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SOME CONSEQUENCES OF MARTIN'S AXIOM AND THE NEGATION OF THE CONTINUUM HYPOTHESIS

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§ 0. W. Sierpisnki [3] demonstrated 82 propositions, called C_1–C_{82}, with the aid of the continuum hypothesis. D. A. Martin and R. M. Solovay remarked in [2] that 48 of these propositions followed from Martin's axiom (MA), 23 were refuted by MA $+ 2^{\aleph_0} > \aleph_1$ and three were independent of MA $+ 2^{\aleph_0} > \aleph_1$. But the relation of the remaining eight propositions to MA $+ 2^{\aleph_0} > \aleph_1$ has been unsettled.

In this paper, we shall show at least five of them (C_8, C_{13}, C_{61}, C_{62} and C_{70}) are also refuted by MA $+ 2^{\aleph_0} > \aleph_1$.

The following table gives the relation of C_1–C_{82} to MA $+ 2^{\aleph_0} > \aleph_1$.

	0	1	2	3	4	5	6	7	8	9
		×	○	○	○	○	×	○	×*	×
10	×	×	×	×*	○	×	○	○	○	○
20	○	○	○	○	○	○	×	×	×	×
30	○	○	×	×	×	○	○	○	○	○
40	○	○	○	○	○	○	○	?	?	○
50	×	×	△	○	○	○	○	○	○	○
60	○	×*	×*	○	○	×	×	×	×	×
70	×*	○	○	○	○	○	○	×	△	×
80	?	△	○							

By ○, we denote the propositions following from MA, by × the propositions refuted by MA $+ 2^{\aleph_0} > \aleph_1$, by △ the propositions independent of MA $+ 2^{\aleph_0} > \aleph_1$ and by ? the propositions whose relation to MA $+ 2^{\aleph_0} > \aleph_1$ we do not know about at present.

Let $\mathscr{P} = \langle P, \leq \rangle$ be a partially ordered set. A subset X of P is said to be dense in $\mathscr{P}$ if, for every $p \in P$, there is $q \in X$ such that $p \leq q$. If $\mathscr{F}$ is a collection of dense subsets of P, a subset G of P is said to be an $\mathscr{F}$-generic filter on $\mathscr{P}$ if G has the following properties:

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(1) if $p, q \in P$, $p \in G$ and $q \leq p$, then $q \in G$;

(2) if $p, q \in G$, then there is $r \in G$ such that $p \leq r$ and $q \leq r$;

(3) if $X \in \mathscr{F}$, then $X \cap G \neq 0$.

If $p, q \in P$, then p and q are said to be compatible if there is $r \in P$ such that $p \leq r$ and $q \leq r$. $\mathscr{P}$ is said to have the countable chain condition if every collection of pairwise incompatible elements of P is countable.

Martin's axiom (MA) is the following statement:

If $\mathscr{P} = \langle P, \leq \rangle$ is a partially ordered set having the countable chain condition and $\mathscr{F}$ is a collection of dense open subsets of P of cardinality $< 2^{\aleph_0}$, then there exists an $\mathscr{F}$-generic filter on $\mathscr{P}$.

§1. In this section, we shall show C_8, C_9, C_{61} and C_{62} are refuted by MA $+ \neg$CH. From [2], we quote the following lemma.

LEMMA 1. *Let A and B be collections of subsets of ω, each of cardinality $< 2^{\aleph_0}$, such that if $x \in B$ and K is a finite subset of A then $x - \cup K$ is infinite. If we assume* MA, *then there exists a subset t of ω such that $x \cap t$ is finite if $x \in A$ and infinite if $x \in B$.*

Let ${}^{\omega}\omega$ be the set of all functions from ω into ω, (more generally, x_y be the set of all functions from x into y). Following Sierpinski [3], we define a partial ordering $<$ on ${}^{\omega}\omega$ as follows:

$$f < g \leftrightarrow (\exists k \in \omega)(\forall n \geq k)[f(n) < g(n)] .$$

The following lemma is due to K. Kunen [1].

LEMMA 2. *Let F be a subset of ${}^{\omega}\omega$ of cardinality $< 2^{\aleph_0}$. If we assume* MA, *then there exists $g \in {}^{\omega}\omega$ such that if $f \in F$ then $f < g$.*

From Lemma 2, we have the following proposition, which is the negation of C_9.

