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SOLUTIONS OF PERIOD FOUR FOR A NON-LINEAR DIFFERENCE EQUATION

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Abstract

The paper extends earlier work by using the factorisation method to discuss solutions of period four for the difference equation

$$x_{n+1} = ax_n^3 + (1-a)x_n \quad (0 < a \leqslant 4).$$

This equation was suggested by R. M. May as a simple mathematical model for the effect of frequency-dependent selection in genetics. It is shown that for a given value of the parameter, a, the identification of solutions of period four can be reduced to finding real roots for a polynomial equation of degree eight. The appropriate values of x_n follow from a quartic equation. By splitting up the problem in this way it becomes relatively straightforward to determine the critical values of a at which the various solutions of period four first appear and to discuss the stability of these solutions. Intervals of stability are tabulated in the paper.

1. Introduction

The present paper is an extension of an earlier one [4] in which the motivation for the work was discussed in the first two paragraphs. This followed May [6, 7] who linked the equation

$$x_{n+1} = F(x_n) = ax_n^3 + (1-a)x_n \tag{1.1}$$

with the phenomenon of frequency-dependent selection in population growth and suggested that it merited further investigation. He noted that the parameter a had to be restricted to an interval $0 < a \leqslant 4$ to agree with the genetics problem and it will be shown later that the interval $2 < a \leqslant 4$ suffices for the purposes of this paper. Similarly, we can restrict x_n to the interval $[-1, 1]$.

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In the papers cited above, May showed that solutions with period two are available for $a > 2$, although these solutions are unstable for $a > 1 + \sqrt{5}$. Experience with the logistic difference equation suggests that solutions with period four should first appear at $a = 1 + \sqrt{5}$ and this is confirmed by the discussion in this paper. As a increases, other families of solutions of period four can be distinguished and it is of interest to determine the critical value of a and the stability interval for each family.

If we refer to a cyclic solution with minimum period four as a C4 solution and write (b_1, b_2, b_3, b_4) for the elements of the solution, then $(-b_1, -b_2, -b_3, -b_4)$ is also a C4 solution because $F(-b_1) = -F(b_1) = -b_2$, and so on. The existence of these "mirror image" solutions simplifies the problem for we can assume to begin with that $\alpha \geqslant 0$, where

$$\alpha = b_1 + b_2 + b_3 + b_4, \tag{1.2}$$

and we can include appropriate mirror image solutions later.

Iterating equation (1.1) four times gives x_{n+4} as a polynomial of degree 81 in x_n and hence the condition $x_{n+4} = x_n$ produces a polynomial equation of degree 81. We can write this equation as $G(x_n) = x_{n+4} - x_n = 0$. However this condition includes solutions with minimum period two (C2 solutions) and the three equilibrium solutions (C1 solutions) as special cases and these special cases contribute a factor of degree 9 to $G(x_n)$. The remaining factor gives an equation $H(x_n) = 0$, where H is of degree 72 in x_n. Solving this equation directly would be a formidable proposition and the aim of the factorisation method is to split the problem into two simpler steps. As the factorisation method has been applied in earlier papers [2, 3, 4], it will be summarised fairly briefly in this paper.

Section 2 introduces some of the notation that is used later and goes on to discuss a special case where a full solution is possible. Section 3 lists a number of equations which are useful in the general case and Section 4 shows how they can be combined to give the key equation for solving the problem. Section 5 gives an expression for the stability criterion which is then applied to the special case mentioned in Section 2. A result for the case where $a = 4$ is also included.

In the factorisation method the value of α is used to identify a particular solution and the main problem is to know which values of α are appropriate for a given value of a. The key equation in Section 4 is an eighth degree polynomial whose roots provide the desired values for α. The values of b_1 to b_4 then come from a quartic equation and Section 6 discusses the solution of this quartic. It is essential to know when it will give real solutions for the elements and in this problem there are special features which make it easier to decide. In effect, the solution of the quartic is achieved by solving two separate quadratics.

Section 7 lists the numerical results that were obtained. This includes the critical values of a at which the different families of solution appear and the upper limits of a for stability in the cases where stable solutions occur.

2. Introductory ideas and a special case

For a C4 solution (b_1, b_2, b_3, b_4), the basic equation is that

$$b_{i+1} = F(b_i) = ab_i^3 + (1 - a)b_i, \tag{2.1}$$

for $i = 1, 2, 3, 4$, with $b_5 = b_1$. This C4 solution contributes a factor $h(x)$ to $H(x)$, where

$$\begin{aligned} h(x) &= (x - b_1)(x - b_2)(x - b_3)(x - b_4) \\ &= x^4 - \alpha x^3 + \beta x^2 - \gamma x + \delta, \end{aligned} \tag{2.2}$$

an equation which defines α, β, γ, δ as symmetrical functions of the b's. At most there can be 18 factors of this type in $H(x)$ and some of them will be related because they correspond to mirror image solutions. If there is a distinct mirror image solution $(-b_1, -b_2, -b_3, -b_4)$ its contribution to $H(x)$ is a factor

$$\begin{aligned} h^*(x) &= (x + b_1)(x + b_2)(x + b_3)(x + b_4) \\ &= x^4 + \alpha x^3 + \beta x^2 + \gamma x + \delta \end{aligned} \tag{2.3}$$

and the factors $h(x)$ and $h^*(x)$ combine to give

$$\begin{aligned} k(x) = h(x)h^*(x) &= (x^2 - b_1^2)(x^2 - b_2^2)(x^2 - b_3^2)(x^2 - b_4^2) \\ &= x^8 - Ax^6 + Bx^4 - Cx^2 + D, \end{aligned} \tag{2.4}$$

where

$$A = \alpha^2 - 2\beta, \quad B = \beta^2 - 2\alpha\gamma + 2\delta, \quad C = \gamma^2 - 2\beta\delta, \quad D = \delta^2. \tag{2.5}$$

It turns out that at most $H(x)$ provides eight factors of type $k(x)$, where the mirror image solutions are distinct, and two factors of type $h(x)$ where the mirror image solution is the same as the original solution.

