RETRACEABLE SETS

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1. Introduction. Let us compare two properties of sets of non-negative integers: (1) the set α has *property* Γ, if there exists an effective procedure which when applied to any element of α different from its maximum (which α does not necessarily possess) yields the next larger element of α; (2) the set α has *property* Δ, if there exists an effective procedure which when applied to any element of α different from its minimum yields the next smaller element of α. It is readily seen that every recursive set has both properties. Let α be any infinite set with property Γ. We define: $f(0)$ = the minimum of $\alpha, f(n + 1)$ = the element obtained when the effective procedure is applied to $f(n)$. It is clear that $f(n)$ is a strictly increasing recursive function ranging over α. The class of all sets with property Γ is therefore the same as the well-known denumerable class of all recursive sets. We now show that there are c non-recursive sets which possess property Δ. Let $\{a_n\}$ be any sequence of numbers chosen from the set $(0, \dots, 9)$, but such that $a_0 \neq 0$. Put

$$\sigma = (a_0, 10\cdot a_0 + a_1, 10^2\cdot a_0 + 10\cdot a_1 + a_2, \dots)$$

$$f(x) = \begin{cases} x \text{ for } 0 \leqslant x \leqslant 9, \\ \left[\dfrac{x}{10}\right] \text{ for } x \geqslant 10. \end{cases}$$

We see that $f(x)$ is a recursive function which maps a_0 onto itself and every other element of σ onto the next smaller element of σ. Thus, σ is an infinite set with property Δ. The sequence $\{a_n\}$ can be chosen in c different ways and different choices of $\{a_n\}$ yield different sets σ. Since there are only c sets (that is, of non-negative integers), it follows that there exist exactly c sets with property Δ; only $\aleph_0$ of these are recursive, hence c are non-recursive.

It is the purpose of this paper to prove several theorems concerning sets with property Δ (henceforth called *retraceable sets*), in particular:

(1) *every degree of unsolvability can be represented by a retraceable set;*

(2) *every degree of unsolvability which can be represented by a r.e.* (*that is, recursively enumerable*) *set can also be represented by a r.e. set with a retraceable complement.*

2. Notations and terminology. A non-negative integer is called a *number*, a collection of numbers is called a *set* and a collection of sets a *class.*

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The Boolean operations of union, intersection and complementation are written $+$, $\cdot$, $'$ respectively; $\subset$ stands for inclusion (proper or not), ϕ for the empty set of numbers, ϵ for the set of all numbers, δf and ρf for the domain and range of the function $f(x)$ respectively, and $fg(x)$ for the function $f(g(x))$. The cardinal of a collection Θ is denoted by card Θ, the minimum of α (in case $\alpha \neq \phi$) by min α and the maximum of α (in case $\alpha \neq \phi$ and is finite) by max α. We write $c_\sigma(x)$ for the characteristic function of σ and $p_\sigma(x)$ for the function with domain σ which maps the minimum of σ onto itself and every other element of σ onto the next smaller element of σ. In the special case $\sigma = \phi$ the function $p_\sigma(x)$ is nowhere defined; if $\sigma = \epsilon$ it is the usual predecessor function. If α is an infinite set, the strictly increasing function ranging over α is called the *principal function of* α and denoted by $h_\alpha(x)$. The function $f(x)$ is *downward* if $f(x) \leqslant x$ for every $x \in \delta f$. The sets α and β are *separable* (written $\alpha|\beta$) if there are disjoint r.e. sets α_1 and β_1 such that $\alpha \subset \alpha_1$ and $\beta \subset \beta_1$.

We assume the reader to be familiar with the following notions: array, discrete array, immune set, simple set, hypersimple set (see, for instance (**1**)). A set is *hyperimmune* if it is infinite and its complement includes at least one row of every discrete array. We shall use the fact due to Rice (**7**, Theorem 21) that the infinite set α is hyperimmune if and only if $h_\alpha(x)$ is not bounded by any recursive function. It is well-known that there is an array in which every finite set occurs exactly once (**6**, p. 304); $\{\rho_n\}$ will denote a specific array of this type which has the additional property $\rho_0 = \phi$.

Definition. The set α is *retraceable*, if $p_\alpha(x)$ has a partial recursive extension. If α is retraceable, every partial recursive extension of $p_\alpha(x)$ is a *retracing function of* α or a *function which retraces* α. A function $r(x)$ is a *retracing function* if it retraces at least one infinite set.

Definition. The number a is an *initial number* of the downward function $f(x)$ if $a \in \delta f$ and $f(a) = a$. The set of all initial numbers of a downward function is the *initial set* of that function.

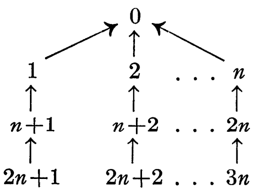
Definition. A set is *introreducible*, if it is (Turing) reducible to each of its infinite subsets.

The notions of retraceability and introreducibility were communicated by R. S. Tennenbaum.

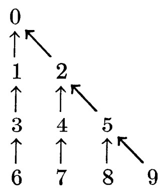
3. Examples. Many downward functions are conveniently described by a diagram. Let n be any number $\geqslant 1$, and let $p_0, p_1, \ldots$ be the sequence of all primes arranged according to size. The following six diagrams (supposed to be extended indefinitely) are self-explanatory.

We denote the functions described in the six diagrams by $r_1(x), \ldots, r_6(x)$ respectively. Each of the functions $r_1(x), \ldots, r_5(x)$ is partial recursive and retraces at least one infinite set; $r_6(x)$ is partial recursive and downward, but retraces only finite sets. For $1 \leqslant i \leqslant 6$, let σ_i stand for the initial set of $r_i(x)$

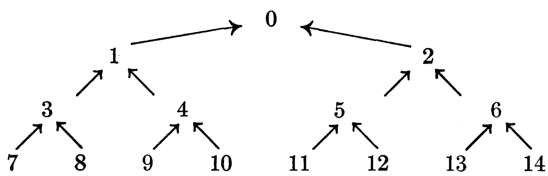
Example 1



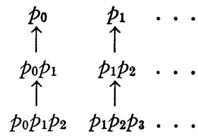
Example 2



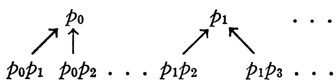
Example 3



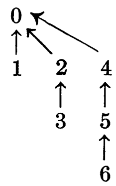
Example 4



Example 5



Example 6



and T_i for the class of all *infinite* sets retraced by $r_i(x)$. Knowing the cardinality of σ_i does not tell us much about the cardinality of T_i or vice versa. For the cardinalities of $\sigma_1, \ldots, \sigma_6$ are 1, 1, 1, $\aleph_0$, $\aleph_0$, 1 respectively and those of $T_1, \ldots, T_6$ are n, $\aleph_0$, c, $\aleph_0$, c, 0 respectively. Note that $r_2(x)$ and $r_4(x)$ have the common property that they retrace exactly denumerably many infinite sets, all of which are recursive; but, while the sets in T_4 are mutually disjoint, the sets in T_2 all contain 0, that is, the only initial number of $r_2(x)$. Examples 3 and 5 illustrate the existence of retracing functions which retrace c mutually almost disjoint infinite sets.

