

A DOMAIN WITH NONPLURISUBHARMONIC d -BALANCED SQUEEZING FUNCTION

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Abstract

In this note, we give an example of a domain whose d -balanced squeezing function is nonplurisubharmonic.

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1. Introduction

We present an example of a domain in \mathbb{C}^2 whose d -balanced squeezing function fails to be plurisubharmonic. Let us first recall some related notions.

For a bounded domain $D \subseteq \mathbb{C}^n$ and $z \in D$, Deng *et al.* [3] introduced the squeezing function S_D on D , defined by

$$S_D(z) := \sup_f \{r : \mathbb{B}^n(0, r) \subseteq f(D), f \in O_u(D, \mathbb{B}^n), f(z) = 0\},$$

where $\mathbb{B}^n(0, r)$ denotes a ball of radius r centred at the origin and $O_u(D, \mathbb{B}^n)$ denotes the collection of injective holomorphic maps from D to the unit ball \mathbb{B}^n .

Rong and Yang [10] extended this idea by replacing the unit ball with a bounded, balanced, convex domain. Recall that a domain $\Omega \subseteq \mathbb{C}^n$ is called balanced if λz belongs to Ω for each z in Ω and λ in the closed unit disc $\overline{\mathbb{D}}$ of the complex plane. Its Minkowski function h_Ω on \mathbb{C}^n is defined by

$$h_\Omega(z) := \inf\{t > 0 : z/t \in \Omega\}$$

and $\Omega(r) = \{z \in \mathbb{C}^n : h_\Omega(z) < r\}$ for $0 < r \leq 1$.

For a bounded domain $D \subseteq \mathbb{C}^n$, the generalised squeezing function S_D^Ω on D is defined by

$$S_D^\Omega(z) := \sup\{r : \Omega(r) \subseteq f(D), f \in O_u(D, \Omega), f(z) = 0\}.$$

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In [5], we introduced the d -balanced squeezing function by replacing a balanced domain with a d -balanced domain.

Let $d = (d_1, d_2, \dots, d_n) \in \mathbb{Z}_n^+$, $n \geq 2$. Then a domain $\Omega \subseteq \mathbb{C}^n$ is said to be d -balanced if $(\lambda^{d_1} z_1, \lambda^{d_2} z_2, \dots, \lambda^{d_n} z_n) \in \Omega$ for each $z = (z_1, z_2, \dots, z_n) \in \Omega$ and $\lambda \in \overline{\mathbb{D}}$.

For a d -balanced domain Ω , the d -Minkowski function $h_{d,\Omega}$ on \mathbb{C}^n is defined by

$$h_{d,\Omega}(z) := \inf \left\{ t > 0 : \left(\frac{z_1}{t^{d_1}}, \frac{z_2}{t^{d_2}}, \dots, \frac{z_n}{t^{d_n}} \right) \in \Omega \right\}.$$

Let $\Omega^d(r) = \{z \in \mathbb{C}^n : h_{d,\Omega}^d(z) < r\}$ for $0 < r \leq 1$.

DEFINITION 1.1. For a bounded domain $D \subseteq \mathbb{C}^n$ and a bounded, convex, d -balanced domain Ω , where $d = (d_1, d_2, \dots, d_n)$, the d -balanced squeezing function $S_{d,D}^\Omega$ is

$$S_{d,D}^\Omega(z) := \sup \{r : \Omega^d(r) \subseteq f(D), f \in \mathcal{O}_u(D, \Omega), f(z) = 0\}.$$

We can easily see that if Ω is balanced, then $d = (1, 1, \dots, 1)$ and $S_{d,D}^\Omega$ reduces to S_D^Ω .

In [4], Fornæss and Scherbina gave an example of a domain whose squeezing function is nonplurisubharmonic. Recently, Rong and Yang [11] gave examples of domains with nonplurisubharmonic generalised squeezing functions. Here we consider the same problem for d -balanced squeezing functions and present an example (see Theorem 3.5).