PROPOSITION 1 (*Assume* MA *and* $2^{\aleph_0} > \aleph_1$). *Let E be an uncountable subset of $\boldsymbol{R}$, the set of reals, and $\langle f_n : n \in \omega \rangle$ be a convergent sequence of functions from E to $\boldsymbol{R}$. Then there exists an uncountable subset N of E such that $\langle f_n : n \in \omega \rangle$ is uniformly convergent on N.*

Proof. We may assume E is of cardinality $\aleph_1$. Let f be the limit of $\langle f_n : n \in \omega \rangle$. Then for any $x \in E$ and $m \in \omega$, there is $k \in \omega$ such that if $n \geq k$ then $|f_n(x) - f(x)| < 1/m + 1$. Take such $k \in \omega$ and denote it by $\varphi_x(m)$. Then we can define $\aleph_1$ functions φ_x from ω into ω. Using

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Lemma 2, we can find $\varphi \in {}^{\omega}\omega$ such that $\varphi_x < \varphi$ for all $x \in E$. For each $x \in E$, let k_x denote the least $k \in \omega$ such that $\varphi_x(m) < \varphi(m)$ for all $m \geq k$. Since E is uncountable, there is $k \in \omega$ and an uncountable subset N of E such that if $x \in N$ then $k_x = k$. Then for any $x \in N$ and $m \geq k$, if $n \geq \varphi(m)$ then $|f_n(x) - f(x)| < 1/m + 1$. This means $\langle f_n : n \in \omega \rangle$ converges uniformly to f on N.

Since C_8 and C_9 are equivalent, C_8 is also refuted by MA + $2^{\aleph_0} > \aleph_1$.

Recall that an F_σ-set is the union of a countable family of closed sets and a G_δ-set is the intersection of a countable family of open sets.

LEMMA 3.[1] *Let X be a separable metric space of cardinality $< 2^{\aleph_0}$. If we assume* MA, *then every subset of X is F_σ and G_δ in X.*

Proof. Let D be any subset of X and $\{B_i : i \in \omega\}$ be a basis for open sets of X such that all B_i are non-empty. For each $x \in X$, let $s_x = \{i \in \omega : x \in B_i\}$. If we put $A = \{s_x : x \in X - D\}$ and $B = \{s_y : y \in D\}$, then A and B are of cardinality $2^{\aleph_0}$. It is easily checked that if $y \in D$ and $x_1, \cdots, x_n \in X - D$ then $s_y - (s_{x_1} \cup \cdots \cup s_{x_m})$ is infinite. By Lemma 1, we can find a subset t of ω such that $s_x \cap t$ is finite if $x \in X - D$ and $s_y \cap t$ is infinite if $y \in D$. For each $n \in \omega$, let

$$K_n = \bigcup_{\substack{i > n \\ i \in t}} B_i .$$

And let $K = \bigcap_{n \in \omega} K_n$. Then K is a G_δ-set of X. In order to prove that D is a G_δ-set of X, it suffices to prove the following (1) and (2):

$$(1) \qquad D \subseteq K$$

$$(2) \qquad (X - D) \cap K = 0 .$$

Let y be an arbitrary element of D and $n \in \omega$. Since $t \cap s_y$ is infinite, there is $i \in t \cap s_y$ such that $i > n$. Then $y \in B_i$ and $B_i \subseteq K_n$, so $y \in K_n$. Since y and n are arbitrary, we have (1). Let x be any element of $X - D$. Since $t \cap s_x$ is finite, there is $n \in \omega$ such that if $i \in t$ and $i > n$ then $i \notin s_x$. For such $n \in \omega$, we have $x \notin K_n$, and so $x \notin K$. Thus we have (2).

Replacing D with $X - D$, we have that $X - D$ is a G_δ-set of X. Hence D is an F_σ-set of X. Therefore D is F_σ and G_σ in X.

[1] This lemma is a slight generalization of that of J. Silver.

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The following proposition is the negation of C_{62}.