When we say that the mirror image solution is not distinct we mean that $(-b_1, -b_2, -b_3, -b_4)$ gives the same C4 solution as (b_1, b_2, b_3, b_4), whether or not the elements are ordered in the same way. This means that we have the same solution if

$$-b_1 = b_1 \quad \text{or} \quad -b_2 = b_1 \quad \text{or} \quad -b_3 = b_1 \quad \text{or} \quad -b_4 = b_1. \tag{2.6}$$

The first of these conditions gives the equilibrium solution $b_1 = 0$, which we can ignore, and the second gives

$$b_3 = F(b_2) = -F(b_1) = -b_2 = b_1.$$

It follows that $b_4 = F(b_3) = F(b_1) = b_2$ and thus we have a C2 solution. In the same way, $b_1 = -b_4$ leads to a C2 solution, so we can ignore both of these cases. This leaves only the case $b_1 = -b_3$ and when this holds

$$b_4 = F(b_3) = -F(b_1) = -b_2,$$

so we have simultaneously

$$b_1 + b_3 = 0, \qquad b_2 + b_4 = 0. \tag{2.7}$$

It follows that $h(x) = (x - b_1)(x - b_2)(x + b_1)(x + b_2)$ and hence

$$\alpha = 0, \quad \beta = -b_1^2 - b_2^2, \quad \gamma = 0, \quad \delta = b_1^2 b_2^2, \tag{2.8}$$

while the basic equations reduce to

$$b_2 = b_1\{ab_1^2 + (1 - a)\}, \tag{2.9}$$

$$b_3 = -b_1 = b_2\{ab_2^2 + (1 - a)\}. \tag{2.10}$$

Multiplying together these equations gives

$$-b_1 b_2 = b_1 b_2\{a^2\delta - a(1 - a)\beta + (1 - a)^2\}.$$

If we assume $b_1 b_2 \neq 0$, to avoid an equilibrium solution, we get

$$-1 = a^2\delta - a(1 - a)\beta + (1 - a)^2. \tag{2.11}$$

Also, if we multiply equation (2.9) by b_1 and equation (2.10) by b_2, then add, the result is

$$0 = a(b_1^4 + b_2^4) + (1 - a)(b_1^2 + b_2^2) = a(\beta^2 - 2\delta) - (1 - a)\beta. \tag{2.12}$$

Eliminating δ gives

$$0 = a^2\beta^2 - 3a(1 - a)\beta + (2a^2 - 4a + 4) \tag{2.13}$$

and hence

$$2a\beta = 3(1 - a) \pm \sqrt{(a^2 - 2a - 7)}. \tag{2.14}$$

This means that the solution for β is not real unless $a^2 - 2a - 7 \geqslant 0$, that is unless $a \geqslant 1 + 2\sqrt{2}$. Thus the critical value for this type of solution is $a^* = 1 + 2\sqrt{2}$. When $a = a^*$, there is a single solution, with $\beta = -(3\sqrt{2})/a^*$, $\delta = 3/(a^*)^2$, $b_1^2 = (3 + \sqrt{3})/(a^*\sqrt{2})$, $b_2^2 = (3 - \sqrt{3})/(a^*\sqrt{2})$.

For $a > a^*$, there are two solutions, one with

$$2a\beta = 3(1 - a) + Q, \qquad 2a^2\delta = (a^2 - 2a - 1) + (1 - a)Q, \tag{2.15}$$

where $Q = \sqrt{(a^2 - 2a - 7)}$, and the other with $+Q$ replaced by $-Q$ in equations (2.15). It can be checked that these relationships lead to real values for b_1 and b_2. The stability of these solutions is discussed later, in Section 5.

It is useful to note that these are the only C4 solutions for which $\alpha = 0$. If we add the four equations (2.1) we get

$$\alpha = \sum b_{i+1} = a\sum b_i^3 + (1 - a)\sum b_i = a\sum b_i^3 + (1 - a)\alpha,$$

where the summation is over $i = 1$ to $i = 4$. Taking $a > 0$, we get

$$\alpha = \sum b_i^3 = \alpha^3 - 3\alpha\beta + 3\gamma$$

and hence

$$\gamma = (\alpha/3)(1 - \alpha^2 + 3\beta). \tag{2.16}$$

This means that if $\alpha = 0$, then γ is also zero. Now $\alpha = 0$ implies that $b_1 + b_3 = -(b_2 + b_4)$ and from this

$$\gamma = b_1 b_3(b_2 + b_4) + b_2 b_4(b_1 + b_3) = (b_1 b_3 - b_2 b_4)(b_2 + b_4). \quad (2.17)$$

Since $\gamma = 0$, we have $b_1 b_3 = b_2 b_4$ or $b_2 + b_4 = 0$. If $b_2 + b_4 = 0$, we are back to equation (2.7) and the C4 solutions already discussed. If $b_1 b_3 = b_2 b_4$, together with $b_1 + b_3 = -(b_2 + b_4)$, then we must have either $b_1 = -b_2$, $b_3 = -b_4$ or $b_1 = -b_4$, $b_3 = -b_2$. Either case leads to a C2 or C1 solution. Thus the only C4 solutions which have $\alpha = 0$ are the special cases considered above.

3. Additional relationship in the general case

The first step in the factorisation method is to obtain equations which allow β, γ and δ to be evaluated for given values of a and α. The basic equations available are equations (2.1) and in obtaining equation (2.16) we have already had an example of how they can be used. In addition we need an equation which allows us to find suitable values of α when a is specified. It turns out that this is a polynomial equation of degree 8 in α^2 and, for a given a, we have to determine the positive real roots of this equation. Each root gives a value of α, which we can take as positive, and from this we get corresponding values of β, γ, δ. Finally, the b_i can be determined as the roots of

$$h(x) = x^4 - \alpha x^3 + \beta x^2 - \gamma x + \delta = 0. \quad (3.1)$$

The laborious part is setting up the required relationships and the equation for α. Once this has been done the computational work is straightforward.

Because equations (2.1) have cyclic symmetry in the elements b_i, rather than full permutational symmetry, it is convenient to subdivide β into two parts, each with cyclic symmetry, and also to use Σ as a cyclic summation symbol. We can write

$$\beta_1 = \sum b_1 b_2 = b_1 b_2 + b_2 b_3 + b_4 b_1, \quad (3.2)$$

$$\beta_2 = b_1 b_3 + b_2 b_4 = (1/2)\sum b_1 b_3, \quad (3.3)$$

with $\beta = \beta_1 + \beta_2$. Purely algebraic expansions, without using equations (2.1), give a number of relationships such as

$$\alpha\gamma - 4\delta - \beta_1\beta_2 = \sum b_1 b_2^2 b_3, \quad (3.4)$$

$$\beta^2 - 2\alpha\gamma + 4\delta - \beta_2^2 = \sum b_1^2 b_2^2, \quad (3.5)$$

$$A\beta_2 = \beta_2 \sum b_1^2 = \sum\left(b_1^3 b_3 + b_1 b_2^2 b_3\right), \quad (3.6)$$

