\phi_\infty(A_n)$ is isomorphic to a C^*-algebras with the form $C(Y, M_d)$, where Y is a compact subset of X, we may simply assume that $A_N = \phi_\infty(A_N)$.

Let $\{e_{ij}\}$ be a set of matrix units for M_d. Set $\varepsilon_{ij} = 1 \otimes e_{ij}$. We view $\varepsilon_{ij} \in A$. Notice that $\varepsilon_{11} A_N e_{11} \cong C(X)$. Let x be a normal element in $\varepsilon_{11} A_N \varepsilon_{11}$ with $\mathrm{sp}(x) = X$ such that x is a generator for $\varepsilon_{11} A_n \varepsilon_{11} \cong C(X)$. It is easy to see that it is sufficient to show that for any $\eta > 0$, there is a normal element $z \in \varepsilon_{11} A \varepsilon_{11}$ which is a direct sum of finitely many normal elements z_i with each $\mathrm{sp}(z_i)$ being homeomorphic to the unit circle such that

$$\|x - z\| < \eta.$$

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By applying 5.2 and 5.3 (and using the notation in 5.2), we have

$$\|x' \oplus y' \oplus y - z'\| < \varepsilon/8$$

where $y \in M_m(qAq)$ (m depends only on $\mathrm{sp}(x)$ and η) and y is a direct sum of m normal elements y_i with $\mathrm{sp}(y_i)$ being homeomorphic to the unit circle, and $y' \in M_{Lm}(qAq)$, $z' \in M_{(L+1)m}(qAq)$ (L depends only on $\mathrm{sp}(x)$ and η) are normal elements with finite spectra $\mathrm{sp}(y')$ $\mathrm{sp}(z') \in X_{\varepsilon/8}$, where

$$X_{\varepsilon/8} = \big\{\xi : \mathrm{dist}(\xi, X) < \varepsilon/8\big\}.$$

There is a normal element $\tilde{y} \in M_m(qAq)$ which is a direct sum of m normal elements $\tilde{y}_i$ with each $\mathrm{sp}(\tilde{y}_i) = \mathrm{sp}(y_i)$ such that

$$\gamma(y \oplus \tilde{y}) = 0.$$

It follows from [Ln10, 4.11] that there is a normal element $y'' \in M_{2m}(A)$ with finite spectrum $\mathrm{sp}(y'') \subset \mathrm{sp}(y)$ such that

$$\|y \oplus \tilde{y} - y''\| < \varepsilon/8.$$

We have

$$\|x' \oplus y' \oplus y'' - z' \oplus \tilde{y}\| < \varepsilon/4.$$

Now we may assume that the integer k (as in 5.2) is larger enough such that

$$(Lm + 2m)[q] < [p_j], \quad j = 1, 2, \ldots, s.$$

We then apply the absorption method used in 2.12 to obtain a unitary $U \in M_{(L+1)m+2m}(A)$ such that

$$\Big\|x - U^*\Big(\sum_{j=1}^{s} \lambda_j p_j' \oplus z' \oplus^* y\Big)U\Big\| < \varepsilon,$$

where $p_j' \le p_j$ are some projections. Take $z = U^*(\sum_{j=1}^{s} \lambda_j p_j' \oplus z \oplus^* y)U$. ■

Using the results established in Sections 2 and 3, we can prove the following lemma in the same way as in 5.2.

LEMMA 5.5. *Let A be a separable unital simple C^*-algebra with real rank zero, stable rank zero and weakly unperforated $K_0(A)$ of countable rank. Let $\phi\colon C(F) \longrightarrow A$ be a monomorphism, where F is a compact subset of S^2 or $S^1 \times S^1$. Suppose that*

$$\phi_*^{(0)}(\ker d) = 0 \textit{ and } \phi_*^{(1)} = 0$$

Then, for any $\varepsilon > 0$, any finitely many $f_1, f_2, \ldots, f_m \in C(F)$ and any integer k, there are mutually orthogonal projections $p_1, p_2, \ldots, p_s \in A$ and $\lambda_1, \lambda_2, \ldots, \lambda_s \in F$ such that

$$\Big\|\phi(f_j) - \Big(y_j + \sum_{i=1}^{s} f_j(\lambda_i)p_i\Big)\Big\| < \varepsilon/8,$$

where $y_j = q\phi(f_j)q$, $j = 1, 2, \ldots, s$ *and* $q = 1 - \sum_{i=1}^{s} p_i$,

$$\|q\phi(f_j) - \phi(f_j)q\| < \varepsilon, \quad j = 1, 2, \ldots, s,$$
$$\|y_j \oplus \psi_1(f_j) - \psi_2(f_j)\| < \varepsilon/8, \quad j = 1, 2, \ldots, s$$

where $\psi_1 \colon C(F) \longrightarrow M_L(qAq)$ *and* $\psi_2 \colon C(F) \longrightarrow M_{L+1}(qAq)$ *are homomorphisms with finite dimensional range and*

$$k(2L+2)[q] < [p_i], \quad i = 1, 2, \ldots, s.$$

THEOREM 5.6. *Let* $A = \lim_{\to}(A_n, \phi_n)$ *be a simple* C^**-algebra of real rank zero, where each* A_n *has the form*

$$A_n = \sum \bigoplus_{i=1}^{k} C(X_n^i) \otimes M_{m_i},$$

where X_n^i *is homeomorphic to* S^2, *or to* $S^1 \times S^1$. *Suppose that* $K_0(A)$ *has countable rank. Then* A *is an inductive limit of finite direct sum of matrix algebras over* $C(S^1)$. *Consequently, those algebras are classified (up to isomorphisms) by their graded ordered group* $\big(K_0(A), K_1(A)\big)$. *In particular,* A *is an AF-algebra if and only if* $K_1(A) = 0$.

COROLLARY 5.7. *Let* $A = \lim_{\to}(A_n, \phi_n)$ *be a simple* C^**-algebra of real rank zero, where each* A_n *is a finite direct sums of matrix algebras over* $C(S^2)$. *If* $K_0(A)$ *has countable rank, then* A *is an AF-algebra.*

PROOF. Under the assumption, $K_1(A) = 0$, since $K_1(A_n) = 0$ for all n. ∎

COROLLARY 5.8 ([EG]). *Let* B *be a Bunce-Deddens algebra. Then* $B \otimes B$ *is an inductive limit of finite direct sums of matrix algebras over* $C(S^1)$ *of real rank zero.*

PROOF. $B \otimes B$ is simple and $K_0(B \otimes B)$ is of finite rank. Furthermore, $B \otimes B$ is of the following inductive limit:

$$C(S^1 \times S^1) \longrightarrow M_4\big(C(S^1 \times S^1)\big) \longrightarrow M_{16}\big(C(S^1 \times S^1)\big) \longrightarrow \cdots. \qquad ∎$$

The proof of 5.6 is a duplicate of that of 5.4. But we have to use 5.5 instead of using 5.2. Instead of 5.3, we can use Theorem 11 in [EGLP2].

While this paper is writing, George A. Elliott and Guihua Gong [GL1] have obtained a more general result than 5.6. We decided not to give all the details.

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