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RELATIVE INJECTIVES AND FREE MONADS

BY
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Introduction. Let Σ be a class of maps in a category $\mathcal{A}$. An object I of $\mathcal{A}$ is Σ-injective if $\mathcal{A}(\sigma, I)$ is an epimorphism for all $\sigma \in \Sigma$. This paper is concerned with the question of finding "enough" Σ-injectives in a functorial way. More precisely, we seek a monad (T, η, μ) on $\mathcal{A}$ such that (1) TA is Σ-injective for all A and (2) every Σ-injective is η-injective (i.e. injective with respect to the class $\{\eta A \mid A \in \mathcal{A}\}$). We call such a monad a Σ-injective monad. For example, if $\mathcal{A}$ is complete, Σ the class of monomorphisms and I an injective cogenerator then it is well known that a Σ-injective monad exists.

If $\mathcal{A}$ is cocomplete and Σ is a set, we show that there exists a pointed endofunctor on $\mathcal{A}$ such that if the pointwise free monad on this endofunctor exists it will be a Σ-injective monad. By applying a theorem of Barr's [1] on the existence of free monads we obtain our main result which roughly can be stated as follows: If $\mathcal{A}$ is complete and cocomplete with suitable factorizations of maps, and Σ is a set of maps whose domains are "small", then there exists a Σ-injective monad on $\mathcal{A}$. We give several applications of our results.

1. **Main results.** A pointed endofunctor on a category $\mathcal{A}$ is a pair (R, ρ) where R is an endofunctor and $\rho : 1 \to R$ is a natural transformation. If (R, ρ) is a pointed endofunctor then an R-algebra is a pair (A, a) where A is an object of $\mathcal{A}$ and $a : RA \to A$ is a map with $a \cdot \rho A = 1$. A morphism from the R-algebra (A, a) to the R-algebra (B, b) is a map $f : A \to B$ such that $f \cdot a = b \cdot Rf$. We thus get a category R-Alg and a forgetful functor $U : R$-Alg $\to \mathcal{A}$.

We use the following notation from [6]. If X is a set and $A \in \mathcal{A}$, we write $X \otimes A$ for the coproduct of X copies of A; so that by definition $\mathcal{A}\,(X \otimes A, B) \approx$ Sets $(X, \mathcal{A}(A, B))$. If $H : \mathcal{C} \to$ Sets is a functor and $A \in \mathcal{A}$, we also write $H \otimes A : \mathcal{C} \to \mathcal{A}$ for the functor sending C to $HC \otimes A$.

We need one final bit of notation. If f is a map we denote by $\partial_0 f$ and $\partial_1 f$ the domain and codomain of f respectively.

PROPOSITION 1.1. *If $\mathcal{A}$ is cocomplete and Σ is a set of maps in $\mathcal{A}$, then there exists a pointed endofunctor (R, ρ) such that*

1. *A is Σ-injective if and only if ρA has a left inverse (i.e. if and only if A has an R-algebra structure.).*

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2. *If* $\sigma \in \Sigma$ *and* $f: \partial_0\sigma \to A$ *is a map, then there exists a map* $g: \partial_1\sigma \to RA$ *with* $\rho A \cdot f = g \cdot \sigma$.

Proof. Define the endofunctor D by

$$D = \coprod \{\mathscr{A}(\partial_0\sigma, -) \otimes \partial_0\sigma \mid \sigma \in \Sigma\}.$$

If $\sigma \in \Sigma$ and $f \in \mathscr{A}(\partial_0\sigma, A)$ we let $d_\sigma: \mathscr{A}(\partial_0\sigma, A) \otimes \partial_0\sigma \to DA$ and $i_f: \partial_0\sigma \to \mathscr{A}(\partial_0\sigma, A) \otimes \partial_1\sigma$ denote the respective natural maps giving the coproduct.

Define a natural transformation $\varepsilon: D \to 1$ by $\varepsilon A \cdot d_\sigma \cdot i_f = f$. Then εA satisfies the following property: if $f: \partial_0\sigma \to A$, there exists a $g: \partial_0\sigma \to DA$ with $\varepsilon A \cdot g = f$.