2. Background and an estimate for a d -balanced squeezing function

Let us first recall the definitions of the Carathéodory pseudodistance and the Carathéodory extremal maps. For a domain $D \subseteq \mathbb{C}^n$ and $z_1, z_2 \in D$, a Carathéodory pseudodistance c_D on D is defined by

$$c_D(z_1, z_2) = \sup_f \{p(0, \mu) : f \in \mathcal{O}(\mathbb{D}, D), f(z_1) = 0, f(z_2) = \mu\},$$

where p denotes the Poincaré metric on the unit disc \mathbb{D} and $\mathcal{O}(\mathbb{D}, D)$ denotes the set of holomorphic maps from \mathbb{D} to D . A function $f \in \mathcal{O}(\mathbb{D}, D)$ at which this supremum is attained is called a Carathéodory extremal function.

We now recall results that will be used in this section. Lempert [8, Theorem 1] and Kosiński and Warszawski [7, Theorem 1.3 and Remark 1.6] yield the following result.

RESULT 2.1. For a convex domain $\Omega \subseteq \mathbb{C}^n$, $c_\Omega = \tilde{k}_\Omega$, where \tilde{k}_Ω denotes the Lempert function on Ω .

Combining Result 2.1 with [2, Theorem 1.6], we get the following result.

RESULT 2.2. For a bounded, convex, $d=(d_1, d_2, \dots, d_n)$ -balanced domain $\Omega \subseteq \mathbb{C}^n$,

$$\tanh^{-1} h_{d,\Omega}(z)^L \leq c_\Omega(0, z) = \tilde{k}_\Omega(0, z) \leq \tanh^{-1} h_{d,\Omega}(z),$$

where $L = \max_{1 \leq i \leq n} d_i$.

RESULT 2.3. Let $\Omega \subseteq \mathbb{C}^n$ be a domain and $K \subseteq \Omega$ be compact such that $\Omega \setminus K$ is connected. Then, each holomorphic function f on $\Omega \setminus K$ extends to a holomorphic function on Ω .

RESULT 2.4 (see [9, Proposition 1]). Let $\Omega \subseteq \mathbb{C}^n$ be a balanced domain and let $h_{d,\Omega}$ be its d -Minkowski function. Then Ω is pseudoconvex if and only if $h_{d,\Omega}$ is plurisubharmonic.

RESULT 2.5 (see [6, Remark 2.2.14]). If $\Omega \subseteq \mathbb{C}^n$ is a d -balanced domain, then:

- (1) $\Omega = \{z \in \mathbb{C}^n : h_{d,\Omega}(z) < 1\}$;
- (2) $h_{d,\Omega}(\lambda^{d_1} z_1, \lambda^{d_2} z_2, \dots, \lambda^{d_n} z_n) = |\lambda| h_{d,\Omega}(z)$ for each $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$;
- (3) $h_{d,\Omega}$ is upper semicontinuous.

For a bounded domain $\Omega \subseteq \mathbb{C}^n$ and a compact subset K of Ω , denote

$$d_{c_\Omega}^K(z) = \min_{w \in K} \tanh(c_\Omega(z, w)).$$

We begin with the following theorem for d -balanced domains, which is analogous to Theorem 2.1 in [11].

THEOREM 2.6. Let $\Omega \subseteq \mathbb{C}^n$ be a bounded, $d=(d_1, d_2, \dots, d_n)$ -balanced, convex, homogeneous domain. If K is a compact subset of Ω such that $D = \Omega \setminus K$ is connected, then

$$S^d(z)^L \leq d_{c_\Omega}^{\partial K}(z) = d_{c_\Omega}^K(z) \leq S^d(z), \tag{2.1}$$

where $L = \max_{1 \leq i \leq n} d_i$.

PROOF. For $z \in D$, let $g \in \text{Aut}(\Omega)$ be such that $g(z) = 0$. Using the convexity of Ω , we have $\{tv + (1 - t)w : 0 \leq t \leq 1, v \in D, w \in K^\circ\} \cap \partial K \neq \emptyset$; therefore, $d_{c_\Omega}^{\partial K}(v) = d_{c_\Omega}^K(v)$ for each $v \in D$.