$$A\beta_1 - \beta_1\beta_2 = \sum\left\{(b_2 + b_4) b_1^3\right\}, \quad (3.7)$$

$$\beta_1\left(\beta_2^2 - 2\delta\right) = \sum\left(b_1^3 b_2 b_3^2 + b_1^2 b_2 b_3^3\right), \quad (3.8)$$

as well as the more familiar relationships

$$\sum b_1^2 = A = \alpha^2 - 2\beta, \tag{3.9}$$

$$\sum b_1^4 = A^2 - 2B = \alpha^4 - 4\alpha^2\beta + 2\beta^2 + 4\alpha\gamma - 4\delta, \tag{3.10}$$

$$\sum b_1^6 = A^3 - 3AB + 3C. \tag{3.11}$$

If we take $b_2 = ab_1^3 + (1-a)b_1$ as our paradigm for equation (2.1) we can obtain the relationships

$$\beta_1 = \sum b_1 b_2 = a\sum b_1^4 + (1-a)\sum b_1^2 = a(A^2 - 2B) + (1-a)A, \tag{3.12}$$

$$A = \sum b_2^2 = a\sum b_1^3 b_2 + (1-a)\beta_1, \tag{3.13}$$

$$\beta_1 = \sum b_2 b_3 = a\sum b_1^3 b_3 + 2(1-a)\beta_2, \tag{3.14}$$

$$2\beta_2 = \sum b_2 b_4 = a\sum b_1^3 b_4 + (1-a)\beta_1, \tag{3.15}$$

$$a^2\sum b_1^6 = \sum \{b_2 - (1-a)b_1\}^2 = A(a^2 - 2a + 2) - 2(1-a)\beta_1. \tag{3.16}$$

If we multiply all four of the basic equations together, we get

$$b_2 b_3 b_4 b_1 = (b_1 b_2 b_3 b_4)\prod\{ab_1^2 + (1-a)\},$$

where $\prod$ is used as a cyclic product symbol. We can take $b_1 b_2 b_3 b_4$ as a nonzero factor (to avoid the equilibrium solution $b_i = 0$) and expand the product on the right-hand side. This gives

$$1 = a^4 D + a^3(1-a)C + a^2(1-a)^2 B + a(1-a)^3 A + (1-a)^4. \tag{3.17}$$

Similarly, we can write

$$b_4 - b_2 = (b_3 - b_1)\{a(b_1^2 + b_1 b_3 + b_3^2) + (1-a)\},$$

$$b_3 - b_1 = (b_2 - b_4)\{a(b_2^2 + b_2 b_4 + b_4^2) + (1-a)\}.$$

and take $b_4 \neq b_2$ and $b_3 \neq b_1$ to avoid a C2 solution. Multiplying the equations together and cancelling a nonzero factor $(b_4 - b_2)(b_3 - b_1)$ leaves

$$\begin{aligned} -1 &= \{a(b_1^2 + b_1 b_3 + b_3^2) + (1-a)\}\{a(b_2^2 + b_2 b_4 + b_4^2) + (1-a)\} \\ &= a^2\left(\delta + \sum b_1^2 b_2^2 + \sum b_1 b_2^2 b_3\right) + a(1-a)(A + \beta_2) + (1-a)^2. \end{aligned} \tag{3.18}$$

In the same way,

$$b_4 + b_2 = (b_3 + b_1)\{a(b_1^2 - b_1 b_3 + b_3^2) + (1-a)\}, \tag{3.19}$$

$$b_1 + b_3 = (b_2 + b_4)\{a(b_2^2 - b_2 b_4 + b_4^2) + (1-a)\}, \tag{3.20}$$

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and assuming $(b_1 + b_3)(b_2 + b_4) \neq 0$ leads to

$$1 = a^2\left(\delta + \sum b_1^2 b_2^2 - \sum b_1 b_2^2 b_3\right) + a(1-a)(A - \beta_2) + (1-a)^2. \quad (3.21)$$

By adding equations (3.18) and (3.21)

$$0 = a^2\left(\delta + \sum b_1^2 b_2^2\right) + a(1-a)A + (1-a)^2, \quad (3.22)$$

and it follows that

$$-1 = a^2 \sum b_1 b_2^2 b_3 + a(1-a)\beta_2. \quad (3.23)$$

Other equations are available but we have enough for the moment.

4. Combining the various relationships

From equation (2.16)

$$3\alpha\gamma = \alpha^2\left(1 - \alpha^2 + 3\beta\right) \quad (4.1)$$

and we can combine this with equations (3.4) and (3.23) to give

$$12a^2\delta = a^2\alpha^2\left(1 - \alpha^2 + 3\beta\right) - 3a^2\beta_1\beta_2 + 3 + 3a(1-a)\beta_2. \quad (4.2)$$

From equations (3.5) and (3.22),

$$a^2(2\alpha\gamma - 5\delta) = a^2\left(\beta_1^2 + 2\beta_1\beta_2\right) + a(1-a)A + (1-a)^2 \quad (4.3)$$

and replacing $\alpha\gamma$ and δ from equations (4.1) and (4.2) leads to

$$\begin{aligned} a^2\left(4\beta_1^2 + 3\beta_1\beta_2\right) = {} & -(4a^2 - 8a + 9) + (5a^2 - 4a)\alpha^2 - a^2\alpha^4 \\ & + \beta_1(8a - 8a^2 + 3a^2\alpha^2) + \beta_2(3a - 3a^2 + 3a^2\alpha^2). \end{aligned} \quad (4.4)$$

We can obtain a second relationship between β_1 and β_2 by using equations (3.14) and (3.23) in equation (3.6). This gives

$$2a^2\left(\beta_1\beta_2 + \beta_2^2\right) = 1 - a\beta_1 + (3a - 3a^2 + a^2\alpha^2)\beta_2. \quad (4.5)$$

In the same way, combining equations (3.7), (3.13) and (3.15) leads to

$$a^2\left(2\beta_1^2 + 3\beta_1\beta_2\right) = -a\alpha^2 + (4a - 2a^2 + a^2\alpha^2)\beta_1. \quad (4.6)$$

Equations (4.4), (4.5) and (4.6) provide three relationships between β_1 and β_2 so it should be possible to eliminate β_1 and β_2 and obtain an equation linking a with α. From the theory of resultants [1], this would lead most directly to a 7×7 determinant, although a smaller determinant (4×4) would be possible [5]. It would be necessary also to solve for β_1 and β_2 from these quadratic forms. Instead of pursuing this approach the available equations were manipulated to give three linear relationships between β_1 and β_2, which made it easy to solve for β_1 and β_2 and to write down the equation for α as a 3×3 determinant.