4. Propositions. Let α be an immune, retraceable set and $f(x)$ a retracing function of α. Then $\alpha \subset \delta f$ and this inclusion must be proper, because δf is r.e. In certain cases we can actually find elements in δf which cannot belong to α. For if $f(x)$ is not downward every element x_1 such that

$$(1) \qquad x_1 \in \delta f \text{ and } x_1 < f(x_1)$$

belongs to α', and if ρf is not included in δf every element x_2 such that

$$(2) \qquad x_2 \in \delta f \text{ and } f(x_2) \notin \delta f$$

belongs to α'. Omitting an element x_1 from δf which satisfies (1) or an element x_2 which satisfies (2) changes $f(x)$, but not the class of all sets retraced by

$f(x)$. We call a function $f(x)$ *special* if $f(x)$ is downward and $\rho f \subset \delta f$. If the function $r(x)$ retraces α, the function $r_1(x)$ defined by

$$r_1(x) = \begin{cases} r(x) \text{ if } x \in \delta r \text{ and } r(x) \leqslant x, \\ \ x \ \text{ if } x \in \delta r \text{ and } r(x) > x, \end{cases}$$

is a downward function which retraces α. The subset δ^* of all elements $x \in \delta r_1$ such that $r_1(x), r_1r_1(x), \ldots$ are all defined (after finitely many elements all elements in this sequence are then the same) is r.e. If $r_2(x)$ is the restriction of $r_1(x)$ to δ^*, $r_2(x)$ is a special retracing function of α. Thus, *a set is retraceable if and only if it has a special retracing function.*

For every special function $f(x)$ we define

$$\text{(3)} \qquad \begin{cases} f^0(x) = x, f^{n+1}(x) = ff^n(x), \\ n(x) = (\mu z)[f^{z+1}(x) = f^z(x)], \\ \rho_{g(x)} = (f^0(x), \ldots, f^{n(x)}(x)) - (f^0(x)). \end{cases}$$

The functions $n(x)$ and $g(x)$ have the same domain as $f(x)$, and if $f(x)$ is partial recursive, so are $n(x)$ and $g(x)$. The function $g(x)$ is called the *associated function* of $f(x)$ and is denoted by $f^*(x)$.

PROPOSITION P1. *The set α is retraceable if and only if there is a partial recursive function $t(x)$ such that*

$$\text{(a)} \qquad x \in \alpha \Rightarrow \begin{cases} t(x) \text{ is defined}, \\ \rho_{t(x)} = \{y | y \in \alpha \text{ and } y < x\}. \end{cases}$$

Proof. We ignore the trivial case $\alpha = \phi$.

(1) Let the function $t(x)$ satisfy (a). Put

$$r(x) = \begin{cases} \min \alpha, \text{ if } \rho_{t(x)} = \phi, \text{ that is, } t(x) = 0, \\ \max \rho_{t(x)}, \text{ if } \rho_{t(x)} \neq \phi, \text{ that is, } t(x) > 0, \end{cases}$$

then $r(x)$ is a retracing function of α with the same domain as $t(x)$.

(2) Let α be retraceable. Then α has a special retracing function, say $r(x)$. If $t(x) = r^*(x)$, $t(x)$ is related to α by (a).

In the special case that α has a r.e. complement condition (a) in P1 can be replaced by

$$\text{(b)} \qquad \begin{cases} \text{there is a partial recursive function } t_1(x) \text{ such that} \\ x \in \alpha \Rightarrow \begin{cases} t_1(x) \text{ is defined}, \\ t_1(x) = \text{card } \{y | y \in \alpha \text{ and } y < x\}. \end{cases} \end{cases}$$

Since (a) obviously implies (b) it suffices to prove the converse. Assume (b). Let $a'(n)$ be a recursive function ranging over α' and put $u(x) = x - t_1(x)$. Let δ^* be the set of all elements x of δt_1 such that α' contains at least $u(x)$ elements $< x$. Then δ^* is a r.e. subset of δt_1 and for every $x \in \delta^*$ we can effectively find the $u(x)$ elements $b_{x1}, \ldots, b_{x,u(x)}$ which are the first $u(x)$ elements $< x$ which show up in the sequence $a'(0), a'(1), \ldots$. Thus there is a partial recursive function $q(x)$ defined on δ^* such that

$$\rho_{q(x)} = (0, \ldots, x - 1) - (b_{x1}, \ldots, b_{x,u(x)}).$$

Now assume $x \in \alpha$. Among the x elements $0, \ldots, x - 1$ exactly $u(x)$ belong to α' and exactly $t_1(x)$ to α. Hence,

$$x \in \alpha \Rightarrow x \in \delta^* \text{ and } \rho_{q(x)} = \{y | y \in \alpha \text{ and } y < x\}.$$

We call the special function $f(x)$ *non-trivial* if the function

$$(4) \qquad m(x) = \text{card } \rho_{f^*(x)}$$

is not bounded. Every non-trivial special function has therefore an infinite domain (the converse is false as is illustrated by the identity function) and every special retracing function is non-trivial. The function described in Example 6 is non-trivial and special, but not a retracing function. Let

$$(5) \qquad \tau_n = \{x | m(x) = n\}.$$

The formulas (3), (4), and (5) associate with every special function $f(x)$ a sequence $\{\tau_n\}$ of sets; this sequence will be called the *sequence associated with* $f(x)$. The sets $\tau_0, \tau_1, \ldots$ are mutually disjoint for any special function $f(x)$; if $f(x)$ is non-trivial they are non-empty. In the special case that $f(x)$ is also partial recursive, the associated sequence $\{\tau_n\}$ is a r.e. sequence of mutually disjoint non-empty r.e. sets; in that case there is a recursive function $t(n, x)$ such that $\tau_n = \rho t(n, x)$ for every n. Finally, if $f(x)$ is a special retracing function, every infinite set retraced by $f(x)$ contains exactly one element of each of the sets $\tau_0, \tau_1, \ldots$.

Proposition P2. *Every recursive set is retraceable and every retraceable set is introreducible.*

Proof. The first part is obvious. Every finite set is trivially introreducible. Let α be an infinite retraceable set, $t(x)$ the function associated with a special retracing function of α and β an infinite subset of α. We claim that α is reducible to β. Put

$$g(x) = (\mu y)[x < y \text{ and } y \in \beta],$$

then $\delta g = \epsilon$, $\rho g \subset \delta t$ and $g(x)$ is recursive in $c_\beta(x)$. Moreover,

$$x \in \alpha \Leftrightarrow x \in \{y | y \in \alpha \text{ and } y < g(x)\} = \rho_{tg(x)};$$

hence, $c_\alpha(x)$ is recursive in $c_\beta(x)$.

