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SPLICING n-CONVEX FUNCTIONS USING SPLINES

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ABSTRACT. It is proved that a regular piecewise n-convex function differs from an n-convex function only by a polynomial spline of degree $n-1$. The argument is given in terms of Peano and de la Vallée Poussin derivatives.

1. Introduction. Let $x_0, x_1, \ldots, x_n$ be $(n+1)$ distinct points from $[a, b]$. By $V_n(F)$ we denote the nth divided difference of F:

$$V_n(F) \equiv \sum_{k=0}^{n} F(x_k)/w_n'(x_k),$$

where $w_n(x) = (x-x_0)(x-x_1)\cdots)x-x_n)$,

If $V_n(F) \geq 0$ for all choices of points $x_0, x_1, \ldots, x_n$ in $[a, b]$ then F is said to be n-convex on $[a, b]$. An n-convex function f on $[a, b]$ will be said to be *regular* if $f_{(n-1),+}(a)$ and $f_{(n-1),-}(b)$ are both finite. A *regular piecewise n-convex* function is a function which on each of the subintervals of a finite partition of a finite interval is regular n-convex.

Let P be a partition $t_0 < t_1 < \cdots < t_n$ of $[a, b]$. A polynomial spline of degree d, $d = 0, 1, 2, \ldots$ and smoothness k ($k \geq -1$, k integral) is any function $s(t) \in C^k[a, b]$ which reduces to a polynomial of degree d on each subinterval (t_{i-1}, t_i) of $[a, b]$ where $C^{-1}[a, b]$ denotes the class of functions with finite discontinuities on $[a, b]$. The set of all such piecewise polynomials is denoted by $S_d(\pi, k)$. (See, for example, [4]).

In the present paper we prove that a regular piecewise n-convex function differs from an n-convex function only by a polynomial spline of degree $n-1$. The argument is given in terms of Peano and de la Vallée Poussin derivatives. (See, e.g. [1]).

2. Splicing n-convex functions.

THEOREM 2.1. *Suppose $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of the finite interval $[a, b]$. Suppose g_i is a regular, n-convex function defined on $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, N$. Then there exists a spline function $L \in C^{-1}[a, b]$ of degree $n-1$*

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with nodes $x_0, x_1, \ldots, x_n$ *such that*

$$G(x) \equiv g_i(x) + L(x), \qquad x \in [x_{i-1}, x_i], \qquad i = 1, 2, \ldots, N,$$

is n*-convex on* $[a, b]$.

(In other words, a regular piecewise n-convex function differs from an n-convex function only by a spline of degree $n-1$).

Proof. Clearly we need prove only the case $N = 2$. Consider the function G defined by

$$G(x) = \begin{cases} g_1(x), & x \in [x_0, x_1], \\ g_2(x) + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 & \\ \qquad \equiv g_2(x) + P_{n-1}(x) & x \in [x_1, x_2], \end{cases}$$

where the n coefficients of $P_{n-1}(x)$ are determined by the n conditions

$$g_1^{(k)}(x_1) - g_2^{(k)}(x_1) = P_{n-1}^{(k)}(x_1), \qquad k = 0, 1, 2, \ldots, n-2,$$

$$g_{1(n-1),-}(x_1) - g_{2(n-1),+}(x_1) = P_{n-1}^{(n-1)}(x_1).$$

Then $G^{(k)}(x_1)$, $k = 0, 1, 2, \ldots, n-2$, and $G_{(n-1)}(x_1)$ exist, and $G(x)$ is n-smooth at x_1 (cf [3], §8).

Since $G(x)$ is n-convex on the subintervals $[x_0, x_1]$ and $[x_1, x_2]$ we have that $G(x)$ possesses derivatives $G_r(x)$, $0 \le r \le n-2$ on $[x_0, x_2]$ and a derivative $G_{(n-1)}(x)$ except at a countable number of points on $[a, b]$ ([1], Theorem 7). It follows ([3], §8) that $G(x)$ is n-smooth except on a countable set in $[a, b]$. This verifies that $G(x)$ satisfies condition A_n^* (cf [2]) on $[a, b]$. Also it is clear that $\bar{D}^n G(x) \ge 0$ in $(a, b) - \{x_1\}$. Since we have shown that $G(x)$ is n-smooth at $x = x_1$, it follows that $G(x)$ is n-convex in $[a, b]$. ([2], Theorem 2.2).

We give now an extension of the result of Theorem 2.1.

Let $\{a_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty}$ be two monotonic sequences in the interval $[a, b]$ such that

$$\cdots < a_k < a_{k-1} < \cdots < a_1 < b_1 < b_2 < \cdots < b_k \cdots$$

where $a_k \to a$ and $b_k \to b$. Let f be defined by

$$f(x) = \begin{cases} g_0(x), & x \in [a_1, b_1), \\ g_i(x), & x \in [a_{i+1}, a_i), \qquad i = 1, 2, \ldots \\ h_i(x), & x \in [b_i, b_{i+1}), \qquad i = 1, 2, \ldots, \end{cases}$$

where $g_0(x)$ is regular n-convex on $[a_1, b_1]$, $g_i(x)$ and $h_i(x)$ are regular n-convex on $[a_{i+1}, a_i]$ and $[b_i, b_{i+1}]$, $i = 1, 2, \ldots$, respectively. Then there exists a function $L_k(x) \in C^{-1}[a_k, b_k]$ which is a polynomial spline of degree n

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such that the function $f(x)+L_k(x)$ is n-convex on $[a_k, b_k]$. If f_k is defined by

$$f_k(x)=\begin{cases} f(x)+L_k(x), & x\in[a_k, b_k], \\ p_k(x), & x\in[a, a_k], \\ q_k(x), & x\in[b_k, b], \end{cases}$$

where $p_k(x)$ and $q_k(x)$ are polynomials of degree $(n-1)$ the coefficients of which are determined so that

$$p_k^{(j)}(a_k-)=f_{k(j)+}(a_k), \qquad g_k^{(j)}(b_k+)=f_{k(j)-}(b_k), \qquad 0\le j\le n-1,$$

then $f_k(x)$ (and hence $\lim_{k\to\infty} f_k(x)$) is n-convex on $[a, b]$.

References

1. P. S. Bullen, *A Criterion For n-Convexity*, Pacific J. Math. vol. **36** (1971) pp. 81–98.
2. G. E. Cross, *The P^n-integral*, Canad. Math. Bull. vol. **18** (1975) pp. 493–497.
3. R. D. James, *Generalized n^{th} primitives*, Trans. Amer. Math. Soc. vol. **76** (1954) pp. 149–176.
4. P. M. Prenter, *Splines and Variational Methods*, John Wiley, 1975.

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