NOTE ON THE ALGEBRA OF S-FUNCTIONS

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Considerable advance has been made recently towards a systematic method of evaluating the "product" $\{\mu\} \otimes \{\lambda\}$, most notably in the methods of Robinson **(3)**, Littlewood **(2)**, and Todd **(5)** and the differential operator technique of H. O. Foulkes.

In this note a formula is derived which expresses $\{\mu\} \otimes \{2r\}$ in terms of products $\{\mu\} \otimes \{n\}$ (where $n < 2r$) and the more easily calculated functions $\{\mu\} \otimes S_N$ **(1**, p. 235**)**.

The products $\{\mu\} \otimes \{r\}$, where (r) denotes the partition of r consisting of a single element, are of particular interest because of their applications in invariant theory.

For brevity we will denote $\{\mu\} \otimes S_i$ by t_i. Then we have **(4**, p. 374**)**

$$\{\mu\} \otimes \{r\} = \sum_{(\alpha)} \frac{1}{\alpha_1! \ldots \alpha_r!} \left(\frac{t_1}{1}\right)^{\alpha_1} \ldots \left(\frac{t_r}{r}\right)^{\alpha_r}.$$

Hence, if we define $\{\mu\} \otimes \{0\} = 1$, we have

$$\prod_{i=1}^{\infty} \exp\left(\frac{t_i}{i} z^i\right) = \sum_{r=0}^{\infty} \{\mu\} \otimes \{r\} z^r.$$

Now

$$\prod_{i=1}^{\infty} \exp\left(\frac{t_i}{i} (+z)^i\right) \cdot \prod_{i=1}^{\infty} \exp\left(\frac{t_i}{i} (-z)^i\right) = \prod_{i=1}^{\infty} \exp\left(\frac{t_{2i}}{i} z^{2i}\right),$$

i.e.,

$$\sum_{r=0}^{\infty} \{\mu\} \otimes \{r\} z^r \cdot \sum_{r=0}^{\infty} \{\mu\} \otimes \{r\} (-z)^r = \prod_{i=1}^{\infty} \exp\left(\frac{t_{2i}}{i} z^{2i}\right).$$

Equating coefficients of z^{2k} and transposing, we have:

$$\{\mu\} \otimes \{2k\} = (\{\mu\} \otimes \{2k-1\})(\{\mu\}) - (\{\mu\} \otimes \{2k-2\})(\{\mu\} \otimes \{2\})$$
$$+ \ldots + \frac{(-1)^{k+1}}{2} (\{\mu\} \otimes \{h\})^2 + \tfrac{1}{2} \sum_{(\beta)} \frac{1}{\beta_1! \ldots \beta_k!} \left(\frac{t_2}{1}\right)^{\beta_1} \ldots \left(\frac{t_{2k}}{k}\right)^{\beta_k}$$

This formula is particularly useful in calculating $\{m\} \otimes \{4\}$, since

$$\{m\} \otimes \{4\} = (\{m\} \otimes \{3\})(\{m\}) - \tfrac{1}{2}(\{m\} \otimes \{2\})^2 + \tfrac{1}{2}(\tfrac{1}{2}t_2^2 + t_4)$$

and explicit formulas **(4**, pp. 380-382**)** are available for $\{m\} \otimes \{3\}$ and $\{m\} \otimes \{2\}$.

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