Next we define the endofunctor H by

$$H = \coprod \{\mathscr{A}(\partial_0\sigma, -) \otimes \partial_1\sigma \mid \sigma \in \Sigma\}.$$

If $\sigma \in \Sigma$ and $f \in \mathscr{A}$ $(\partial_0\sigma, A)$ we let $h_\sigma: \mathscr{A}(\partial_0\sigma, A) \otimes \partial_1\sigma \to HA$ and $j_f: \partial_1\sigma \to \mathscr{A}(\partial_0\sigma, A) \otimes \partial_1\sigma$ denote the respective natural maps giving the coproduct.

The natural transformation $\delta: D \to H$ is defined by $\delta A \cdot d_\sigma \cdot i_f = h_\sigma \cdot j_f \cdot \sigma$. The map δA satisfies the following: given $f: \partial_0\sigma \to A$ there is a $g: \partial_0\sigma \to DA$ and $l: \partial_1\sigma \to HA$ with $\varepsilon A \cdot g = f$ and $\delta A \cdot g = l \cdot \sigma$.

Finally, we form the pointed endofunctor (R, ρ) so that the following is a pushout.

$$\begin{array}{ccc} D & \xrightarrow{\delta} & H \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\tau} \\ 1 & \xrightarrow[\rho]{} & R \end{array}$$

Using the properties of ε and δ it is easy to show that conditions 1 and 2 of the proposition hold for (R, ρ).

Remarks. 1. The construction of proposition 1.1 was suggested by a construction in [3].

2. An object I is Σ-injective when for each $\sigma \in \Sigma$ and each $f: \partial_0\sigma \to I$ there is $g: \partial_1\sigma \to I$ with $g \cdot \sigma = f$. There are situations where one is concerned with "Σ-injectives" which satisfy the extension property not with respect to all maps $\partial_0\sigma \to I$, but only certain allowable maps (see below examples 2.4 and 2.5). We can take this situation into account in the following way. Let Ψ be a collection of maps such that $f \cdot \psi \in \Psi$ for all f and all $\psi \in \Psi$. An object I is then (Ψ, Σ)-injective if for all $\sigma \in \Sigma$ and all $\psi: \partial_0\sigma \to I$ in Ψ there is a $g: \partial_1\sigma \to I$ with $g \cdot \sigma = \psi$. The construction in proposition 1.1 works where we replace $\mathscr{A}(\partial_0\sigma, A)$ by the set $\{\psi: \partial_0\sigma \to A \mid \psi \in \Psi\}$. This produces a pointed endofunctor (R, ρ) with the corresponding properties for (Ψ, Σ)-injectives and maps in Ψ. What we do in the rest of the paper works for this situation also.

Given a pointed endofunctor (S, δ), the free monad on (S, δ) is a monad (T, η, μ) together with a natural tranformation $\gamma: S \to T$ such that (a) $\gamma \cdot \delta = \eta$ and (b) for every monad (T', η', μ') for which there is a natural transformation $\gamma': S \to T'$ with $\gamma' \cdot \delta = \eta'$, there is a unique monad map $\beta: T \to T'$ with $\beta \cdot \gamma = \gamma'$. The existence of a free monad on (S, δ) is related to the category of S-algebras. The forgetful functor $U: S\text{-Alg} \to \mathcal{A}$ is monadic if it has a left adjoint. In this case, the monad generated is the free monad on (S, δ) and we say (following Kelly) that the free monad exists pointwise. If $\mathcal{A}$ is complete then Barr [1] has shown that the existence of the free monad on (S, δ) is equivalent to the existence of a left adjoint to U.

If Σ is a class of maps, then a Σ-injective monad is a monad (T, η, μ) such that TA is Σ-injective for all A and all Σ-injectives are injective with respect to $\{\eta A \mid A \in \mathcal{A}\}$. If we are concerned with (Ψ, Σ)-injectives for some class Ψ, then a (Ψ, Σ)-injective monad is defined in the same manner as a Σ-injective monad but with (Ψ, Σ) replacing Σ throughout. In the situation of Proposition 1.1, if the free monad on (R, ρ) exists pointwise, it will be a Σ-injective monad. So we seek conditions for the free monad on (R, ρ) to exist pointwise.

The most general result about the existence of pointwise free monads that we know is due to Kelly (unpublished, but see [8] and [11]). His result includes the results of Barr [1] and Dubuc [4] as special cases. However, for our purposes Barr's results will suffice. So we describe his result below and leave the application of Kelly's result (or Dubuc's) to the reader.