The first step was to solve for β_1^2, $\beta_1\beta_2$ and β_2^2 as linear functions of β_1 and β_2. This gave

$$2a^2\beta_1^2 = -a^2\alpha^4 + (5a^2 - 3a)\alpha^2 - (4a^2 - 8a + 9) + (4a - 6a^2 + 2a^2\alpha^2)\beta_1 + (3a - 3a^2 + 3a^2\alpha^2)\beta_2, \tag{4.7}$$

$$3a^2\beta_1\beta_2 = a^2\alpha^4 + (2a - 5a^2)\alpha^2 + (4a^2 - 8a + 9) + (4a^2 - a^2\alpha^2)\beta_1 + (3a^2 - 3a - 3a^2\alpha^2)\beta_2, \tag{4.8}$$

$$6a^2\beta_2^2 = -2a^2\alpha^4 + (10a^2 - 4a)\alpha^2 - (8a^2 - 16a + 15) + (2a^2\alpha^2 - 3a - 8a^2)\beta_1 + (15a - 15a^2 + 9a^2\alpha^2)\beta_2. \tag{4.9}$$

By using these equations it was straightforward to express β^2, δ, B, C, D, A^2, AB and A^3 as linear functions of β_1 and β_2 and thus make use of equations (3.11), (3.12), (3.16) and (3.17). Equation (3.12) did not give any additional information, although it served as a check. Equations (3.11) and (3.16) gave the linear equation

$$\beta_1 P_1 + \beta_2 Q_1 = R_1, \tag{4.10}$$

where

$$P_1 = 4a^3\alpha^4 + (9a^2 - 20a^3)\alpha^2 + (16a^3 - 45a^2 + 9a), \tag{4.11}$$

$$Q_1 = 15a^3\alpha^4 + (45a^2 - 30a^3)\alpha^2 + (15a^3 - 45a^2), \tag{4.12}$$

$$R_1 = 4a^3\alpha^6 + (17a^2 - 24a^3)\alpha^4 + (36a^3 - 85a^2 + 54a)\alpha^2 + (72 - 99a + 68a^2 - 16a^3). \tag{4.13}$$

A second linear equation came from equation (3.8), where b_1^3 and b_3^3 can be replaced by $(1/a)\{b_2 - (1-a)b_1\}$ and $(1/a)\{b_4 - (1-a)b_3\}$, respectively. This leads to the equation

$$a\beta_1(\beta_2^2 - 2\delta) = \beta_1^2 - \alpha\gamma + a\beta_1\beta_2. \tag{4.14}$$

Some straightforward algebra, using equations (4.7) to (4.9), produced an equation

$$\beta_1 P_2^* + \beta_2 Q_2^* = R_2^*, \tag{4.15}$$

where

$$P_2^* = 6a^3\alpha^4 + (24a^2 - 30a^3)\alpha^2 + (24a^3 - 78a^2 + 54a), \tag{4.16}$$

$$Q_2^* = 12a^3\alpha^4 + (45a^2 - 30a^3)\alpha^2 + (18a^3 - 63a^2 + 45a), \tag{4.17}$$

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$$R_2^* = 4a^3\alpha^6 + (21a^2 - 26a^3)\alpha^4 + (46a^3 - 111a^2 + 75a)\alpha^2 - (24a^3 - 108a^2 + 174a - 135). \tag{4.18}$$

In the later work, equation (4.15) was replaced by

$$\beta_1 P_2 + \beta_2 Q_2 = R_2, \tag{4.19}$$

with

$$P_2 = P_1 - P_2^*, \qquad Q_2 = Q_1 - Q_2^*, \qquad R_2 = R_1 - R_2^*, \tag{4.20}$$

since this gave slightly simpler terms in the linear equation.

Equation (3.17) also leads to a linear equation in β_1 and β_2 but it turned out that this was a linear combination of equations (4.10) and (4.19). It served as a check on the coefficients in these equations and in the numerical calculations equation (3.17) was used as a check on the values obtained for A, B, C and D.

A third linear equation was obtained by combining equations (4.8), (4.9) and (4.19). From equations (4.8) and (4.9), we can write

$$6a^2\beta_1\beta_2 = S_1 + T_1\beta_1 + V_1\beta_2, \qquad 6a^2\beta_2^2 = S_2 + T_2\beta_1 + V_2\beta_2,$$

where S_1, T_1, V_1, S_2, T_2 and V_2 are functions of a and α^2, and from this

$$\begin{aligned} 6a^2R_2\beta_2 &= 6a^2(P_2\beta_1 + Q_2\beta_2)\beta_2 \\ &= P_2(S_1 + T_1\beta_1 + V_1\beta_2) + Q_2(S_2 + T_2\beta_1 + V_2\beta_2). \end{aligned}$$

This gives a linear equation

$$\beta_1 P_3^* + \beta_2 Q_3^* = R_3^*, \tag{4.21}$$

with

$$P_3^* = P_2T_1 + Q_2T_2, \qquad Q_3^* = P_2V_1 + Q_2V_2 - 6a^2R_2, \qquad R_3^* = -P_2S_1 - Q_2S_2.$$

In practice, equation (4.21) was replaced by

$$\beta_1 P_3 + \beta_2 Q_3 = R_3, \tag{4.22}$$

where

$$P_3 = (2/a)P_3^* + (5a - 3 - 5a\alpha^2)P_1, \tag{4.23}$$

with similar definitions for Q_3 and R_3. In more detail,

$$P_3 = -15a^3\alpha^4 + (30a^3 - 72a^2)\alpha^2 + (-15a^3 + 72a^2 + 243a), \tag{4.24}$$

$$Q_3 = 3a^4\alpha^6 + (72a^3 - 33a^4)\alpha^4 + (57a^4 - 144a^3 + 27a^2)\alpha^2 - 27a^4 + 72a^3 + 189a^2 - 54a, \tag{4.25}$$

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$$R_3 = 3a^3\alpha^6 - (18a^3 + 39a^2)\alpha^4$$
$$+ (27a^3 + 60a^2 + 18a)\alpha^2 - 12a^3 - 21a^2 + 9a + 54. \qquad (4.26)$$