Proposition P3. *Every introreducible set is recursive or immune.*

Proof. Let α be introreducible, but not immune. Either α is finite, hence recursive, or α has an infinite r.e. subset. In the latter case α also has an infinite recursive subset, say η. In that case α is recursive, because it is reducible to η.

Corollary. *Every retraceable set is recursive or immune.*

Proposition P4. *The family of all principal functions of infinite retraceable sets is closed under composition.*

Proof. Let γ be the range of $h_\alpha h_\beta$, where α and β are infinite retraceable sets. Since both h_α and h_β are strictly increasing, so is $h_\alpha h_\beta$; thus $h_\gamma = h_\alpha h_\beta$. Let $f(u)$ be the function defined on $\gamma - (h_\gamma(0))$ which maps $h_\gamma(x+1)$ onto $h_\gamma(x)$. To prove that γ is retraceable it suffices to show that $f(u)$ has a partial recursive extension. Let $t(x)$ be the function associated with a special retracing function of α and let $r(x)$ be a retracing function of β. Put

$$m(u) = \text{card } \rho_{t(u)} \text{ for } u \in \delta t,$$
$$\sigma = \{u | u \in \delta t \text{ and } t(u) > 0 \text{ and } m(u) \in \delta r\}.$$

Then $m(u)$ is a partial recursive funcion and σ a r.e. set. We write $m = m(u)$, keeping in mind that m depends on u. For every element u in σ the m elements $a_u(0), \ldots, a_u(m-1)$ such that

$$\rho_{t(u)} = (a_u(0), \ldots, a_u(m-1))$$
$$a_u(0) < a_u(1) < \ldots < a_u(m-1)$$

can be effectively found. Let $g(u) = a_u r(m)$ for $u \in \sigma$, then $g(u)$ is a partial recursive function. In the special case $u = h_\gamma(x+1)$ we have:

$$a_u(0) = h_\alpha(0), \ldots, a_u(m-1) = h_\alpha(m-1),$$
$$m = h_\beta(x+1), \; r(m) = h_\beta(x),$$
$$g(u) = a_u r(m) = h_\alpha r(m) = h_\alpha h_\beta(x) = h_\gamma(x) = f(u).$$

Thus $g(u)$ is a partial recursive extension of $f(u)$.

COROLLARY. *Every infinite retraceable set has c infinite retraceable subsets.*

Proof. Let α and σ be infinite retraceable sets and $\alpha_\sigma = \rho h_\alpha h_\sigma$, then α_σ is an infinite retraceable subset of α. We know from the introduction that σ can be chosen in c different ways. The desired result now follows from the fact that different choices of σ yield different sets α_σ.

PROPOSITION P5. *Every retraceable set with a r.e. complement has a recursive special retracing function.*

Proof. Let α be retraceable and α' r.e. If α is recursive, the function $p_\alpha(x)$ is partial recursive and the function $f(x)$ such that

$$f(x) = p_\alpha(x) \text{ for } x \in \alpha; \; f(x) = x \text{ for } x \notin \alpha,$$

is a recursive special retracing function of α. Now, assume α is not recursive; in this case both α and α' are infinite. Let $r(x)$ be a special retracing function of α and let $a'(x)$ and $d(x)$ be one-to-one recursive functions ranging over α' and δr respectively. Put $b(2n) = a'(n)$, $b(2n+1) = d(n)$, then every number occurs at least once in the sequence $b(0), b(1), \ldots$. Let us call the number x an a'-number, if $(\mu n)[x = b(n)]$ is even, otherwise a d-number. Observe that every element of α is a d-number, while α' contains both d-numbers and a'-numbers. Let

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$$f(x) = \begin{cases} r(x) \text{ if } x \text{ is a } d\text{-number}, \\ x \text{ if } x \text{ is an } a'\text{-number}. \end{cases}$$

Since we can effectively determine for every number whether it is a d-number or an a'-number, we see that $f(x)$ is a recursive retracing function of α. The function $f(x)$ is downward, because $r(x)$ is downward; $\rho f \subset \delta f$ since $\delta f = \epsilon$. Thus $f(x)$ is a special function.

PROPOSITION P6. *Two disjoint retraceable sets have a common retracing function if and only if they are separable.*

Proof. Let α_1 and α_2 be disjoint and retraceable, $r_1(x)$ and $r_2(x)$ retracing functions of α_1 and α_2 respectively. We may assume without loss of generality that α_1 and α_2 are non-empty; put $m_1 = \min \alpha_1$, $m_2 = \min \alpha_2$. Assume $\alpha_1|a_2$, say $\alpha_1 \subset \beta_1$, $\alpha_2 \subset \beta_2$, where β_1 and β_2 are disjoint and r.e. Define

$$r_3(x) = \begin{cases} r_1(x) \text{ for } x \in \beta_1 \cdot \delta r_1, \\ r_2(x) \text{ for } x \in \beta_2 \cdot \delta r_2, \end{cases}$$

then $r_3(x)$ retraces both α and β. To prove the converse, assume α_1 and α_2 have a common retracing function. It is readily seen that this implies that α_1 and α_2 have a common special retracing function say $r(x)$. Let $t(x) = r^*(x)$. Put

$$\beta_1 = \{x|x \in \delta t \text{ and } \min \rho_{t(x)} = m_1\} + (m_1),$$
$$\beta_2 = \{x|x \in \delta t \text{ and } \min \rho_{t(x)} = m_2\} + (m_2),$$

then α_1 and α_2 are separated by the disjoint r.e. sets β_1 and β_2.