PROPOSITION 2. (*Suppose* MA *and* $2^{\aleph_0} > \aleph_1$). *Let E be any uncountable set of reals and f be any function from E into $\boldsymbol{R}$, the set of reals. Then there exists an uncountable subset N of E such that $f \restriction N$, the restriction of f to N, is continuous on N.*

Proof. We may assume E is of cardinality $\aleph_1$. Let F be an arbitrary closed set in $\boldsymbol{R}$. Then, by Lemma 3, $f^{-1}(F)$, the inverse image of F, is a G_δ-set of E. Thus f_a is Baire function of class $\leqq 1$. As is well-known, every Baire function of class ≤ 1 whose range is a subset of $\boldsymbol{R}$ is the limit of a sequence of continuous functions. Let $\langle f_n : n \in \omega \rangle$ be a sequence of continuous functions from E to $\boldsymbol{R}$ which converges to f. Then, by Proposition 1, there exists an uncountable subset N of E such that $\langle f_n : n \in \omega \rangle$ converges uniformly to f on N. Since each $f_n \restriction N$ is continuous on N, so is $f \restriction N$.

This proposition implies the following proposition, which is the negation of C_{61}.

PROPOSITION 3. (*Suppose* MA *and* $2^{\aleph_0} > \aleph_1$). *There is a subset F of ${}^{R}\boldsymbol{R}$ of cardinality $2^{\aleph_0}$ such that if $g \in {}^{R}\boldsymbol{R}$ then for some $f \in F$ the set $\{x \in \boldsymbol{R} : f(x) = g(x)\}$ is uncountable.*

Proof. Let F be the set of Baire functions from $\boldsymbol{R}$ into $\boldsymbol{R}$. Then clearly, F is of cardinality $2^{\aleph_0}$. By Proposition 2, if $g \in {}^{R}\boldsymbol{R}$, then there exists an uncountable subset N of $\boldsymbol{R}$ such that $g \restriction N$ is continuous on N. The following is a well-known theorem.

Let X be an arbitrary metric space, let Y be a complete separable space and A be a subset of X. Then every Baire function from A to Y can be extended to a Baire function from X into Y.

Since $f \restriction N$ is a Baire function on N, by this theorem, there exists $f \in F$ such that $f \restriction N = g \restriction N$. Thus the set $\{x \in \boldsymbol{R} : f(x) = g(x)\}$ includes N, and is uncountable.

§ **2.** Let $[\omega]^{\aleph_0}$ denote the set of all infinite subsets of ω. We define a relation $\subseteq^*$ on $[\omega]^{\aleph_0}$ as follows:

$$a \subseteq^* b \leftrightarrow a - b \text{ is finite, where } a, b \in [\omega]^{\aleph_0} .$$

Intuitively $a \subseteq^* b$ iff $a \subseteq b$ almost everywhere.

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LEMMA.[1] *Suppose MA. Let Θ be an ordinal such that $\Theta < 2^{\aleph_0}$, and let $\langle a_\alpha : \alpha < \Theta\rangle$ be a sequence of elements of $[\omega]^{\aleph_0}$ such that if $\alpha < \beta < \Theta$ then $a_\beta \subseteq^* a_\alpha$. Then there exists $a \in [\omega]^{\aleph_0}$ such that if $\alpha < \Theta$ then $a \subseteq^* a_\alpha$.*

Proof. Let $A = \{\omega - a_\alpha : \alpha < \Theta\}$ and $B = \{a_\alpha : \alpha < \Theta\}$. Then clearly, A and B are of cardinality $< 2^{\aleph_0}$. If $\alpha, \alpha_1, \cdots, \alpha_n < \Theta$, then

$$a_\alpha - \bigcup_{i=1}^{n} (\omega - a_{\alpha_i}) = a_\alpha \cap a_{\alpha_1} \cap \cdots \cap a_{\alpha_n} .$$

It is easily checked the intersection of finite elements of B is an element of $[\omega]^{\aleph_0}$. Thus A and B satisfy the condition of Lemma 1 of § 1. Therefore there is a subset a of ω such that $a - a_\alpha$ is finite and $a \cap a_\alpha$ is infinite for any $\alpha < \Theta$. For such $a \subseteq \omega$, we have $a \in [\omega]^{\aleph_0}$ and $a \subseteq^* a_\alpha$.

From this lemma, we obtain the following proposition, which is the negation of C_{13}.