Before stating Barr's theorem we recall some of the relevant definitions from his paper. Let α be a limit ordinal, we use $\mathcal{O}_\alpha$ to denote the ordered category of ordinals $<\alpha$. A functor $\mathcal{O}_\alpha \to \mathcal{A}$ is called an α-sequence in $\mathcal{A}$. Let $\mathcal{M}$ be a class of monomorphisms and α a limit ordinal. An α-sequence $D: \mathcal{O}_\alpha \to \mathcal{A}$ is called an $(\mathcal{M}, \alpha)$-sequence of subobjects of $A \in \mathcal{A}$ if there is a natural tranformation ξ from D to the constant functor A such that $\xi i: Di \to A$ is a morphism in $\mathcal{M}$ for each $i \in \alpha$. If $T: \mathcal{A} \to \mathcal{A}$ is a functor, we say that T is $(\mathcal{M}, \alpha)$-small if whenever $D: \mathcal{O}_\alpha \to \mathcal{A}$ is an $(\mathcal{M}, \alpha)$-sequence, the natural map colim $TD \to T$ colim D is an isomorphism. Finally we say that $\mathcal{A}$ has small $\mathcal{M}$ factorizations if for every $A \in \mathcal{A}$ there is a set ΓA of objects of $\mathcal{A}$ such that any $A \to B$ in $\mathcal{A}$ factors as $A \to C \to B$ with $C \in \Gamma A$ and $C \to B$ in $\mathcal{M}$. With the above concepts we can now state a version of Barr's theorem which we need.

THEOREM B. *Let $\mathcal{A}$ be complete and cocomplete, (S, δ) a pointed endo-functor. If there is a class $\mathcal{M}$ of monomorphisms and a limit ordinal α such that $\mathcal{A}$ has small $\mathcal{M}$ factorizations and S is $(\mathcal{M}, \alpha)$-small, then the free monad on (S, δ) exists.*

While Theorem B is different from Barr's Theorem 5.5 in that it deals with a functor S together with a natural transformation $\delta . 1 \to S$ instead of just a functor, an easy modification of Barr's proofs will give the result above.

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In applying Barr's result to our situation, we need the following definition. Let $\mathcal{A}$ be a category, $\mathcal{M}$ a class of monomorphisms in $\mathcal{A}$ and α a limit ordinal. Then an object A is $(\mathcal{M}, \alpha)$-Barr-small if $\mathcal{A}(A, -)$ preserves colimits of $(\mathcal{M}, \alpha)$-sequences.

THEOREM 1.2. *Let $\mathcal{A}$ be complete and cocomplete and suppose that $\mathcal{A}$ has small $\mathcal{M}$ factorizations. Let Σ be a set of maps in $\mathcal{A}$ such that there is an ordinal α with $\partial_0\sigma$ being $(\mathcal{M}, \alpha)$-Barr-small for all $\sigma \in \Sigma$. Then there exists a Σ-injective monad on $\mathcal{A}$.*

Proof. We show that the free monad on the pointed endofunctor (R, ρ) of Proposition 1.1 exists. To this end we note that since for each $\sigma \in \Sigma$ the functor $\mathcal{A}(\partial_0\sigma, -)$ preserves colimits of $(\mathcal{M}, \alpha)$-sequences of subobjects, the functor $\mathcal{A}(\partial_0\sigma, -)\otimes A$ is $(\mathcal{M}, \alpha)$-Barr-small for any A in $\mathcal{A}$. Since coproducts preserve colimits we get that the functors D and H of Propositional 1.1 are $(\mathcal{M}, \alpha)$-small. Consequently R is $(\mathcal{M}, \alpha)$-small and the result follows from Barr's theorem.

We now look at the question of when η is a monomorphism.

PROPOSITION 1.3. *Let $\mathcal{A}$ be complete and cocomplete with small $\mathcal{M}$ factorizations. Suppose $\mathcal{M}$ satisfies the following properties*

1. *Any coproduct of maps in $\mathcal{M}$ is in $\mathcal{M}$.*
2. *Any pushout of a map in $\mathcal{M}$ is in $\mathcal{M}$.*
3. *There is an ordinal α such that for every γ-sequence, where $\gamma \leq \alpha$, for which the connecting maps are in $\mathcal{M}$, the natural maps to the colimit are also in $\mathcal{M}$.*

Let Σ be a set of maps in $\mathcal{M}$ such that $\partial_0\sigma$ is $(\mathcal{M}, \alpha)$-Barr-small for all $\sigma \in \Sigma$. Then there is a Σ-injective monad on $\mathcal{A}$ with η a monomorphism.