5. Theorems.

THEOREM T1. *There are exactly c retraceable sets; among these* $\aleph_0$ *are recursive, c hyperimmune and c immune, but not hyperimmune.*

Proof. Let us call a sequence $\{a_n\}$ of numbers *decimal* if $1 \leqslant a_0 \leqslant 9$ and $0 \leqslant a_n \leqslant 9$ for $n \geqslant 1$. In the introduction we defined a one-to-one correspondence between the family of all decimal sequences and a certain class of c infinite retraceable sets; we denote this correspondence by Φ. If $\sigma = \Phi < a_n >$,

$$h_\sigma(n) = 10^n \cdot a_0 + 10^{n-1} \cdot a_1 + \ldots + a_n.$$

This implies that $\Phi < a_n >$ is a recursive set if and only if a_n is a recursive function of n. There are c decimal sequences $\{a_n\}$ in which a_n is not a recursive function of n; the images of these sequences under Φ are therefore retraceable sets which are immune. None of these c sets is, however, hyperimmune, since each contains exactly one element of $(0, \ldots, 9)$, exactly one element of $(10, \ldots, 99)$ etc. We proceed to prove that there exist c retraceable sets which are hyperimmune. It suffices to prove the existence of a single retraceable set which is hyperimmune. For if θ is such a set, θ has c infinite retraceable subsets by the Corollary of P4, and every infinite subset of a hyperimmune set is again hyperimmune. With every sequence $\{a_n\}$ of numbers we associate a set

$$\alpha = \Psi < a_n > = (a_0, a_0a_1, a_0a_1a_2, \ldots).$$

Let $\mathfrak{F}$ denote the family of all strictly increasing sequences of primes. If $\alpha = \Psi < a_n >$, where $\{a_n\} \in \mathfrak{F}$, we can for every element $x \in \alpha - (a_0)$ obtain $p_\alpha(x)$ from x by dividing x by its greatest prime factor. Thus Ψ maps $\mathfrak{F}$ onto a class of infinite retraceable sets. Let $p_0, p_1, \ldots$ be the sequence of all primes arranged according to size. Suppose $t(n)$ is the principal function of a hyperimmune set. Put $q_n = p_{t(n)}$, $\theta = \Psi < q_n >$, $r_n =$ the principal function of θ, then $\{q_n\} \in \mathfrak{F}$ and θ is an infinite retraceable set. The function $t(n)$ is not bounded by any recursive function, because it is the principal function of a hyperimmune set **(7**, Theorem 21**)**. Taking into account that $t(n) < q_n \leqslant r_n$, it follows that r_n is not bounded by any recursive function. This implies that θ is hyperimmune.

In the following we write $\alpha \equiv_T \beta$ if α and β are (Turing) reducible to each other, that is, if $c_\alpha(x)$ and $c_\beta(x)$ are recursive in each other. We shall use the well-known functions $j(x, y)$, $k(z)$, $l(z)$ defined by

$$\begin{aligned} j(x, y) &= x + \tfrac{1}{2}(x + y)(x + y + 1), \\ k(z) &= (\mu x)(\exists y)[j(x, y) = z], \\ l(z) &= (\mu y)(\exists x)[j(x, y) = z]. \end{aligned}$$

THEOREM T2. *Every degree of unsolvability can be represented by a retraceable set.*

Proof. Let α be any set. We wish to associate with α a retraceable set β such that $\beta \equiv_T \alpha$. If α is finite we can take $\beta = \alpha$; if $0 \in \alpha$ we can replace α by $\alpha - (0)$, since $\alpha \equiv_T \alpha - (0)$. We may therefore assume without loss of generality that α is infinite and does not contain 0. Let $a_n = h_\alpha(n)$, then $a_0 > 0$, and hence $a_n > n$. Put

(6) $$b_0 = a_0, \; b_{n+1} = j(b_n, a_{n+1}), \; \beta = \rho b_n.$$

Before proving that β satisfies the requirements we first observe that

(7) $$0 < b_0 < b_1 < \ldots; \; b_n > n.$$

For it follows from the definition of $j(x, y)$ that $x < j(x, y)$ for $y \neq 0$. Thus, since $a_{n+1} > n + 1 > 0$, we have $b_n < j(b_n, a_{n+1}) = b_{n+1}$; moreover, $b_0 = a_0 > 0$. Hence the first part of (7) holds and this implies $b_n > n$. The function

$$f(x) = \begin{cases} b_0 & \text{for } x = b_0 \\ k(x) & \text{for } x \neq b_0 \end{cases}$$

is a recursive extension of $p_\beta(x)$ in view of (6). This shows that β is retraceable.

The relation $a_x > x$ implies

(8) $$x \in \alpha \Leftrightarrow x \in (a_0, a_1, \ldots, a_x),$$
$$x \in \alpha \Leftrightarrow x \in (b_0, l(b_1), \ldots, l(b_x)).$$

The elements $b_0, \ldots, b_x$ can be found by computing $c_\beta(0), c_\beta(1), \ldots$ until the value 1 has shown up $x + 1$ times. Hence $c_\alpha(x)$ is recursive in $c_\beta(x)$ by (8). From $b_y > y$ we infer

(9) $$y \in \beta \Leftrightarrow y \in (b_0, b_1, \ldots, b_y).$$

In view of (6) the elements $b_0, b_1, \ldots, b_y$ can be effectively computed from $a_0, a_1, \ldots, a_y$; but $a_0, a_1, \ldots, a_y$ can be determined by computing $c_\alpha(0)$, $c_\alpha(1), \ldots$ until the value 1 has shown up $y + 1$ times. Thus $c_\beta(x)$ is recursive in $c_\alpha(x)$ by (9) and β is a retraceable set such that $\beta \equiv_T \alpha$.

COROLLARY. *Every degree of unsolvability higher than the lowest degree (that is, than the degree consisting of all recursive sets) can be represented by an immune set.*

Remark. The proposition that there exist exactly c retraceable steps can also be obtained as a corollary of T2. For every degree of unsolvability consists of $\aleph_0$ sets and the total number of sets is c. Thus there are c degrees of unsolvability, hence, at least c (and therefore exactly c) retraceable sets.

Among the degrees of unsolvability those which can be represented by r.e. sets are of special interest. These degrees can, of course, also be defined as those which can be represented by sets which a r.e. complement, because $\alpha \equiv_T \alpha'$ for every set α. We shall see in T3 that every degree of this type can be represented by a retraceable set with a r.e. complement. For every function $f(x)$ we define

$$\zeta_f = \{x | (\exists y)[x < y \text{ and } f(x) > f(y)]\}.$$

The set σ is a *deficiency set*, if $\sigma = \zeta_f$ for some one-to-one recursive function $f(x)$; σ is a *deficiency set of the infinite r.e. set* α if $\sigma = \zeta_a$ for some one-to-one recursive function $a(x)$ ranging over α.