Clearly, $h = g|_D : D \rightarrow \Omega$ is injective and holomorphic with $h(z) = 0$. For notational convenience, let us denote $d_{c_\Omega}^K(z)$ by α . We claim that $\Omega^d(\alpha) \subseteq h(D)$. Let $h_{d,\Omega}(v) < \alpha$, which upon using Result 2.2 implies $\tanh(c_\Omega(0, v)) < \alpha$. Since g is an automorphism, we get $\tanh(c_{g(\Omega)}(g(z), g(v))) < \alpha$ for $v \in \Omega$. Therefore, $\tanh(c_\Omega(z, v')) < \alpha = \min_{w \in K} \tanh(c_\Omega(z, w))$. Thus, we get $v' \notin K$ and therefore, $v = g(v') \in g(D)$. This proves our claim and hence, we obtain

$$S^d(z) \geq \alpha = d_{c_\Omega}^K(z).$$

For the other inequality, consider an injective holomorphic map $f : D \rightarrow \Omega$ such that $f(z) = 0$. By Result 2.3, there exists a holomorphic function $F : \Omega \rightarrow \mathbb{C}^n$ such that $F|_D = f$. Using Result 2.4 and following the argument as in [11, Theorem 2.1], we obtain $F(\Omega) \subseteq \Omega$. Observe that $F(\partial K) \cap F(D) \neq \emptyset$. Let $r > 0$ be such that $\Omega^d(r) \subseteq F(D)$. If possible, let $\tanh(c_\Omega(0, F(\partial K)))^{1/L} < r$, then upon using Result 2.2, we get $h_{d,\Omega}(F(\partial K)) < r$. This implies that $F(\partial K) \in \Omega^d(r) \subseteq F(D)$, which is a contradiction. Therefore, $r < \tanh(c_\Omega(0, F(\partial K)))^{1/L}$ which, upon using the decreasing property of c_Ω , implies that $r < \tanh(c_\Omega(z, \partial K))^{1/L}$. Finally, we can conclude that $S^d(z)^L \leq d_{c_\Omega}^K(z)$. \square

REMARK 2.7. A careful look at the above proof makes it clear that the left-hand side of inequality (2.1) holds even if Ω is not homogeneous.

3. Nonplurisubharmonic d -balanced squeezing functions

Let $G_2 \subseteq \mathbb{C}^2$ be the domain defined by

$$G_2 = \{(z_1 + z_2, z_1 z_2) : z_1, z_2 \in \mathbb{D}\}.$$

The domain G_2 is called the symmetrised bidisc. Its genesis lies with the problem of ‘robust stabilisation’ in control engineering. Although it is closely related to the bidisc, its geometry is very different to that of the bidisc. It is polynomially convex, hyperconvex and starlike about the origin but not convex ($(2, 1), (2i, -1) \in G_2$ but $(1 + i, 0) \notin G_2$). Another point to note here is that it is not homogeneous (there is no automorphism of G_2 , which maps any $(a, 0)$ with $0 < a < 1$ to $(0, 0)$). For many equivalent characterisations of G_2 , see [1].

The domain G_2 has several interesting properties. For example, Lempert’s theorem holds for G_2 even though it is neither convex nor can it be exhausted by domains biholomorphic to convex domains.

We will require Carathéodory extremal maps for G_2 to prove our result. Agler and Young [1] proved that for each $z_1, z_2 \in G_2$, there exists $\lambda \in \mathbb{C}, |\lambda| = 1$ such that ϕ_λ is a Carathéodory extremal function, where ϕ_λ is defined by

$$\phi_\lambda(z_1, z_2) = \frac{2\lambda z_2 - z_1}{2 - \lambda z_2}.$$

It is easy to check that G_2 is $(1, 2)$ -balanced. Let us denote by Ω the set of all possible linear combinations of elements of G_2 , that is, Ω is the convex hull of G_2 . We begin with the following lemma.