PROPOSITION. (*Assume* MA *and* $2^{\aleph_0} > \aleph_1$). *Let $\langle f_n : n \in \omega\rangle$ be a sequence of functions from $\boldsymbol{R}$ to $\boldsymbol{R}$. Then there exists a sequence $\langle m_k : k \in \omega\rangle$ of natural numbers such that $m_0 < m_1 < \cdots < m_k < \cdots$ and the set $\{x \in \boldsymbol{R} : \langle f_{m_k}(x) : k \in \omega\rangle$ converges to a finite or infinite value$\}$ is uncountable.*

Proof.[2] For each $a \in [\omega]^{\aleph_0}$, let a' denote the sequence $\langle n_k : k \in \omega\rangle$ such that $n_0 < n_1 < \cdots < n_k < \cdots < \cdots$ and $a = \{n_k : k \in \omega\}$. By the limit of the sequence $\langle f_n(x) : n \in a\rangle$, we mean the limit of the sequence $\langle f_{n_k}(x) : k \in \omega\rangle$ in the usual sense, where $\langle n_k : k \in \omega\rangle = a'$. Let E be a subset of $\boldsymbol{R}$ of cardinality $\aleph_1$. Order E into a transfinite sequence of type ω_1 as follows:

$$x_0, x_1, \cdots, x_\alpha, \cdots \qquad (\alpha < \omega_1)$$

By transfinite induction on α, we define a sequence $\langle a_\alpha : \alpha < \omega_1\rangle$ of elements of $[\omega]^{\aleph_0}$ such that $a_\beta \subseteq^* a_\alpha$ if $\alpha < \beta < \omega_1$ and the sequences $\langle f_n(x_\alpha) : n \in a_\alpha\rangle$ with $\alpha \in \omega_1$ are convergent. The sequence $\langle f_n(x_0) : n \in \omega\rangle$ includes a convergent subsequence $\langle f_{n_k}(x_0) : k \in \omega\rangle$, whose limit is finite or infinite. So, we define a_0 to be $\{n_k : k \in \omega\}$. Assume that a_β with $\beta < \alpha$ are defined and $a_\gamma \subseteq^* a_\beta$ if $\beta < \gamma < \alpha$. Then, by the above lemma, we can find $a \in [\omega]^{\aleph_0}$ such that $a \subseteq^* a_\beta$ for all $\beta < \alpha$. The sequence $\langle f_i(x_\alpha) :$

[1] It was pointed out by the referee that this lemma could be proved from Lemma 2 of § 1.

[2] This proof was suggested to the author by Professor Kanji Namba.

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$i \in a\rangle$ includes a convergent subsequence $\langle f_{i_k}(x_\alpha): k \in \omega\rangle$. So, we define a_α to be $\{i_k: k \in \omega\}$.

By the lemma of this section, let b be an element of $[\omega]^{\aleph_0}$ such that $b \subseteq^* a_\alpha$ for all $\alpha < \omega_1$. For every $\alpha < \omega_1$, since $b \subseteq^* a_\alpha$, the sequence $\langle f_m(x_\alpha); m \in b\rangle$ is convergent. If we put $\langle m_k: k \in \omega\rangle = b'$, then the set $\{x: \langle f_{m_k}(x): k \in \omega\rangle$ is convergent$\}$ includes E, and is uncountable.

§ **3.** Let E be a subset of $\boldsymbol{R}$ and $a \in \boldsymbol{R}$. By $E(a)$ we denote the set $\{x + a: x \in E\}$.

Without MA, we can prove the following proposition.

PROPOSITION. (*Suppose* $2^{\aleph_0} > \aleph_1$). *If* E *is an uncountable subset of* $\boldsymbol{R}$ *such that its complement is of cardinality* $2^{\aleph_0}$, *then there exists* $a \in \boldsymbol{R}$ *such that* $E(a)\triangle E$, *the symmetric difference of* $E(a)$ *and* E, *is uncountable.*

Proof. Suppose, on the contrary, that for any $a \in \boldsymbol{R}$, $E(a)\triangle E$ is countable. Let N be a subset E of cardinality $\aleph_1$. Then we show $\bigcap_{x \in N}[\boldsymbol{R} - E(-x)] \neq 0$. If $\bigcap_{x \in N}[\boldsymbol{R} - E(-x)] = 0$, then $\boldsymbol{R} = \bigcup_{x \in N} E(-x)$. On the other hand

$$\bigcup_{x \in N} E(-x) = \bigcup_{x,y \in N}[E(-x)\triangle E(-y)] \cup \bigcap_{x \in N} E(-x) .$$