THEOREM T3. *Every degree of unsolvability which can be represented by a r.e. set can also be represented by a r.e. set with a retraceable complement.*

Proof. Let α be a r.e. set. We wish to associate with α a r.e. set β such that $\beta \equiv_T \alpha$ and β' is retraceable. If α is recursive, α' is also recursive and therefore retraceable. We may therefore assume that α is not recursive. Let a_n be a one-to-one recursive function ranging over α and let $\beta = \zeta_a$. Then β is a r.e. set such that $\beta \equiv_T \alpha$ by **(2**, Theorem 1**)**. We claim that β' is retraceable. Let us denote the finite sequence $\{a_0, \ldots, a_m\}$ by $\Sigma(m)$. Our proof is based on

(10) $$x \in \beta' \Rightarrow \left[z < x \text{ and } z \in \beta' \Leftrightarrow \begin{cases} \text{(a)} \ a_z \text{ occurs in } \Sigma(x-1), \\ (b) \ a_z < \text{each of its successors in } \Sigma(x) \end{cases} \right]$$

To establish (10) assume $x \in \beta'$, that is, $(\forall y)[x < y \Rightarrow a_x < a_y]$. If $x = 0$ we interpret $\Sigma(x - 1)$ as the empty sequence. In that case (a) is false for every z and so is $z < x$ and $z \in \beta'$. We may therefore assume $x > 0$. Suppose $z < x$ and $z \in \beta'$. Then $z \leqslant x - 1$ and a_z occurs in $\Sigma(x - 1)$. Moreover, since $z \in \beta'$, a_z is less than each of its successors in $\{a_n\}$, in particular less than each of its successors in $\Sigma(x)$. Thus z satisfies both (a) and (b). Conversely, assume that z satisfies (a) and (b). Then $z < x$ because z satisfies (a), and

(11) $$a_z < a_{z+1}, a_z < a_{z+2}, \ldots, a_z < a_x,$$

because z satisfies (b). Also

(12) $$a_x < a_{x+i} \text{ for } i \geqslant 1,$$

in view of $x \in \beta'$. Combining (11) and (12) we see that $z \in \beta'$. This completes the proof of (10). Whether $x \in \beta$ or $x \notin \beta$, the set

$$\{z | a_z \text{ satisfies (a) and (b)}\}$$

can be effectively obtained from x. Thus there is a recursive function $t(x)$ such that

$$\rho_{t(x)} = \{z | a_z \text{ satisfies (a) and (b)}\}.$$

By (10)

$$x \in \beta' \Rightarrow \rho_{t(x)} = \{z | z < x \text{ and } z \in \beta'\}.$$

We conclude by P1 that β' is retraceable.

Let α be any r.e., but not recursive set and let β be one of the deficiency sets of α. We have seen in the proof of T3 that β' is retraceable and it was shown in **(2)** that β is hypersimple. The set β' is therefore an example of a hyperimmune retraceable set with a r.e. complement. By P1 there exist retraceable sets which are immune, but not hyperimmune. The question arises whether such sets can have a r.e. complement. The answer is negative according to the following theorem.

THEOREM T4. *Every retraceable set with a r.e. complement is recursive or hyperimmune.*

Proof. Let α be retraceable and α' r.e. If α is finite, it is recursive. We may therefore assume that α is infinite. All we have to show is:

(13) $$\alpha \text{ not hyperimmune} \Rightarrow \alpha \text{ recursive}.$$

The conclusion of (13) is equivalent to the assertion that α is not immune, in view of the retraceability of α. Since α is infinite, α is not immune if and only if α has an infinite r.e. subset. Moreover, α has an infinite r.e. subset if and only if there is an effective procedure which, given any number k, enables us to find k distinct elements of α. Instead of proving (13) we can therefore prove:

(14) *If α is not hyperimmune there is an effective procedure such that given any number k we can find k distinct elements of α.*

Assume α is not hyperimmune. Using the fact that α is infinite we infer that there is a discrete array each of whose rows contains at least one element of α, say $\{\delta_n\}$. Let k be any number. Put $n_k = \max(\delta_0 + \ldots + \delta_k)$, then $\delta_0 + \ldots + \delta_k$ contains at least $k+1$ elements of α which are $\leqslant n_k$. Each of the elements $0, 1, \ldots, n_k - 1$ occurs in at most one row of $\{\delta_n\}$; thus, at most finitely many rows of $\{\delta_n\}$ contain an element $< n_k$ and we can effectively find the first row of $\{\delta_n\}$ all of whose elements are $\geqslant n_k$, say

$$\delta = (d(0), \ldots, d(s)), \text{ where } d(0) < d(1) < \ldots < d(s).$$

Note that the elements of δ and s depend on k. Row δ contains at least one element and at most $s+1$ elements of α. Let $d(w) = \min \alpha \cdot \delta$. The set α has a recursive special retracing function, since α' is r.e., say $r(x)$; the function $t(x) = r^*(x)$ is therefore also recursive. Put

$$\sigma_0 = \rho_{td(0)} \cdot \rho_{td(1)} \cdot \ldots \cdot \rho_{td(s)}.$$

We now infer from

$$d(w) \in \alpha \Rightarrow \rho_{td(w)} = \{y | y \in \alpha \text{ and } y < d(w)\}$$
$$d(w) \in \delta \Rightarrow d(w) \geqslant n_k \Rightarrow \text{card } \rho_{td(w)} \geqslant k$$

that

$$\sigma_0 \subset \rho_{td(w)} \subset \alpha \text{ and card } \rho_{td(w)} \geqslant k.$$

If card $\sigma_0 \geqslant k$ we are through, because $\sigma_0 \subset \alpha$. If card $\sigma_0 < k$ at least one of the sets $\rho_{td(0)}, \rho_{td(1)}, \ldots, \rho_{td(s)}$ does not contain the k smallest elements of α; since $d(0), \ldots, d(s)$ are all $\geqslant n_k$ this means that at least one of these $s+1$ elements does not belong to α. Let $a'(n)$ be a recursive function generating α'; by computing $a'(0), a'(1), \ldots$ we can effectively find the first element of $\{a'(n)\}$ which belongs to δ, say $d(p)$. Put

$$\sigma_1 = \prod \rho_{td(i)},$$

i ranging over $(0, \ldots, s) - (p)$, then $\sigma_1 \subset \rho_{td(w)} \subset \alpha$ and card $\rho_{td(w)} \geqslant k$. Again, if card $\sigma_1 \geqslant k$ we are through; if card $\sigma_1 < k$ we generate $\alpha' = (a'(0), a'(1), \ldots)$ until an element of $\alpha' \cdot \delta - (d(p))$ is obtained, say $d(q)$; we then define

$$\sigma_2 = \prod \rho_{td(i)},$$

i ranging over $(0, \ldots, s) - (p, q)$, etc. The set δ contains only $s+1$ elements, hence the procedure must terminate. This means that after a finite number of steps we obtain a set σ_u such that $\sigma_u \subset \rho_{td(w)} \subset \alpha$ and card $\sigma_u \geqslant k$. Then we have found k elements of α, though in general, we don't know whether $\sigma_u = \rho_{td(w)}$. Thus (14) is proved.

COROLLARY. *There exist immune sets which are not retraceable, but have a r.e. complement.*

Proof. If ζ is simple, but not hypersimple, ζ' satisfies the requirements.