LEMMA 3.1. *Let Ω be the convex hull of G_2 . Then:*

- (1) Ω is $(1, 2)$ -balanced;
- (2) $|z_1| < 2$ and $|z_2| < 1$ for each point $(z_1, z_2) \in \Omega$.

PROOF. (1) Let $\sum_{i=1}^k \alpha_i z_i \in \Omega$, $\sum_{i=1}^k \alpha_i = 1$, $\alpha_i \geq 0$ and $z_i = (z_i^{(1)}, z_i^{(2)}) \in G_2$ for each i . Let $|\lambda| \leq 1$. Each $z_i \in G_2$ so that $(\lambda z_i^{(1)}, \lambda^2 z_i^{(2)}) \in G_2$. Therefore, $\sum_{i=1}^k \alpha_i (\lambda z_i^{(1)}, \lambda^2 z_i^{(2)}) = (\lambda \sum_{i=1}^k \alpha_i z_i^{(1)}, \lambda^2 \sum_{i=1}^k \alpha_i z_i^{(2)}) \in \Omega$ and hence, Ω is $(1, 2)$ -balanced.

(2) Follows from the structure of G_2 . □

Choose r , with $0 < r < 1$, such that $\overline{\mathbb{D}^2(0, r)}$, the closure of the polydisk in \mathbb{C}^2 of radius r centred at the origin, is contained in Ω . Take the point $Q = (0, r) \in \overline{\mathbb{D}^2(0, r)} \subseteq \Omega$ and let $\epsilon > 0$ be such that a ball $\mathbb{B}^2(Q, r)$ of radius $\epsilon < r$ centred at Q is contained in Ω . Let us take $K = \partial\mathbb{D}^2(0, r) \setminus \mathbb{B}^2(Q, \epsilon)$ and $H = \{z \in \mathbb{C}^2 : z_2 = 0\}$. It can be seen that K is compact and $D = \Omega \setminus K$ is connected. We will show that $S_{d,D}^\Omega$ (denoted by S^d for notational convenience) is not plurisubharmonic. For this, we will show that $h = S^d|_{\mathbb{D}^n(0,r) \cap H}$ does not satisfy the maximum principle.

In particular, such a restriction is not subharmonic; this, in turn, will imply that S^d is not plurisubharmonic. This is proved via the following steps.

- We begin by showing that $S^d(0) \geq \frac{1}{2}$.
- We use Theorem 2.6 to show that

$$S^d(z)^2 \leq \frac{r - |z|}{2 - r|z|}$$

for $z \in \mathbb{D}^n(0, r) \cap H$, $z \neq 0$ (observe Remark 2.7).

- We then show that $S^d(z) \leq S^d(0)$ for $z = (z_1, 0) \in \mathbb{D}^n(0, r) \cap H$ for $r > |z| > \beta$, where $\beta = r(4 - r)/(4 - r^3)$ (note that $\beta < r$).
- Now we restrict h to $A = \mathbb{B}(0, \beta)$ to obtain a maximum at some $a \in A$.
- We conclude by combining all these points along with the observation that $S^d(z) \rightarrow 0$ as $|z| \rightarrow r$.

LEMMA 3.2. $S^d(0) \geq \frac{1}{2}$.

PROOF. Consider the identity map $\text{id} : D \rightarrow \Omega$. Clearly, id is injective holomorphic with $\text{id}(0) = 0$. We claim that $\Omega^d(r/2) \subseteq \text{id}(D) = D$. To see this, take z such that $h_{d,\Omega}(z) < r/2$. Upon using Result 2.5, we first get $z \in \Omega$ and then $(2z_1/r, 4z_2/r^2) \in \Omega$. Now using Lemma 3.1(2), $|z_1| < r$ and $|z_2| < r^2/4 < r$. Thus, $z \notin \partial\mathbb{D}^2(0, r)$ and therefore, $z \notin K$. This proves our claim and shows that $S^d(0) \geq 1/2$. □

We need the following elementary lemma to prove our next proposition.