Proof. Let (R, ρ) be the pointed endofunctor constructed in Proposition 1.1. Then because of the properties of $\mathcal{M}$ we have that ρA is in $\mathcal{M}$ for all A. We show that the free monad on (R, ρ) has ηA a monomorphism for all A. It suffices to show that for each A there is a monomorphism from A to a Σ-injective. To this end we define the following α-sequence $L : \mathcal{O}_\alpha \to \mathcal{A}$. Set $L(0) = A$, $L(1) = RA$ and $L(0, 1) = \rho A$. For a non-limit ordinal $\beta + 1$ set $L(\beta + 1) = R(L(\beta))$ and $L(\beta, \beta + 1) = \rho L(\beta)$. For a limit ordinal γ set $L(\gamma) = \lim_{\beta<\alpha} L(\beta)$ and $L(\beta, \gamma)$ the natural map to the colimit. This map is in $\mathcal{M}$. Hence we get an α-sequence all of whose connecting morphisms are in $\mathcal{M}$. Set $I = \lim L$, with $\{\Pi_\beta : L(\beta) \to I \mid \beta < \alpha\}$ the natural maps to the colimit. We claim that I is Σ-injective. For let $\sigma \in \Sigma$ and $f : \partial_0\sigma \to I$. Then since the α-sequence L forms an $(\mathcal{M}, \alpha)$-sequence of subobjects of I, there is a $\beta < \alpha$ and a $g : \partial_0\sigma \to L(\beta)$ such that $\Pi_\beta \cdot g = f$. Then by the properties of (R, ρ) there is an $h : \partial_1\sigma \to RL(\beta) = L(\beta + 1)$ with $h \cdot \sigma = \rho L(\beta) \cdot g$. Hence $\Pi_{\beta+1} h \cdot \sigma = \Pi_{\beta+1} \cdot \rho L(\beta) \cdot g = \Pi_\beta \cdot g = f$. So I is Σ-injective. Finally we note that $\Pi_0 : A \to I$ is in $\mathcal{M}$ so we are done.

REMARK. If $\mathcal{M}$ is such that if $f \cdot g \in \mathcal{M}$, we have $g \in \mathcal{M}$, then under the conditions of Proposition 1.3 we get that $\eta A \in \mathcal{M}$.

2. EXAMPLES (2.1) Let $\mathcal{A}$ be complete and cocomplete with a proper $(\mathcal{E}, \mathcal{M})$ factorization system in the sense of Freyd and Kelly [6]. They call an object A α-ordinally bounded for a limit ordinal α if $\mathcal{A}(A, -)$ preserves the union of $(\mathcal{M}, \alpha)$-sequences. They show that if A is α-ordinally bounded then it is $(\mathcal{M}, \alpha)$-Barr-small. The category $\mathcal{A}$ is ordinally bounded if for each A there is an α so that $\mathcal{A}$ is α-ordinally bounded. There are many examples of ordinally bounded categories. These include the locally presentable categories of Gabriel and Ulmer [7] where we use the factorization $\mathcal{E}$ = extremal epimorphisms and $\mathcal{M}$ = monomorphisms. Examples of locally presentable categories (from [6]) are: the category of Sets; the category of algebras in Sets over a theory with rank; the category of small categories; the category of sheaves of sets on a Grothendieck topology; an AB5 category with a generator. The category of topological spaces is bounded for the factorization $\mathcal{E}$ = surjections and $\mathcal{M}$ = the inclusion of subspaces. The category of Hausdorff spaces with the same factorization is bounded. For more examples see [6].

If $\mathcal{A}$ is ordinally bounded and Σ is any set of maps then there is a fixed α with $\partial_0\sigma$ being α-ordinally bounded for all $\sigma \in \Sigma$. Hence if $\mathcal{A}$ is $\mathcal{E}$ – cowell powered (as is the case in the explicit examples mentioned above), then there is a Σ-injective monad on $\mathcal{A}$.

(2.2) An $AB5$ category with a generator with its unique proper factorization $\mathcal{E}$ = epimorphisms, $\mathcal{M}$ = monomorphisms is cowell-powered and ordinally bounded. It is well known (see for example [9]) that conditions 1,2, and 3 of Proposition 3 hold for $\mathcal{M}$. By taking Σ = set of subobjects of the generator we get the existence of a Σ-injective monad with η a monomorphism. As is well known, the Σ-injectives are the injectives, hence we get the well known result that an AB5 category with a generator has enough injectives.