Remark. Let us call a simple set σ *extendible* if there is a simple set τ such that $\sigma \subset \tau$ and $\tau - \sigma$ is infinite. A simple, but not hypersimple, set is always extendible. For suppose $\{\delta_n\}$ is a discrete array each of whose rows contains at least one element of the complement of the simple set σ. Put $\tau = \sigma + \Sigma_0^\infty \delta_{2n}$, then τ is a simple set which includes σ and is such that $\tau - \sigma$ is infinite. The question was raised (**9**, p. 215, Problem 9), whether there exists a simple set which is not extendible. We have just shown that if such a simple set exists, it must be hypersimple. We now prove that the hypersimple sets discussed above, namely, those with a retraceable complement, are not candidates for non-extendibility. For, assume ζ is hypersimple and ζ' is retraceable. Let $r(x)$ be a recursive special retracing function of ζ' and let $\{\tau_n\}$ be the sequence of sets associated with $r(x)$. Put $\zeta^* = \zeta + \Sigma_0^\infty \tau_{2n}$, then ζ^* is a hypersimple set such that $\zeta \subset \zeta^*$ and $\zeta^* - \zeta$ is infinite.

THEOREM T5. *There exist two retraceable immune sets α and β such that* (1) $\alpha|\beta$, (2) $\alpha + \beta$ *is not retraceable, and* (3) $\alpha + \beta$ *is introreducible.*

Proof. Let $\{f_n\}$ be a sequence of numbers chosen from $(1, \ldots, 9)$ with $f_0 = 2$ and such that f_n is not a recursive function of n. Put

$$a_0 = f_0,\ a_{n+1} = 10 \cdot a_n + f_{n+1},\ \alpha = \rho a_n,\ b_n = 10^{a_n},\ \beta = \rho b_n.$$

We claim that α and β satisfy the requirements. The set α is clearly retraceable and immune; 10^x is a strictly increasing recursive function which maps α onto β and hence β is also retraceable and immune. Put $\eta = (10, 10^2, \ldots)$, then α and β can be separated by the recursive sets η' and η; this proves (1). We define

$$\delta_n = (10^n + 1, 10^n + 2, \ldots, 10^{n+1} - 1).$$

Note that α has exactly one element in common with each row of the discrete array $\{\delta_n\}$. From $x < 10^{x-1}$ for $x \geqslant 2$ we conclude: *For every number $x \geqslant 2$ there is exactly one number y such that*

$$x < 10^{x-1} < y < 10^x \text{ and } y \in \alpha. \tag{15}$$

Suppose $\alpha + \beta$ were retraceable and $g(x)$ were one of its retracing functions. Put

$$y_0 = 2,\ y_1 = g(10^2),\ y_{n+1} = g(10^{y_n}).$$

We wish to prove that y_n is a strictly increasing recursive function all of whose values belong to α. First of all, $2 \in \alpha$ and $10^2 \in \beta \subset \alpha + \beta$. Thus y_1 is defined; in fact, since 10^2 is the minimum of β, y_1 is the unique element of α which lies between 10 and 10^2. Hence $y_0 < y_1$, where $y_0, y_1 \in \alpha$. Now assume $2 < y_n \in \alpha$, then $10^{y_n} \in \beta$; by (15) there is exactly one number $z \in \alpha$ such that

$$y_n < 10^{y_n - 1} < z < 10^{y_n}. \tag{16}$$

There is no number $z \in \beta$ which satisfies (16); the unique number $z \in \alpha$ which satisfies (16) is therefore

$$g(10^{y_n}).$$

Thus y_n is defined and $y_n < y_{n+1} \in \alpha$. The function y_n generates therefore an infinite recursive subset of the immune set α; this is a contradiction and hence (2) is correct. The sets σ_1 and σ_2 are *recursively equivalent* if σ_2 is the image of σ_1 under some partial recursive one-to-one function. We recall that β is the image of α under the recursive one-to-one function 10^x. To prove (3) it is therefore sufficient to prove the following lemma: *if the retraceable sets σ_1 and σ_2 are separable and recursively equivalent, their sum is introreducible.* Assume the hypothesis of the lemma. Let $r_1(x)$ and $r_2(x)$ be special retracing functions of σ_1 and σ_2 respectively, $t_1(x) = r_1^*(x)$, $t_2(x) = r_2^*(x)$. Suppose θ_1 and θ_2 are disjoint r.e. sets such that $\sigma_1 \subset \theta_1$ and $\sigma_2 \subset \theta_2$; assume finally that $p(x)$ is a partial recursive one-to-one function related to σ_1 and σ_2 by $\sigma_1 \subset \delta p$ and $\sigma_2 = p(\sigma_1)$. To show that $\sigma_1 + \sigma_2$ is introreducible, assume that γ is an infinite subset of $\sigma_1 + \sigma_2$. Then $\gamma \subset \theta_1 + \theta_2$ and $\gamma = \gamma \cdot \theta_1 + \gamma \cdot \theta_2$. Define

$$\gamma_1 = \gamma \cdot \theta_1 + p^{-1}(\gamma \cdot \theta_2), \quad \gamma_2 = \gamma \cdot \theta_2 + p(\gamma \cdot \theta_1).$$

It follows that γ_1 is an infinite subset of σ_1 and γ_2 an infinite subset of σ_2. If we could compute $c_\gamma(0), c_\gamma(1), \ldots$ we could also generate the sets $\gamma \cdot \theta_1$, $\gamma \cdot \theta_2$, $p(\gamma \cdot \theta_1)$, $p^{-1}(\gamma \cdot \theta_2)$ and hence γ_1 and γ_2. For any number x, let the smallest numbers y and z such that

$$y \in \gamma_1 \text{ and } y > x \text{ and } z \in \gamma_2 \text{ and } z > x$$

be denoted by $n_1(x)$ and $n_2(x)$ respectively. Thus $n_1(x)$ and $n_2(x)$ are everywhere defined functions recursive in $c_\gamma(x)$ such that

$$n_1(x) \in \gamma_1 \text{ and } x < n_1(x) \text{ and } n_2(x) \in \gamma_2 \text{ and } x < n_2(x).$$

This implies

$$x < n_1(x) \in \sigma_1 \quad \text{and} \quad x \in \sigma_1 \Leftrightarrow x \in \rho_{t_1 n_1(x)},$$
$$x < n_2(x) \in \sigma_2 \quad \text{and} \quad x \in \sigma_2 \Leftrightarrow x \in \rho_{t_2 n_2(x)}.$$

We conclude that the characteristic function of $\sigma_1 + \sigma_2$ is recursive in $c_\gamma(x)$.

COROLLARY 1. *The product of two retraceable sets is again retraceable, but the sum of two separable retraceable sets is not necessarily retraceable.*

COROLLARY. 2 *There exist introreducible sets which are not retraceable.*

A sufficient condition that two distinct retraceable sets have a common retracing function is that they are separable (by P6), but this condition is not necessary. For the infinite recursive sets $(0, 1, 2, 3, 5, 7, 9, \ldots)$ and $(0, 1, 2, 3, 4, 6, 8, \ldots)$ have a common retracing function and are not disjoint. Let us for any set σ and any number m denote the set $\{x | x \in \sigma \text{ and } x \leqslant m\}$ by $\sigma < m >$.