Equations (4.10), (4.19) and (4.22) are linearly independent and the determinant

$$\Delta = \begin{vmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{vmatrix} \qquad (4.27)$$

is a polynomial of degree 8 in α^2. If we put $Y = a\alpha^2$, the equation which determines α^2 for a given valule of a can be written as $M(Y, a) = 0$, where

$$M(Y, a) = \sum_{n=0}^{8} c_n Y^{8-n}, \qquad (4.28)$$

with

$$c_0 = 1, \qquad c_1 = 21(1-a), \qquad c_2 = 216 - 330a + 162a^2,$$
$$c_3 = 1359 - 2403a + 1902a^2 - 602a^3,$$
$$c_4 = 4617 - 10728a + 9558a^2 - 5160a^3 + 1173a^4,$$
$$c_5 = 10611 - 25839a + 27108a^2 - 16740a^3 + 7053a^4 - 1233a^5,$$
$$c_6 = 18144 - 32724a + 37260a^2 - 26496a^3 + 13662a^4 - 4710a^5 + 664a^6,$$
$$c_7 = 7290 - 7614a + 14418a^2 - 18630a^3$$
$$+ 11673a^4 - 4617a^5 + 1224a^6 - 144a^7,$$
$$c_8 = -37908 + 48114a - 21465a^2 - 4698a^3$$
$$+ 8424a^4 - 2916a^5 + 324a^6. \qquad (4.29)$$

For a given value of a, the coefficients c_1 to c_8 can be evaluated and we then want the positive real roots of $M(Y, a) = 0$. For each of these roots we get a corresponding value of α, which we can take as positive. Then β_1 and β_2 can be determined from equations (4.10) and (4.19) and β is simply $\beta_1 + \beta_2$. Equation (2.16) gives γ and δ could be obtained from equation (4.2), although in practice the equation used was

$$6a^2\delta = -a^2\alpha^4 + (3a^2 - a)\alpha^2 - (2a^2 - 4a + 3)$$
$$+ 2a^2(\alpha^2 - 1)\beta_1 + (3a - 3a^2 + 3a^2)\beta_2, \qquad (4.30)$$

which arises from combining equations (4.2) and (4.8). A, B, C and D can then be evaluated from equation (2.5).

5. The stability criterion

Apart from the check provided by equation (3.17), A, B, C and D are useful in testing the local stability of a C4 solution. For local stability we must have $|S| < 1$, where

$$S = \prod\{F'(b_1)\} = \prod\{3ab_1^2 + (1-a)\}$$
$$= 81a^4D + 27a^3(1-a)C + 9a^2(1-a)^2B + 3a(1-a)^3A + (1-a)^4. \tag{5.1}$$

In the special case considered in Section 2, both α and γ are zero, so $A = -2\beta$, $B = \beta^2 + 2\delta$, $C = -2\beta\delta$, $D = \delta^2$. In one family of solutions

$$2a\beta = 3(1-a) + Q = Q - 3c, \tag{5.2}$$

$$2a^2\delta = a^2 - 2a - 1 + (1-a)Q = c^2 - 2 - cQ, \tag{5.3}$$

where $c = a - 1 \geqslant 2\sqrt{2}$ and $Q = \sqrt{(a^2 - 2a - 7)} = \sqrt{(c^2 - 8)}$. When these values are substituted into S the result is

$$S = 10c^4 - 90c^2 + 81 + (54c - 6c^3)Q. \tag{5.4}$$

As $a \to 1 + 2\sqrt{2}$ from above, $c \to 2\sqrt{2}$, $Q \to 0$ and $S \to 640 - 720 + 81 = 1$. Thus we have a familiar situation, that the critical value of a corresponds to a cycle which is at the upper limit for stability. If we form the derivative dS/dc, the dominant term as $a \to 1 + 2\sqrt{2}$ from above is $6c^2(9 - c^2)/Q$ and this tends to $+\infty$ as $Q \to 0$. For $a = 4$, we have $c = 3$, $Q = 1$, $S = 81$ and $dS/dc = 432$, so there is a strong presumption that this family of solutions is unstable, with $S > 1$ for $a > a^* = 1 + 2\sqrt{2}$. (This can be shown more rigorously, as indicated below.)

In the other family of solutions, we have to replace Q by $-Q$ in equations (5.2), (5.3) and (5.4). With this change, the dominant term in dS/dc also shows a change in sign and tends to $-\infty$ as $c \to 2\sqrt{2}$ from above. On the other hand, for $a = 4$ we again have $S = 81$ and dS/dc is now 648, which suggests that S decreases to a minimum value between $c = 2\sqrt{2}$ and $c = 3$ and then increases rapidly after the minimum. Numerical evidence confirms this and suggests that the minimum is zero (as happened with one family of C2 solutions). To check on this, put $c = (2\sqrt{2})\cosh u$ and take $u = 0$ as corresponding to $c = 2\sqrt{2}$, with $u = u_0$ corresponding to $c = 3$. Then $\cosh u_0 = 3/(2\sqrt{2})$, $\sinh u_0 = 1/(2\sqrt{2})$ and $\exp u_0 = \sqrt{2}$, with $\cosh 2u_0 = 5/4$ and $\sinh 2u_0 = 3/4$. Note that $Q = (2\sqrt{2})\sinh u$. Our expression for S in this case is

$$S = 10c^4 - 90c^2 + 81 - (54c - 6c^3)Q, \tag{5.5}$$

and in terms of u this becomes

$$S = -39 - 120\sinh 2u - 40\cosh 2u + 48\sinh 4u + 80\cosh 4u$$
$$= -39 - (80\sqrt{2})\sinh(2u + u_0) + 64\cosh(4u + 2u_0). \tag{5.6}$$

Hence

$$dS/du = 32\cosh(2u + u_0)\{16\sinh(2u + u_0) - 5\sqrt{2}\}. \tag{5.7}$$

It is easy to verify that $dS/du = -48$ at $u = 0$ and $dS/du = +648$ at $u = u_0$, with a single zero between 0 and u_0. This corresponds to a minimum value of S and at this minimum

$$\sinh(2u + u_0) = (5\sqrt{2})/(16), \qquad \cosh(4u + 2u_0) = 89/64,$$

which gives $S = 0$ at the minimum. The minimum occurs when $c^2 = (27 + 9\sqrt{17})/8$, which corresponds to $a = 3.8308$.

For this family of solutions there is a small range of values of a for which the C4 solutions are stable. The upper limit occurs when S is again equal to 1, after passing through the minimum. If we replace $2u + u_0$ by U in equation (5.6) we get $S = 1$ when

$$0 = -40 - (80\sqrt{2})\sinh U + 64(1 + 2\sinh^2 U),$$

an equation whose solutions are

$$\sinh U = 1/(2\sqrt{2}) \text{ and } \sinh U = (3\sqrt{2})/8. \tag{5.8}$$

The first of these corresponds to $u = 0$ and the second gives $c^2 = \{13 + 3\sqrt{(41)}\}/4$. If we use a^{**} for the corresponding value of a, the stability interval is $a^* < a < a^{**}$, with $a^* = 3.828427$, $a^{**} = 3.837665$.