LEMMA 3.3. For each $z = (z_1, 0)$, $z_1 = a + ib$ with $0 < |z_1| < r$ and $w_0 = (sz_1, 0)$, where $s = r/\sqrt{a^2 + b^2}$, we have

$$\left| \frac{\phi_\tau(z) - \phi_\tau(w_0)}{1 - \phi_\tau(z)\phi_\tau(w_0)} \right| \leq \frac{r - |z_1|}{1 - r|z_1|},$$

where $\tau \in \mathbb{C}$ with $|\tau| = 1$.

PROOF. First note that $|w_0| = r$ and $|(2 - \tau z_1)(2 - \tau w_0)| \geq (2 - r)^2 > 1$. Now consider

$$\begin{aligned} \left| \frac{\phi_\tau(z) - \phi_\tau(w_0)}{1 - \phi_\tau(z)\phi_\tau(w_0)} \right| &= \left| \frac{\frac{z_1}{2 - \tau z_1} - \frac{w_0}{2 - \tau w_0}}{1 - \frac{z_1}{2 - \tau z_1} \frac{w_0}{2 - \tau w_0}} \right| = \left| \frac{\frac{2(z_1 - w_0)}{(2 - \tau z_1)(2 - \tau w_0)}}{1 - \frac{z_1}{2 - \tau z_1} \frac{w_0}{2 - \tau w_0}} \right| \\ &\leq \frac{2|z_1 - w_0|}{1 - |z_1||w_0|} = \frac{r - |z_1|}{1 - |z_1|r}. \end{aligned} \quad \square$$

PROPOSITION 3.4. For each $z = (z_1, 0) \in \mathbb{D}^n(0, r) \cap H$, $z \neq 0$, we have

$$S^d(z)^2 \leq \frac{r - |z_1|}{1 - r|z_1|}.$$

PROOF. Let $z_1 = a + ib$. Take $w_0 = (sz_1, 0)$, where $s = r/\sqrt{a^2 + b^2}$ so that $w \in K$ because $Q < r$. Now, using Theorem 2.6 and Remark 2.7, we obtain

$$\begin{aligned} S^d(z)^2 &\leq d_{c_\Omega}^K(z) = \min_{w \in K} \tanh(c_\Omega(z, w)) \\ &\leq \tanh(c_\Omega(z, w_0)) \\ &\leq \tanh(c_{G_2}(z, w_0)) \quad (\text{since } G_2 \subseteq \Omega) \\ &= \left| \frac{\phi_\tau(z) - \phi_\tau(w_0)}{1 - \phi_\tau(z)\phi_\tau(w_0)} \right| \quad (\text{for some } |\tau| \leq 1) \\ &\leq \frac{r - |z_1|}{1 - r|z_1|} \quad (\text{using Lemma 3.3}). \quad \square \end{aligned}$$

This proposition, in particular, implies $S^d(z) \rightarrow 0$ as $|z_1| \rightarrow r$. We summarise these results with the following theorem.

THEOREM 3.5. *For D and Ω as considered above, S^d is not plurisubharmonic.*

PROOF. It is easy to see that

$$\frac{r - |z_1|}{1 - r|z_1|} < \frac{r^2}{4}$$

if and only if $|z_1| > \beta$. Using Proposition 3.4 and Lemma 3.2, we get $h(z) = S^d(z) < r/2 \leq S^d(0) = h(0)$ for $z = (z_1, 0)$ with $|z_1| > \beta$. Consider the restriction $h|_{\overline{\mathbb{B}(0, \beta)}}$ and let $h(a)$ be its maximum for some $a \in \overline{\mathbb{B}(0, \beta)}$. Then, we have $h(z) \leq \max(h(a), h(0))$ on $\mathbb{D}^n(0, r) \cap H$ proving that $S^d|_{\mathbb{D}^n(0, r) \cap H}$ does not satisfy the maximum principle and S^d is not plurisubharmonic. \square

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