THEOREM T6. *Let α and β be distinct retraceable sets. Then α and β have a common retracing function if and only if either*

(a) $\alpha \cdot \beta = \phi$ and $\alpha | \beta$, *or*
(b) $\alpha \cdot \beta \neq \phi$ and $\max(\alpha \cdot \beta) = k \Rightarrow$
$$[\alpha < k > = \beta < k > \text{ and } \alpha - \alpha < k > | \beta - \beta < k >].$$

Proof. Let $\alpha \neq \beta$. Suppose α and β have a common special retracing function, say $r(x)$; put $t(x) = r^*(x)$. We claim that $\alpha \cdot \beta$ must be finite. For if $\alpha \cdot \beta$ is infinite there is a strictly increasing function ranging over $\alpha \cdot \beta$ say $c(n)$. Since $c(n) \in \alpha$ for every n, $\rho_{tc(n)} \subset \alpha$ for every n; moreover, for every $x \in \alpha$ there is an n_x such that $x < c(n_x) \in \alpha$. Hence $\alpha = \Sigma_0^\infty \rho_{tc(n)}$. Similarly one proves $\beta = \Sigma_0^\infty \rho_{tc(n)}$. Hence $\alpha = \beta$, contrary to the assumption $\alpha \neq \beta$. We conclude that $\alpha \cdot \beta$ is finite. Either $\alpha \cdot \beta = \phi$ or $\alpha \cdot \beta \neq \phi$. In the former case $\alpha | \beta$ by P6. In the latter case we put

$$(17) \qquad k = \max(\alpha \cdot \beta),\ a_1 = \alpha - \alpha <k>,\ \beta_1 = \beta - \beta <k>,$$
$$a_0 = \min \alpha_1,\ b_0 = \min \beta_1,$$

and

$$r_1(x) = \begin{cases} r(x) & \text{for } x \in \delta r - (0, \ldots, k, a_0, b_0), \\ a_0 & \text{for } x = a_0, \\ b_0 & \text{for } x = b_0. \end{cases}$$

Every set obtained from a retraceable set by omitting finitely many of its elements is again retraceable. Thus α_1 and β_1 are disjoint retraceable sets. They have the common retracing function $r_1(x)$. Hence $\alpha_1 | \beta_1$ by P6. To prove the converse we assume that α and β are retraceable sets satisfying (a) or (b). If they satisfy (a) we are through. If they satisfy (b) we define $k, \alpha_1, \beta_1, a_0$, and b_0 as in (17). The sets α_1 and β_1 are retraceable, because α and β are retraceable; since $\alpha_1 | \beta_1$ they have a common retracing function, say $r(x)$. Let $\alpha <k> = (c_0, \ldots, c_p)$, where $c_0 < c_1 < \ldots < c_p$. Put

$$r_0(x) = \begin{cases} r(x) & \text{for } x \in \delta r - (0, \ldots, k, a_0, b_0), \\ c_p & \text{for } x \in (a_0, b_0), \\ c_n & \text{for } x = c_{n+1} \text{ and } 0 \leqslant n \leqslant p - 1, \\ c_0 & \text{for } x = c_0. \end{cases}$$

Then $r_0(x)$ is a common retracing function of α and β.

Let $r(x)$ be any retracing function and T_r the class of all infinite sets retraced by $r(x)$. We know from the Examples 1–5 that given any of the cardinalities $1, 2, 3, \ldots, \aleph_0, c$, the retracing function $r(x)$ can be chosen in such a manner that T_r has the given cardinality. Assuming the continuum hypothesis, these are obviously the only values which card T_r can assume.

THEOREM T7. *It can be proved without the continuum hypothesis that the class of all infinite sets retraced by a retracing function $r(x)$ is either finite or denumerable, or has cardinality c.*

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Proof. All references in this proof are to Sierpinski's book **(8)**. We use the numbers 0, 1, . . . as indices of the elements of a sequence, while Sierpinski uses 1, 2, . . . for the same purpose; this difference in notation is, however, non-essential for the theorems in Sierpinski's book which we shall use. Throughout this proof the agreement that only collections of non-negative integers are referred to as sets is suspended; any collection of points in a metric space is called a set. Let $\mathfrak{E}_\omega$ be the space consisting of all sequences of real numbers, where for $p = \{p_x\}$ and $q = \{q_x\}$,

$$\rho(p, q) = \sum_{x=0}^{\infty} \frac{|p_x - q_x|}{x!(1 + |p_x - q_x|)}.$$

Also let $\mathfrak{N}_\omega$ be the space consisting of all sequences of non-negative integers with the same distance function as $\mathfrak{E}_\omega$. $\mathfrak{E}_\omega$ is a metric space **(8**, p. 134**)** and $\mathfrak{N}_\omega$ is a subspace of $\mathfrak{E}_\omega$. We need three lemmas.

Lemma 1. *The sequence* $\{p^n\} = \{\{p_x^n\}\}$ *of points in* $\mathfrak{N}_\omega$ *converges to the point* p *of* $\mathfrak{E}_\omega$ *if and only if*

$$(\forall x)(\exists t)(\forall n)[n > t \Rightarrow p_x^n = p_x].$$

Proof. The sequence $\{p^n\}$ of points in $\mathfrak{E}_\omega$ converges to the point p of $\mathfrak{E}_\omega$ if and only if for every x,

$$\lim_{n\to\infty} p_x^n = p_x$$

(8, p. 135**)**. The desired result follows from the fact that $p^n \in \mathfrak{N}_\omega$ means that p_x^n is a non-negative integer for every x.

Lemma 2. *It can be proved without the continuum hypothesis that every closed set in* $\mathfrak{N}_\omega$ *is finite, denumerable or has cardinality* c.