Therefore,

$$A \cup \bigcap_{x \in N} E(-x) = \boldsymbol{R} , \quad \text{where} \quad A = \bigcup_{x,y \in N}[E(-x)\triangle E(-y)] .$$

Since A and $\bigcap_{x \in N} E(-x)$ are disjoint, we have $\boldsymbol{R} - \bigcap_{x \in N} E(-x) = A$. Let x be an arbitrary element of N. Then we have $\boldsymbol{R} - E(-x) \subseteq A$. Note that each $E(a)\triangle E(b)$ is countable because $E(a)\triangle E(b) = J(a) \cup K(b)$, where $J = E(b - a)\triangle E$, $K = E(a - b)\triangle E$. Therefore A is of cardinality $\leq \aleph_1$. This contradicts the hypothesis that the complement of E is of cardinality $2^{\aleph_0}$. Thus $\bigcap_{x \in N}[\boldsymbol{R} - E(-x)] \neq 0$.

Let $a \in \bigcap_{x \in N}[\boldsymbol{R} - E(-x)]$, then $N \subseteq \boldsymbol{R} - E(-a)$ because $a \notin E(-x)$ iff $x \notin E(-a)$. Therefore $E(-a)\triangle E$ includes N, and is uncountable.

The following corollary is the negation of C_{70}.

COROLLARY. (*suppose* MA *and* $2^{\aleph_0} > \aleph_1$). *Let* E *be a non-measurable set of reals. Then for some* $a \in \boldsymbol{R}$, $E(a)\triangle E$ *is uncountable.*

Proof. If we assume MA, then every set of reals of cardinalily

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$< 2^{\aleph_0}$ is of Lebesgue measure 0 ([2, § 4]). Hence, if E is non-measurable, the E and its complement are of cardinality $2^{\aleph_0}$. Thus E satisfies the condition of the proposition.

§ **4.** A set E of reals is said to have the property (M)[1] if, for any collection $\mathscr{U}$ of open sets satisfying the condition

$$(*) \qquad (\forall x \in E)(\forall \varepsilon > 0)(\exists U \in \mathscr{U})[\delta(U) < \varepsilon \wedge x \in U]$$

where $\delta(U)$ is the diameter of U, there is a sequence $\langle U_n : n \in \omega \rangle$ of elements of $\mathscr{U}$ such that $E \subseteq \bigcup_{n \in \omega} U_n$ and $\lim_{n \to \infty} \delta(U_n) = 0$.

As a direct application of MA, we have the following proposition.

PROPOSITION. (*Suppose* MA). *Every set of reals of cardinality* $< 2^{\aleph_0}$ *has the property* (M).

Proof. Let E be a set of reals of cardinality $< 2^{\aleph_0}$, and $\mathscr{U}$ be a collection of open sets satisfying the condition $(*)$. For each $n \in \omega$, there is a sequence $\langle U_{nm} : m \in \omega \rangle$ of elements of $\mathscr{U}$ such that $E \subseteq \bigcup_{m \in \omega} U_{nm}$ and $\delta(U_{nm}) < 1/n + 1$ for all $m \in \omega$. We define a partially ordered set $\mathscr{P} = \langle P, \leq \rangle$ as follows:

$$P = \{p : p \text{ is a finite function with } \operatorname{dom}(p) \cup \operatorname{rang}(p) \subseteq \omega\}, ,$$

$$p \leq q \leftrightarrow p \subseteq q .$$

Then clearly, $\mathscr{P}$ satisfies the countable chain condition. For each $x \in E$, if we put $X_x = \{p \in P : x \in \bigcup_{n \in \operatorname{dom}(p)} U_{np(n)}\}$, then X_x is dense in $\mathscr{P}$. Let $\mathscr{F} = \{X_x : x \in E\}$. Then $\mathscr{F}$ is of cardinality $< 2^{\aleph_0}$, so there is an $\mathscr{F}$-generic filter G on $\mathscr{P}$. If we put $f = \bigcup G$, then f is a function with $\operatorname{dom}(f) \subseteq \omega$ and $\operatorname{rang}(f) \subseteq \omega$. We define U_n as follows:

$$U_n = \begin{cases} U_{nf(n)} & \text{if } n \in \operatorname{dom}(f) \\ U_{n0} & \text{otherwise} \end{cases}$$