When β is given by equation (5.2) and S by equation (5.4), we can use the same substitution, $c = (2\sqrt{2})\cosh u$, to discuss the change in S as u increases from 0 to u_0. In this case

$$S = -39 + (80\sqrt{2})\sinh(2u - u_0) + 64\cosh(4u - 2u_0), \tag{5.9}$$

$$dS/du = 32\cosh(2u - u_0)\{16\sinh(2u - u_0) + 5\sqrt{2}\}, \tag{5.10}$$

which gives $dS/du > 0$ for $0 \leqslant u \leqslant u_0$.

Another result which arises from equation (5.1) is that $S = \pm 81$ for $a = 4$. It was noted previously [4] that for $a = 4$, equation (1.1) has a general solution $x_n = \cos(3^n\phi)$, where $x_0 = \cos\phi$, and this leads to cyclic solutions for suitable values of ϕ. In particular, $x_{n+4} = x_n$ when

$$\text{(i) } \phi = N\pi/41, \quad \text{or} \quad \text{(ii) } \phi = N\pi/40, \tag{5.11}$$

for any integer N. As it stands, this includes the C1 and C2 solutions as special cases but we can pick out the independent C4 solutions by choosing suitable values of N. For $\phi = N\pi/41$, there are five independent solutions with $\alpha > 0$ and they can be obtained by taking $N = 1, 2, 4, 7$ and 8. Each of these solutions has a mirror image solution, making ten distinct solutions in all. For $\phi = N\pi/40$, there are three independent solutions with $\alpha > 0$, obtained by taking $N = 1, 2$ and 7. Again, each of these has a mirror image solution. The C4 solutions which have $\alpha = 0$ correspond to $\phi = 4\pi/40$ and $\phi = 5\pi/40$. Thus there are in all eighteen of these trigonometric C4 solutions for $a = 4$, as foreshadowed in the earlier discussion.

These trigonometric solutions were invaluable for testing the equations in Sections 3 and 4 and for testing computer programmes in the numerical work.

If we take $a = 4$ and $b_1 = \cos\phi$, then

$$3ab_1^2 + (1 - a) = 12\cos^2\phi - 3 = 3(\sin 3\phi)/(\sin\phi). \tag{5.12}$$

(Alternatively, equation (5.12) can be obtained by using

$$F'(b_1) = db_2/db_1 = (db_2/d\phi)/(db_1/d\phi).)$$

In the same way

$$3ab_2^2 + (1 - a) = 3(\sin 9\phi)/(\sin 3\phi),$$

and so on, with the result that

$$S = 81(\sin 81\phi)/(\sin\phi). \tag{5.13}$$

For $\phi = N\pi/40$, $81\phi = 2N\pi + \phi$ and $\sin 81\phi = \sin\phi$, so $S = +81$ for $\phi = N\pi/40$. A similar argument gives $81\phi = 2N\pi - \phi$ and $S = -81$ when $\phi = N\pi/41$. These results help to identify which families of solutions should have stability intervals.

A result which does not arise from equation (5.1) but which relates to stability is that for $0 < a \leqslant 2$ and $|x_0| < 1$ all solutions of equation (1.1) converge to zero as $n \to \infty$. For $0 < a \leqslant 1$ this global property is fairly obvious, since convergence is from one side only. For $1 < a \leqslant 2$, x_n and x_{n+1} can have opposite signs but for $0 < |x_n| < 1$

$$r_n = x_{n+1}/x_n = ax_n^2 + (1 - a) = (1 + c)x_n^2 - c. \tag{5.14}$$

As before, $c = a - 1$ and the restriction on a gives $0 < c \leqslant 1$. It follows that

$$1 - r_n = (1 + c)(1 + x_n^2) > 0,$$
$$1 + r_n = (1 + c)x_n^2 + (1 - c) > 0.$$

Thus $|r_n| < 1$ and this ensures that $|x_{n+1}| < |x_n|$. Hence if $0 < |x_0| < 1$ the values of x_n either move closer to zero at each step or jump to zero and stay there. Because of this global convergence property for $0 < a \leqslant 2$, any C4 solutions must occur for $a > 2$ and the numerical work was restricted accordingly.

6. Solution for elements of C4 cycle

Although S can be evaluated from equation (5.1) without knowing b_1 to b_4 explicitly, it is useful to have some way of testing whether or not the values found for α, β_1, β_2, β, γ and δ lead to a real solution for the b's. One approach to this is to use equations (3.19) and (3.20), which can be rewritten as

$$(b_2 + b_4)/(b_1 + b_3) = a(b_1 + b_3)^2 - 3ab_1b_3 + 1 - a, \qquad (6.1)$$

$$(b_1 + b_3)/(b_2 + b_4) = a(b_2 + b_4)^2 - 3ab_2b_4 + 1 - a. \qquad (6.2)$$

This assumes that $\beta_1 = (b_1 + b_3)(b_2 + b_4) \neq 0$, which means that we are excluding the solutions for which $\alpha = 0$ (Section 2). These cases are easy to deal with separately.

If we assume $b_1 + b_3 = b_2 + b_4 \neq 0$, then equations (6.1) and (6.2) give $b_1b_3 = b_2b_4$ and as a result *either* $b_1 = b_2$, with $b_3 = b_4$ *or* $b_1 = b_4$ with $b_2 = b_3$. In either case we get an equilibrium solution. Thus we can deduce that for a C4 solution, with $\alpha \neq 0$, $b_1 + b_3$ and $b_2 + b_4$ are unequal. We shall assume that $b_1 + b_3 > b_2 + b_4$ for the rest of this section. Since $\alpha = (b_1 + b_3) + (b_2 + b_4)$ and $\beta_1 = (b_1 + b_3)(b_2 + b_4)$, we can obtain $b_1 + b_3$ and $b_2 + b_4$ as the roots of

$$Z^2 - \alpha Z + \beta_1 = 0. \qquad (6.3)$$

If $\alpha^2 < 4\beta_1$, this equation has complex roots and we can stop immediately. For unequal real roots we must have $\alpha^2 > 4\beta_1$ and we can write

$$b_1 + b_3 = (1/2)(\alpha + R_1), \qquad b_2 + b_4 = (1/2)(\alpha - R_1), \qquad (6.4)$$

where $R_1 = \sqrt{(\alpha^2 - 4\beta_1)}$. This means that

$$(b_1 + b_3)^2 - (b_2 + b_4)^2 = \alpha R_1, \qquad (6.5)$$

$$\{(b_1 + b_3)/(b_2 + b_4)\} - \{(b_2 + b_4)/(b_1 + b_3)\} = \alpha R_1/\beta_1. \qquad (6.6)$$