Proof. Let $\mathfrak{N}_0$ denote the set of all points $\{q_x\}$ in $\mathfrak{N}_\omega$ which are ultimately vanishing sequences (that is, for which $q_x = 0$ for almost all x). Let $p = \{p_x\} \in \mathfrak{N}_\omega$. Put

$$q_x^n = \begin{cases} p_x \text{ for } x \leqslant n, \\ 0 \text{ for } x > n, \end{cases} \qquad q^n = \{q_x^n\},$$

then $\{q^n\}$ is a sequence of points in $\mathfrak{N}_0$ which converges to p. Thus $\mathfrak{N}_\omega$ is the closure of its denumerable subset $\mathfrak{N}_0$; hence $\mathfrak{N}_\omega$ is separable. It follows from Lemma 1 that if a sequence of points in $\mathfrak{N}_\omega$ converges in $\mathfrak{E}_\omega$, its limit belongs to $\mathfrak{N}_\omega$. Thus $\mathfrak{N}_\omega$ is a closed set in the metric space $\mathfrak{E}_\omega$. However, $\mathfrak{E}_\omega$ is complete **(8**, pp. 190, 191**)**, hence $\mathfrak{N}_\omega$ is also complete. Let $\mathfrak{B}$ be a closed set in $\mathfrak{N}_\omega$, then $\mathfrak{B}$ is a Borel set in a separable complete space, hence

$$\text{card } \mathfrak{B} > \aleph_0 \Rightarrow \text{card } \mathfrak{B} = c \tag{18}$$

by **(8**, p. 228, Corollary 2, Theorem 120**)**. This means that $\mathfrak{B}$ is finite, denumerable or has cardinality c. Moreover, (18) can be proved without using the continuum hypothesis.

LEMMA 3. *Let Δ be a denumerable collection, let $\mathfrak{A}$ be a family of finite sequences of elements of Δ and let $\mathfrak{B}$ be a family of infinite sequences of elements of Δ. Assume, furthermore, that $g = \{g_x\}$ belongs to $\mathfrak{B}$ if and only if all its initial segments $\{g_x\}_{x \leqslant n}$ belong to $\mathfrak{A}$. Then it can be proved without the continuum hypothesis that $\mathfrak{B}$ is finite, denumerable or has cardinality c.*

Proof. We may clearly restrict ourselves to the special case $\Delta = \epsilon$; in this case $\mathfrak{B}$ is a set in $\mathfrak{N}_\omega$ and by Lemma 2 it suffices to prove that $\mathfrak{B}$ is closed. Let the point $g = \{g_x\}$ in $\mathfrak{N}_\omega$ be a limit point of $\mathfrak{B}$ and let $\{g_x^n\}$ be a sequence of points in $\mathfrak{B}$ which converges to g. By Lemma 1 there is a function $t(x)$ such that

(19) $$n > t(x) \Rightarrow g_x^n = g_x.$$

If g were not in $\mathfrak{B}$, it would have an initial segment not belonging to $\mathfrak{A}$, say $\{g_x\}_{x \leqslant m}$. Thus

(20) $$\{g_x\}_{x \leqslant m} \notin \mathfrak{A}.$$

Let t be the number which exceeds the maximum of $t(0), \ldots, t(m)$ by 1, then $t > t(x)$ for $x \leqslant m$. Hence, by (19), $g_x^t = g_x$ for $x \leqslant m$, that is,

(21) $$\{g_x^t\}_{x \leqslant m} = \{g_x\}_{x \leqslant m}.$$

We now have a contradiction. For, since the left side of (21) is an initial segment of the point g^t in $\mathfrak{B}$, it belongs to $\mathfrak{A}$; on the other hand, the right side of (21) does not belong to $\mathfrak{A}$ by (20). We conclude that $g \in \mathfrak{B}$. Hence $\mathfrak{B}$ is a closed set in $\mathfrak{N}_\omega$.

We claim that T7 follows from Lemma 3. Assume that $f(x)$ is any function with an infinite domain and B the class of all infinite sets β of non-negative integers such that $f(x)$ is an extension of $p_\beta(x)$. Since T7 concerns the special case that $f(x)$ is partial recursive and B non-empty, it suffices to prove that B is finite, denumerable or has cardinality c. Let $\mathfrak{B}$ denote the family of the principal functions of all sets in B and $\mathfrak{A}$ the family of all strictly increasing finite sequences $\{x_0, \ldots, x_n\}$ such that

$$\alpha = (x_0, \ldots, x_n) \Rightarrow f(x) \text{ is an extension of } p_\alpha(x).$$

Clearly, $g = \{g_x\} \in \mathfrak{B}$ if and only if all its initial segments belong to $\mathfrak{A}$. Thus it follows by Lemma 3 that the family $\mathfrak{B}$ is finite, denumerable, or has cardinality c; the same is therefore true for the class B.

Remark. Lemma 3 can be used to establish the following theorem about graphs. *Let Γ be a graph with denumerably many edges; then it can be proved without the continuum hypothesis that the number of one-way infinite paths of* Γ *is finite,* $\aleph_0$ *or c.* For let Δ denote the collection of all edges of Γ, $\mathfrak{A}$ the family of all finite paths of Γ and $\mathfrak{B}$ the family of all one-way infinite paths of Γ. Then Lemma 3 is applicable, since the infinite sequence $\{p_0p_1, p_1p_2, \ldots\}$ of edges belongs to $\mathfrak{B}$ if and only if all its initial segments $\{p_0p_1, \ldots, p_np_{n+1}\}$ belong to $\mathfrak{A}$.

6. Concluding remarks. We have not been able to characterize all partial recursive functions which are retracing functions, that is, which retrace at least one infinite set. Let us denote by T_f the class of all infinite sets retraced by $f(x)$. For every retracing function $f(x)$ there is a non-trivial special partial recursive function $r(x)$ which is a restriction of $f(x)$ and which has the property $T_r = T_f$. We can therefore restate the problem as follows: find a necessary and sufficient condition that a non-trivial special partial recursive function be a retracing function. We mention a sufficient condition which is not necessary. *If a non-trivial, special, partial recursive function has a finite initial set and is finite-to-one, it is a retracing function.* For assume $f(x)$ satisfies the hypotheses. Let $\{\tau_n\}$ be the infinite sequence of (mutually disjoint non-empty) sets associated with $f(x)$, then $\{\tau_n\}$ is a sequence of finite sets. Define a binary relation R by: yRx if $y = f(x)$. Since there corresponds with every element x of τ_{n+1} at least one (in fact, exactly one) element y of τ_n such that yRx, it follows by König's Lemma (**4**, p. 121 or **5**, p. 81) that there exists an infinite sequence $\{a_n\}$ such that for every n, $a_n R a_{n+1}$. Then $f(x)$ retraces the infinite set $(a_0, a_1, \ldots)$ and the proof is completed. This immediately raises the question: "Does every function $f(x)$ which satisfies this sufficient condition retrace at least one infinite *recursive* set?" One might be tempted to conjecture that the answer is affirmative, because Brouwer's proof (**3**, p. 42) of his fan theorem, that is, the intuitionistic form of König's lemma, is in some sense constructive. Though it can hardly be doubted that a close connection exists between König's lemma and the subject of the present paper, the authors have, however, been unable to substantiate this conjecture, even in the special case that given any $x \in \rho f$ the cardinality of the set $f^{-1}(x)$ can effectively be found. In this case $\{\tau_n\}$ is a discrete array.

Added May 31, 1958. It is proved in a paper of R. M. Friedberg which will appear in the Journal of Symbolic Logic that there exists a simple set which is not extendible.

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