Then, clearly, $U_n \in \mathscr{U}$ and $\lim_{n \to \infty} \delta(U_n) = 0$. Let x be an arbitrary element of E. Since $X_x \cap G \neq 0$, there is $p \in G$ such that $x \in \bigcup_{n \in \operatorname{dom}(p)} U_{np(n)}$. Since $P \in G$, we have $\bigcup_{n \in \operatorname{dom}(p)} U_{np(n)} \subseteq \bigcup_{n \in \omega} U_n$, so $x \in \bigcup_{n \in \omega} U_n$. Therefore E has the property (M).

[1] See [3, p. 48]

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§5. A set E of reals is said to have the property (λ)[1] if every countable subset of E is a G_δ-set of E.

In this section, we shall show there is a non-measurable set of reals of cardinality $2^{\aleph_0}$ with the property (λ).

A set E of reals is said to have the property $(\boldsymbol{S}^*)$[2] if, for every set N of Lebesgue measure 0, $E \cap N$ is of cardinality $< 2^{\aleph_0}$. If a set E is measurable and has positive measure, then E includes a set of measure 0 and cardinality $2^{\aleph_0}$. If we assume MA, then every set of reals of cardinality $< 2^{\aleph_0}$ is of Lebesgue measure 0. Therefore every set of reals of cardinality $2^{\aleph_0}$ with the property $(\boldsymbol{S}^*)$ is non-measurable. The existence of a non-measurable set of reals of cardinality $2^{\aleph_0}$ with the property (λ) follows from the following proposition.

PROPOSITION. (*Suppose* MA). *There is a set E of reals of cardinality $2^{\aleph_0}$ with the property $(\boldsymbol{S}^*)$ such that every subset of E of cardinality $<$ $2^{\aleph_0}$ is G_δ in E.*

Proof. Order the set of all G_δ-sets of measure 0 into a transfinite sequence of type $2^{\aleph_0}$ as follows:

$$N_0, N_1, \cdots, N_\xi, \cdots, (\xi < 2^{\aleph_0}) .$$

By transfinite induction on α, we define a sequence $\langle x_\alpha : \alpha < 2^{\aleph_0} \rangle$ of reals and a sequence $\langle K_\alpha : \alpha < 2^{\aleph_0} \rangle$ of G_δ-sets of measure 0. Let $K_0 = N_0$ and x_0 be an arbitrary element of $\boldsymbol{R}$. Suppose x_β and K_β with $\beta < \alpha$ are defined, and let

$$S_\alpha = \bigcup_{\beta<\alpha} K_\beta \cup \{x_\beta : \beta < \alpha\} \cup N_\alpha .$$

Then, by MA, S_α is of measure 0, so $\boldsymbol{R} - S_\alpha \neq 0$. Let x_α be an arbitrary element of $\boldsymbol{R} - S_\alpha$ and K_α be the first N_ξ such that $S_\alpha \cup \{x_\alpha\} \subseteq N_\xi$.

Let E be the set $\{x_\alpha : \alpha < 2^{\aleph_0}\}$. Then we have

(1) E is of cardinality $2^{\aleph_0}$;

(2) for each $\alpha < 2^{\aleph_0}$, $E \cap N_\alpha$ is of cardinality $< 2^{\aleph_0}$;

(3) $K_\alpha \subseteq K_\beta$ if $\alpha < \beta < 2^{\aleph_0}$.

From (1) and (2), E is a set of cardinality $2^{\aleph_0}$ with the property $(\boldsymbol{S}^*)$.

Let D be an arbitrary subset of E of cardinality $< 2^{\aleph_0}$. Since $2^{\aleph_0}$ is a regular cardinal, there is $\alpha < 2^{\aleph_0}$ such that $D \subseteq \{x_\beta : \beta \leq \alpha\}$. Put

[1] See [3, p. 94]

[2] Cf. [3, p. 81]

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$X = \{x_\beta : \beta \leq \alpha\}$. Then, by Lemma 3 of §1, D is a G_δ-set in X. Since $X = E \cap K_\alpha$ and K_α is G_δ in R, X is G_δ in E. Therefore D is a G_δ-set in E.

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