If we subtract equation (6.2) from (6.1) and use equations (6.5) and (6.6)

$$3a(b_1b_3 - b_2b_4) = \{a + (1/\beta_1)\}\alpha R_1. \qquad (6.7)$$

Taking $\alpha > 0$, we get the result that

$$b_1b_3 > b_2b_4 \text{ provided } a + (1/\beta_1) > 0. \qquad (6.8)$$

This latter condition held in all the cases that were considered, so we can take b_1b_3 as the larger root of the equation

$$Z^2 - \beta_2 Z + \delta = Z^2 - (b_1b_3 + b_2b_4)Z + b_1b_2b_3b_4 = 0. \qquad (6.9)$$

The roots of this equation must be real, since $b_1b_3 + b_2b_4 = \beta_2$ is real and $b_1b_3 - b_2b_4$ is real, from equation (6.7). Hence we can write

$$b_1b_3 = (1/2)(\beta_2 + R_2), b_2b_4 = (1/2)(\beta_2 - R_2), \tag{6.10}$$

where

$$R_2 = b_1b_3 - b_2b_4 = \sqrt{(\beta_2^2 - 4\delta)}. \tag{6.11}$$

From equation (6.7)

$$R_2 = (\alpha R_1/3)\{1 + (a\beta_1)^{-1}\}. \tag{6.12}$$

We can now solve for b_1 and b_3, using $b_1 + b_3 = (1/2)(\alpha + R_1)$ and $b_1b_3 = (1/2)(\beta_2 + R_2)$. If the roots are real and we take $b_1 > b_3$, then

$$b_1 = (1/4)(\alpha + R_1 + D_1), \qquad b_3 = (1/4)(\alpha + R_1 - D_1), \tag{6.13}$$

where

$$\begin{aligned} D_1^2 &= (\alpha + R_1)^2 - 8(\beta_2 + R_2) \\ &= 2\alpha^2 - 4\beta_1 - 8\beta_2 - (2\alpha R_1/3)\{1 + (4/a\beta_1)\}. \end{aligned} \tag{6.14}$$

Unless $D_1^2 > 0$ the roots will not be real and distinct, since $D_1^2 = 0$ gives a C2 solution and $D_1^2 < 0$ gives complex values. In the same way, b_2 and b_4 are given by $(1/4)(\alpha - R_1 \pm D_2)$, where

$$D_2^2 = 2\alpha^2 - 4\beta_1 - 8\beta_2 + (2\alpha R_1/3)\{1 + (4/a\beta_1)\}, \tag{6.15}$$

provided $D_2^2 > 0$. Thus a sufficient condition for the solution to be real is that

$$W = \mathrm{Min}(\alpha^2 - 4\beta_1, D_1^2, D_2^2) > 0. \tag{6.16}$$

Two minor notes are that $1 + (4/a\beta_1)$ is sometimes positive and sometimes negative, so $D_2^2 - D_1^2$ is not always positive, and that the conditions $b_1 + b_3 > b_2 + b_4$, with $b_1 > b_3$, do not imply that b_1 is the largest root. However, once the roots are known it is easy to arrange them in a suitable order.

7. Numerical results

The numerical work was carried out on a Univac 1100 computer, using double precision. An outline of the steps involved is given at the end of Section 4 and equations for b_1, b_2, b_3, b_4 are given in Section 6. Only values of a between 2 and 4 were used. The first step was to evaluate $M(Y, a)$ over a suitable range of values of Y and a, to see how $M(Y, a)$ behaved and in particular to look for zeros. It was a surprise to find that $M(Y, 2) = 0$ had a positive root and this root, which we shall call Y_4, could be followed throughout the interval $2 \leqslant a \leqslant 4$. Two additional roots appeared between $a = 3.55$ and $a = 3.6$ and a smaller root came in about 3.85. Finally, four larger roots appeared, in pairs, between $a = 3.95$ and $a = 4.0$. For $a = 4$, all eight roots were recorded and they agreed with the values expected from the trigonometrical solutions. The occurrence of a single root for a polynomial of even degree, as recorded for $2 \leqslant a \leqslant 3.55$, looked a little bit strange but in fact it was accompanied by a negative root which we can label Y_1. For $a = 2$, Y_1 was -2.68 and it increased smoothly with a, passing through zero around $a = 3.85$ and being identified after that as a small positive root. We can think of Y_1 and Y_4 as a pair of roots which occur throughout the interval [2, 4] even although Y_1 is negative for part of this range and hence of no interest as far as real C4 solutions are concerned. Indeed it turns out that the Y_4 root does not always give real solutions either and it is this property which saves us from having C4 solutions for $a < 1 + \sqrt{5}$.

Apart from these families of roots, there were indications of an isolated zero at $a = 2.535$, $Y = 6.354$. Regarded as a function of two variables, $M(Y, a)$ has a minimum in this area and it appeared that the minimum was zero. This was checked more carefully and values of M were obtained which were zero to 9 or 10 decimal places. The smallest value recorded on the computer print-outs was -1×10^{-11} for $a = 2.535092575$, $Y = 6.3542741$. As Y^8 equals 2.66×10^6 in this case, the fact that all the terms cancelled to an accuracy of $10^{-16} Y^8$ makes a good case for treating the minimum as zero. For these values of a and Y, equations (4.10), (4.19) and (4.22) gave essentially the same linear relationship between β_1 and β_2, that is the ratios

$$P_1 : Q_1 : R_1, \quad P_2 : Q_2 : R_2, \quad P_3 : Q_3 : R_3,$$

were the same to about eight decimal places. Because of this the usual method of determining β_1 and β_2 was ruled out. Instead, equation (4.10) was used to express β_2 as a linear function of β_1 and this expression was substituted for β_2 in equation (4.7), which gave a quadratic equation for β_1. The quadratic equation proved to have complex roots and this verified that the isolated zero did not lead to a real C4 solution.