(2.3) Let R be an ordered ring which is directed (see [10]) and let $\mathcal{A}$ be the category of ordered R-modules and order preserving R-homomorphisms. $\mathcal{A}$ is complete and cocomplete. If $\mathcal{M}$ is an ordered module M_+ will denote the positive cone. A morphism $f: A \to B$ is an 0-monomorphism if $\operatorname{Ker}(f) = 0$, and $f^{-1}(B_+) = A_+$. Taking $\mathcal{M}$ = class of 0-monomorphisms it is easy to see that $\mathcal{A}$ has small $\mathcal{M}$ factorizations. Furthermore, $\mathcal{M}$ satisfies conditions 1, 2 and 3 of Proposition 1.1. Conditions 1 and 2 are shown in [10]. Condition 3 follows from the fact that the required colimit is the same as the colimit in R-modules with the order given in the natural way.

Now, Ribenboim shows that $\mathcal{A}$ has no injectives other than zero, so interest has centered on the $\aleph$-injectives. An ordered module Q is an $\aleph$-injective ($\aleph$ an infinite regular cardinal) if given any ordered modules M and N, with cardinal number less than $\aleph$ and any 0-monomorphism $f: M \to N$ and any ordered preserving map $g: M \to Q$ there is a $h: N \to Q$ with $h \cdot f = g$. Letting Σ be the set of 0-monomorphisms with domain and codomain of cardinality less than $\aleph$ we see that $\partial_0 \sigma$ is (M, α) small for any limit ordinal of cardinality greater than or equal to $\aleph$. Hence we get a Σ-injective monad on $\mathscr{A}$ with η a pointwise 0-monomorphism. This extends a result in [10].

(2.4) Let $\mathscr{A}$ be complete and cocomplete with a zero object and small $\mathscr{M}$ factorizations. In [5] Eilenberg and Moore call

$$(*) A' \xrightarrow{i} A \xrightarrow{j} A''$$

a sequence if $ji = 0$. An object Q is injective with respect to (*) if for all $f: A \to Q$ with $fi = 0$ there is a $g: A'' \to Q$ with $g \cdot j = f$. If $\mathscr{S}$ is a class of sequences then $I(\mathscr{S})$ is the class of all objects which are injective with respect to each sequence in S. Similarly, given a class of objects $\mathscr{I}$ let $S(I)$ be the class of all sequences with respect to which each $I \in \mathscr{I}$ is injective. A class of sequences is closed if $S(I(S)) = \mathscr{S}$. A closed class is an injective class if for each $A \to A'$ there is $Q \in I(\mathscr{S})$ with $A \to A' \to Q$ in S.

Now if $\mathscr{S}$ is a set of sequences, we let Σ = set of all $\sigma: A \to B$ such that there is $i: C \to A$ with $C \xrightarrow{i} A \xrightarrow{\sigma} B$ in $\mathscr{S}$. Let Ψ be the class of all maps f such that there is a sequence $C \xrightarrow{i} A \xrightarrow{\sigma} B$ with $\partial_0 f = A$ and $fi = 0$. If $\partial_0 \sigma$ is $(\mathscr{M}, \alpha)$ small for all $\sigma \in \Sigma$, then there exists a (Ψ, Σ)-injective monad on $\mathscr{A}$. So if we take the closed class $S(I(\mathscr{S}))$ this will be an injective class for if $f: A \to B$ is any map, we let $k: B \to C$ be the cokernel of f. Then

$$A \xrightarrow{f} B \xrightarrow{\eta_C \cdot k} TC$$

(2.5) Let $\mathscr{A}$ be the category of left R-modules where R is a ring with unit. In [2] Beachy makes the following definition. Given preradicals T and S on $\mathscr{A}$. A module Q is (T, S)-injective if each map $f: N \to Q$ such that (1) N is T-dense submodule of M and (2) Ker (f) is an S-dense submodule of M, can be extended to M. He proves the following extension of Baer's condition: Q is (T, S)-injective if and only if each homomorphism $f: A \to Q$ such that A is T-dense left ideal of R and ker (f) is S-dense in R can be extended to R. So if we take Σ = set of inclusions of T-dense left ideals and Ψ = all maps with domain a T-dense ideal whose kernel is S-dense, then there exists a (Ψ, Σ)-injective monad on $\mathscr{A}$ with η a monomorphism.

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