Once the general pattern of the roots had been established, solutions were carried out for the C4 cycle corresponding to pairs (Y, a) which gave $M(Y, a) = 0$. The Y-values for $a = 4$ were arranged in ascending order and labelled $Y_{0,1}$, $Y_{0,2}$, Y_1, $Y_2, \ldots, Y_8$, where $Y_{0,1} = Y_{0,2} = 0$, and we can use these labels to identify corresponding families of roots. We can take $Y_{0,1}$ to refer to solutions with $\alpha = 0$ and $2a\beta = 3(1 - a) - Q$, while $Y_{0,2}$ refers to solutions with $\alpha = 0$ and $2a\beta = 3(1 - a) + Q$. From Sections 2 and 5, the $Y_{0,1}$ solutions are stable for $a^* < a < a^{**}$, where the appropriate values for a^* and a^{**} are listed in Table 1. The $Y_{0,2}$ solutions are unstable for $a > a^*$.

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TABLE 1. Critical values for solutions of period four and data for solutions with $a = 4$.

Family of solutions	a^* a^{**}	Stable sequence	Data for solutions with $a = 4$			
			Root	Numerical Value	ϕ	S
$Y_{0,1}$, $Y_{0,2}$	3.828427 3.837665	$Y_{0,1}$	$Y_{0,1}$ $Y_{0,2}$	0.0 0.0	$4\pi/40$ $5\pi/40$	+81 +81
Y_1	3.837665 3.842116	Y_1	Y_1	0.119006	$4\pi/41$	−81
Y_2, Y_3	3.547835 3.548831	Y_3	Y_2 Y_3	0.513167 1.430098	$7\pi/40$ $7\pi/41$	+81 −81
Y_4	3.236068 3.288032	Y_4	Y_4	1.479355	$8\pi/41$	−81
Y_5, Y_6	3.967535 3.967556	Y_5	Y_5 Y_6	9.379146 10.0	$2\pi/41$ $2\pi/40$	−81 +81
Y_7, Y_8	3.991929 3.991938	Y_8	Y_7 Y_8	19.486833 20.512395	$\pi/40$ $\pi/41$	+81 −81

For the Y_1 solutions, the critical value, a^*, occurs when $c_8 = M(0, a) = 0$. For $a < a^*$, c_8 is negative and $M(Y, a)$ has a negative root. For $a > a^*$, c_8 is positive and Y_1 takes positive values. It was noted that a^* is slightly less than 3.85 and this suggests that the Y_1 solutions come in when the $Y_{0,1}$ solutions become unstable. For $a = 3.84$, the Y_1 solution gave

$$b_1 = 0.520350, \qquad b_2 = -0.936768, \qquad b_3 = -0.496237, \qquad b_4 = 0.940069,$$

and it will be seen that $b_1 + b_3$ and $b_2 + b_4$ are both close to zero. For the $Y_{0,1}$ solutions, $b_1 + b_3$ and $b_2 + b_4$ are exactly zero, so we can guess that the Y_1 solution and the $Y_{0,1}$ solution are similar. (For comparison, the $Y_{0,1}$ and $Y_{0,2}$ solutions for $a = 3.84$ are (0.511212, −0.938822, −0.511212, 0.938822) and (0.456836, −0.931304, −0.456836, 0.931304). It will be seen that all three solutions are similar but the Y_1 solution is slightly closer to the $Y_{0,1}$ than to the $Y_{0,2}$ solution.) The numerical checks confirmed that $c_8 = 0$ when $a = 3.837665$ and we can take this as the critical value for the Y_1 solutions. For this critical value the appropriate solution is the $Y_{0,1}$ solution and this gives $S = 1$. As a increases, S decreases, passes through the value −1 and becomes −81 for $a = 4$. The upper limit of a for stability occurs when $S = -1$ and an appropriate value for a was determined by interpolating from a table of values of S.

For the Y_4 family of roots, the corresponding C4 solutions are complex for $a < 1 + \sqrt{5}$. For $a > 1 + \sqrt{5}$ the solutions are real, with $S < 1$. For $a = 1 + \sqrt{5}$ there is a C2 solution, with $b_1 = 1/\sqrt{2}$ and $b_2 = (1 - \sqrt{5})/(2\sqrt{2})$ and with

$$\{3ab_1^2 + (1 - a)\}\{3ab_2^2 + (1 - a)\} = -1. \tag{7.1}$$

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If we think of this as the limiting case of a C4 solution, with $b_3 \to b_1$ and $b_4 \to b_2$ as $a \to 1 + \sqrt{5}$ from above, then this limiting C4 solution has $S = 1$ and also $a\beta_1 = -4$, $2\alpha^2 = 4\beta_1 + 8\beta_2$. It follows that D_1^2 and D_2^2 are both zero (from equations (6.14) and (6.15)). For $a > 1 + \sqrt{5}$, the C4 solutions are real, with D_1^2 and D_2^2 both positive. Something of this kind was to be anticipated, for equations (2.1) imply that b_2 and b_4 must be real when b_1 and b_3 are real. (Indeed as soon as one of the elements is real the others must be real.) For $a > 1 + \sqrt{5}$, S decreases as a increases and the limiting value of a for stability occurs when $S = -1$.

The Y_2 and Y_3 roots have a maximum of M between them (for a given value of a) and the critical value, a^*, occurs when this maximum is zero and Y_2, Y_3 coincide. For $a \geqslant a^*$, the C4 solutions are real, with $S = 1$ at $a = a^*$. As a increases, S decreases for the Y_3 family of solutions and there is a small interval within which the solutions are stable. For the Y_2 family of solutions, S increases with a and the C4 solutions are unstable for $a > a^*$.

For the Y_5 and Y_6 roots the pattern is similar. For a given value of a, there is a minimum of M between them and the critical value occurs when the minimum is zero. For this critical value, Y_5 and Y_6 coincide and $S = 1$. For larger values of a, Y_5 and Y_6 are distinct and correspond to real C4 solutions. For the Y_5 solutions there is a small interval of stability but the Y_6 solutions become unstable for $a > a^*$.

The Y_7 and Y_8 roots also come in as a pair, with a minimum of M between the two roots. At the critical value of a, the two roots coincide and $S = 1$. Each root leads to a real C4 solution. For the Y_8 solutions, S decreases as a increases and there is a small stability interval. For the Y_7 solutions, S increases with a and the solutions are unstable.

The appropriate values of a^* and a^{**} are listed in Table 1. It will be seen that the largest stability interval has a width of 0.052, which gives a very sharp decrease in width compared with the largest interval for C2 solutions. The table also gives some information for the solutions which correspond to $a = 4$. In particular, there is a column to indicate the range of values of Y that is involved. The value of 10 for Y_6 is an exact result and the values for Y_2 and Y_7 are $10 - 3\sqrt{10}$ and $10 + 3\sqrt{10}$. This means that $Y - 10$ and $Y^2 - 20Y + 10$ are factors of $M(Y, 4)$ and this was verified as a check on the coefficients in $M(Y, a)$.

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